PART I GAME THEORY

1 The Rules of the Game

1.1: Definitions

Game theory is concerned with the actions of decision makers who are conscious that their actions affect each other. When the only two publishers in a city choose prices for their newspapers, aware that their sales are determined jointly, they are players in a game with each other. They are not in a game with the readers who buy the newspapers, because each reader ignores his effect on the publisher. Game theory is not useful when decisionmakers ignore the reactions of others or treat them as impersonal market forces.

The best way to understand which situations can be modelled as games and which cannot is to think about examples like the following:

1. OPEC members choosing their annual output;
2. General Motors purchasing steel from USX;
3. two manufacturers, one of nuts and one of bolts, deciding whether to use metric or American standards;
4. a board of directors setting up a stock option plan for the chief executive officer;
5. the US Air Force hiring jet fighter pilots;
6. an electric company deciding whether to order a new power plant given its estimate of demand for electricity in ten years.

The first four examples are games. In (1), OPEC members are playing a game because Saudi Arabia knows that Kuwait's oil output is based on Kuwait's forecast of Saudi output, and the output from both countries matters to the world price. In (2), a significant portion of American trade in steel is between General Motors and USX, companies which realize that the quantities traded by each of them affect the price. One wants the price low, the other high, so this is a game with conflict between the two players. In (3), the nut and bolt manufacturers are not in conflict, but the actions of one do affect the desired actions of the other, so the situation is a game none the less. In (4), the board of directors chooses a stock option plan anticipating the effect on the actions of the CEO.

Game theory is inappropriate for modelling the final two examples. In (5), each individual pilot affects the US Air Force insignificantly, and each pilot makes his employment decision without regard for the impact on the Air Force's policies. In (6), the electric company faces a complicated decision, but it does not face another rational agent. These situations are more appropriate for the use of **decision theory** than game theory, decision theory being the careful analysis of how one person makes a decision when he may be faced with uncertainty, or an entire sequence of decisions that interact with each other, but when he is not faced with having to interact strategically with other single decision makers. Changes in the important economic variables could,however, turn examples (5) and (6) into games. The appropriate model changes if the Air Force faces a pilots' union or if the public utility commission pressures the utility to change its generating capacity.

Game theory as it will be presented in this book is a modelling tool, not an axiomatic system. The presentation in this chapter is unconventional. Rather than starting with mathematical definitions or simple little games of the kind used later in the chapter, we will start with a situation to be modelled, and build a game from it step by step.

Describing a Game

The essential elements of a game are **players**, **actions**, **payoffs**, and **information**– PAPI, for short. These are collectively known as the **rules of the game**, and the modeller's objective is to describe a situation in terms of the rules of a game so as to explain what will happen in that situation. Trying to maximize their payoffs, the players will devise plans known as **strategies** that pick actions depending on the information that has arrived at each moment. The combination of strategies chosen by each player is known as the **equilibrium**. Given an equilibrium, the modeller can see what actions come out of the conjunction of all the players' plans, and this tells him the **outcome** of the game.

This kind of standard description helps both the modeller and his readers. For the modeller, the names are useful because they help ensure that the important details of the game have been fully specified. For his readers, they make the game easier to understand, especially if, as with most technical papers, the paper is first skimmed quickly to see if it is worth reading. The less clear a writer's style, the more closely he should adhere to the standard names, which means that most of us ought to adhere very closely indeed.

Think of writing a paper as a game between author and reader, rather than as a single-player production process. The author, knowing that he has valuable information but imperfect means of communication, is trying to convey the information to the reader. The reader does not know whether the information is valuable, and he must choose whether to read the paper closely enough to find out.[1]

To define the terms used above and to show the difference between game theory and decision theory, let us use the example of an entrepreneur trying to decide whether to start a dry cleaning store in a town already served by one dry cleaner. We will call the two firms "NewCleaner" and "OldCleaner." NewCleaner is uncertain about whether the economy will be in a recession or not, which will affect how much consumers pay for dry cleaning, and must also worry about whether OldCleaner will respond to entry with a price war or by keeping its initial high prices. OldCleaner is a well-established firm, and it would survive any price war, though its profits would fall. NewCleaner must itself decide whether to

[1]Once you have read to the end of this chapter: What are the possible equilibria of this game?

initiate a price war or to charge high prices, and must also decide what kind of equipment to buy, how many workers to hire, and so forth.

Players *are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.*

In the Dry Cleaners Game, let us specify the players to be NewCleaner and OldCleaner. Passive individuals like the customers, who react predictably to price changes without any thought of trying to change anyone's behavior, are not players, but environmental parameters. Simplicity is the goal in modelling, and the ideal is to keep the number of players down to the minimum that captures the essence of the situation.

Sometimes it is useful to explicitly include individuals in the model called **pseudo-players** whose actions are taken in a purely mechanical way.

Nature *is a pseudo-player who takes random actions at specified points in the game with specified probabilities.*

In the Dry Cleaners Game, we will model the possibility of recession as a move by Nature. With probability 0.3, Nature decides that there will be a recession, and with probability 0.7 there will not. Even if the players always took the same actions, this random move means that the model would yield more than just one prediction. We say that there are different **realizations** of a game depending on the results of random moves.

An **action** *or* **move** *by player i, denoted a_i, is a choice he can make.*

Player i's **action set**, $A_i = \{a_i\}$, *is the entire set of actions available to him.*

An **action combination** *is a list* $a = \{a_i\},\ (i = 1, \ldots, n)$ *of one action for each of the n players in the game.*

Again, simplicity is our goal. We are trying to determine whether Newcleaner will enter or not, and for this it is not important for us to go into the technicalities of dry cleaning equipment and labor practices. Also, it will not be in Newcleaner's interest to start a price war, since it cannot possibly drive out Oldcleaners, so we can exclude that decision from our model. Newcleaner's action set can be modelled very simply as $\{Enter, Stay\ Out\}$. We will also specify Oldcleaner's action set to be simple: it is to choose price from $\{Low, High\}$.

By player i's **payoff** $\pi_i(s_1, \ldots, s_n)$, *we mean either:*
(1) The utility player i receives after all players and Nature have picked their strategies and the game has been played out; or
(2) The expected utility he receives as a function of the strategies chosen by himself and the other players.

For the moment, think of "strategy" as a synonym for "action". Definitions (1) and (2) are distinct and different, but in the literature and this book the term "payoff" is used for both the actual payoff and the expected payoff. The context will make clear which is meant. If one is modelling a particular real-world situation, figuring out the payoffs is often the hardest part of constructing a model. For this pair of dry cleaners, we will pretend we have looked over all the data and figured out that the payoffs are as given by Table 1a if the economy is normal, and that if there is a recession the payoff of each player who operates in the market is 60 thousand dollars lower, as shown in Table 1b.

Table 1a: The Dry Cleaners Game: Normal Economy

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-100, -50	100, 100
	Stay Out	0,50	0,300

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Table 1b: The Dry Cleaners Game: Recession

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-160, -110	40, 40
	Stay Out	0,-10	0,240

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Information is modelled using the concept of the **information set**, a concept which will be defined more precisely in Section 2.2. For now, think of a player's information set as his knowledge at a particular time of the values of different variables. The elements of the information set are the different values that the player thinks are possible. If the information set has many elements, there are many values the player cannot rule out; if it has one element, he knows the value precisely. A player's information set includes not only distinctions between the values of variables such as the strength of oil demand, but also knowledge of what actions have previously been taken, so his information set changes over the course of the game.

Here, at the time that it chooses its price, OldCleaner will know NewCleaner's decision about entry. But what do the firms know about the recession? If both firms know about the recession we model that as Nature moving before NewCleaner; if only OldCleaner knows, we put Nature's move after NewCleaner; if neither firm knows whether there is a recession at the time they must make their decisions, we put Nature's move at the end of the game. Let us do this last.

It is convenient to lay out information and actions together in an **order of play**. Here is the order of play we have specified for the Dry Cleaners Game:

1 Newcleaner chooses its entry decision from $\{Enter, Stay\ Out\}$.
2 Oldcleaner chooses its price from $\{Low, High\}$.
3 Nature picks demand, D, to be *Recession* with probability 0.3 or $Normal$ with probability 0.7.

The purpose of modelling is to explain how a given set of circumstances leads to a particular result. The result of interest is known as the outcome.

The **outcome** *of the game is a set of interesting elements that the modeller picks from the values of actions, payoffs, and other variables after the game is played out.*

The definition of the outcome for any particular model depends on what variables the modeller finds interesting. One way to define the outcome of the Dry Cleaners Game would be as either *Enter* or *Stay Out*. Another way, appropriate if the model is being constructed to help plan NewCleaner's finances, is as the payoff that NewCleaner realizes, which is, from Tables 1a and 1b, one element of the set {*0, 100, -100, 40, -160*}.

Having laid out the assumptions of the model, let us return to what is special about the way game theory models a situation. Decision theory sets up the rules of the game in much the same way as game theory, but its outlook is fundamentally different in one important way: there is only one player. Return to NewCleaner's decision about entry. In decision theory, the standard method is to construct a **decision tree** from the rules of the game, which is just a graphical way to depict the order of play.

Figure 1 shows a decision tree for the Dry Cleaners Game. It shows all the moves available to NewCleaner, the probabilities of states of nature (actions that NewCleaner cannot control), and the payoffs to NewCleaner depending on its choices and what the environment is like. Note that although we already specified the probabilities of Nature's move to be 0.7 for $Normal$, we also need to specify a probability for OldCleaner's move, which is set at probability 0.5 of *Low price* and probability 0.5 of *High price*.

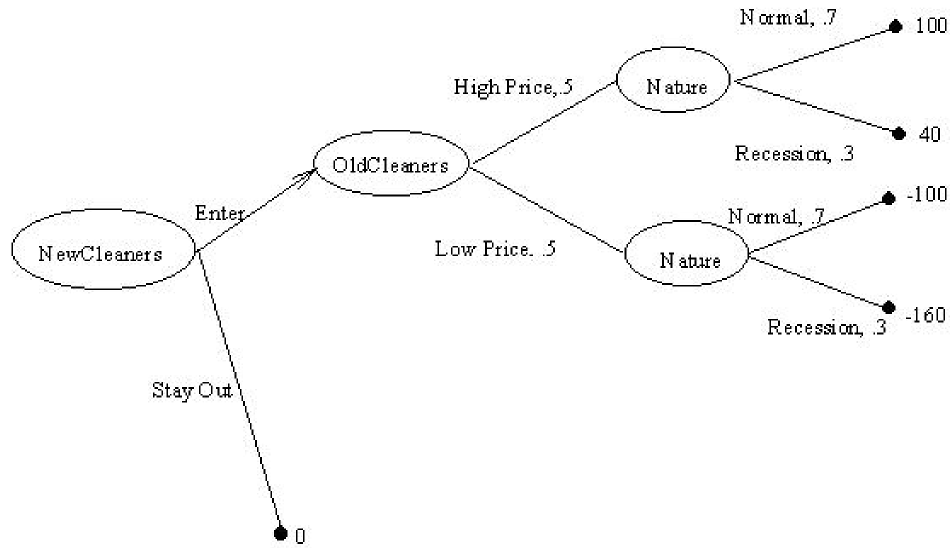

Figure 1: The Dry Cleaners Game as a Decision Tree

Once a decision tree is set up, we can solve for the optimal decision which maximizes the expected payoff. Suppose NewCleaner has entered. If OldCleaner chooses a high price, then NewCleaner's expected payoff is 82, which is 0.7(100) + 0.3(40). If OldCleaner chooses a low price, then NewCleaner's expected payoff is -118, which is 0.7(-100) + 0.3(-160). Since there is a 50-50 chance of each move by OldCleaner, NewCleaner's overall expected payoff from *Enter* is -18. That is worse than the 0 which NewCleaner could get by choosing *stay out*, so the prediction is that NewCleaner will stay out.

That, however, is wrong. This is a game, not just a decision problem. The flaw in the reasoning I just went through is the assumption that OldCleaner will choose *High price* with probability 0.5. If we use information about OldCleaner' payoffs and figure out what moves OldCleaner will take in solving its own profit maximization problem, we will come to a different conclusion.

First, let us depict the order of play as a **game tree** instead of a decision tree. Figure 2 shows our model as a game tree, with all of OldCleaner's moves and payoffs.

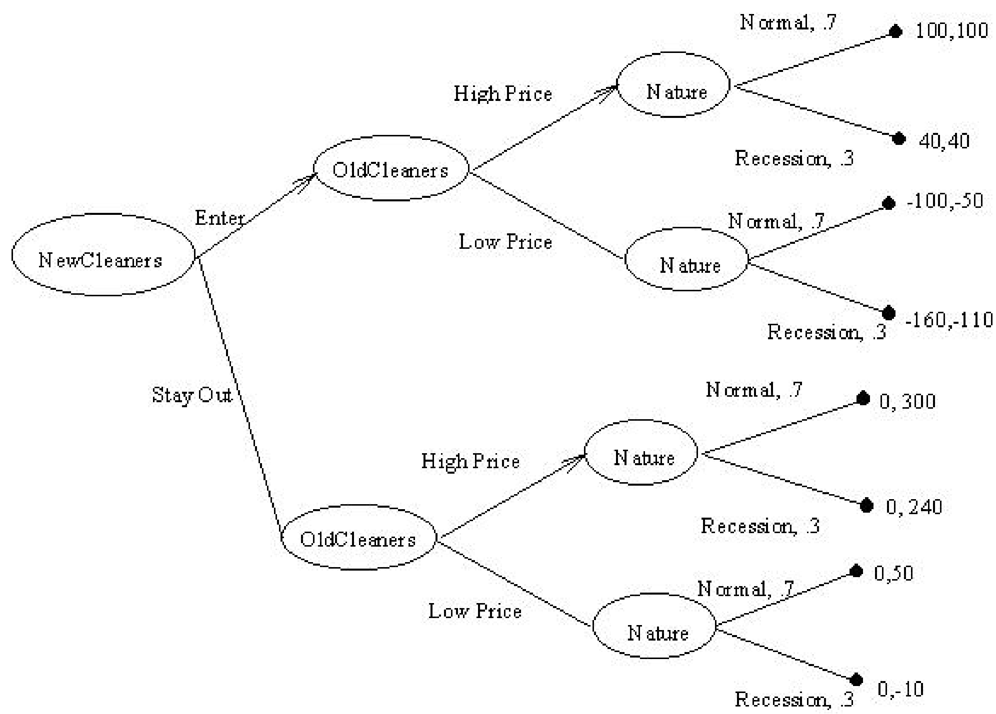

Figure 2: The Dry Cleaners Game as a Game Tree

Viewing the situation as a game, we must think about both players' decision making. Suppose NewCleaner has entered. If OldCleaner chooses *High price*, OldCleaner's expected profit is 82, which is 0.7(100) + 0.3(40). If OldCleaner chooses *Low price*, OldCleaner's expected profit is -68, which is 0.7(-50) + 0.3(-110). Thus, OldCleaner will choose *High price*, and with probability 1.0, not 0.5. The arrow on the game tree for *High price* shows this conclusion of our reasoning. This means, in turn, that NewCleaner can predict an expected payoff of 82, which is 0.7(100) + 0.3(40), from *Enter*.

Suppose NewCleaner has not entered. If OldCleaner chooses *High price*, OldCleaner' expected profit is 282, which is 0.7(300) + 0.3(240). If OldCleaner chooses *Low price*, OldCleaner's expected profit is 32, which is 0.7(50) + 0.3(-10). Thus, OldCleaner will choose *High price*, as shown by the arrow on *High price*. If NewCleaner chooses *Stay out*, NewCleaner will have a payoff of 0, and since that is worse than the 82 which NewCleaner can predict from *Enter*, NewCleaner will in fact enter the market.

This switching back from the point of view of one player to the point of view of another is characteristic of game theory. The game theorist must practice putting himself in *everybody* else's shoes. (Does that mean we become kinder, gentler people? – Or do we just get trickier?)

Since so much depends on the interaction between the plans and predictions of different players, it is useful to go a step beyond simply setting out actions in a game. Instead, the modeller goes on to think about **strategies**, which are action plans.

Player i's **strategy** s_i *is a rule that tells him which action to choose at each instant of the game, given his information set.*

Player i's **strategy set** *or* **strategy space** $S_i = \{s_i\}$ *is the set of strategies available to him.*

A **strategy profile** $s = (s_1, \ldots, s_n)$ *is a list consisting of one strategy for each of the* n *players in the game.* [2]

Since the information set includes whatever the player knows about the previous actions of other players, the strategy tells him how to react to their actions. In The Dry Cleaners Game, the strategy set for NewCleaner is just { *Enter, Stay Out* } , since NewCleaner moves first and is not reacting to any new information. The strategy set for OldCleaner, though, is

$$\left\{ \begin{array}{l} \text{High Price if NewCleaner Entered, Low Price if NewCleaner Stayed Out} \\ \text{Low Price if NewCleaner Entered, High Price if NewCleaner Stayed Out} \\ \text{High Price No Matter What} \\ \text{Low Price No Matter What} \end{array} \right\}$$

The concept of the strategy is useful because the action a player wishes to pick often depends on the past actions of Nature and the other players. Only rarely can we predict a player's actions unconditionally, but often we can predict how he will respond to the outside world.

Keep in mind that a player's strategy is a complete set of instructions for him, which tells him what actions to pick in every conceivable situation, even if he does not expect to reach that situation. Strictly speaking, even if a player's strategy instructs him to commit suicide in 1989, it ought also to specify what actions he takes if he is still alive in 1990. This kind of care will be crucial in Chapter 4's discussion of "subgame perfect" equilibrium. The completeness of the description also means that strategies, unlike actions, are unobservable. An action is physical, but a strategy is only mental.

Equilibrium

To predict the outcome of a game, the modeller focusses on the possible strategy profiles, since it is the interaction of the different players' strategies that determines what happens. The distinction between strategy profiles, which are sets of strategies, and outcomes, which

[2] I used "strategy combination" instead of "strategy profile" in the third edition, but "profile" seems well enough established that I'm switching to it.

are sets of values of whichever variables are considered interesting, is a common source of confusion. Often different strategy profiles lead to the same outcome. In The Dry Cleaners Game, the single outcome of *NewCleaner Enters* would result from either of the following two strategy profiles:

$$\left\{ \begin{array}{l} \textit{High Price if NewCleaner Enters, Low Price if NewCleaner Stays Out} \\ \textit{Enter} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \textit{Low Price if NewCleaner Enters, High Price if NewCleaner Stays Out} \\ \textit{Enter} \end{array} \right\}$$

Predicting what happens consists of selecting one or more strategy profiles as being the most rational behavior by the players acting to maximize their payoffs.

An **equilibrium** $s^* = (s_1^*, \ldots, s_n^*)$ *is a strategy profile consisting of a best strategy for each of the* n *players in the game.*

The **equilibrium strategies** are the strategies players pick in trying to maximize their individual payoffs, as distinct from the many possible strategy profiles obtainable by arbitrarily choosing one strategy per player. Equilibrium is used differently in game theory than in other areas of economics. In a general equilibrium model, for example, an equilibrium is a set of prices resulting from optimal behavior by the individuals in the economy. In game theory, that set of prices would be the **equilibrium outcome**, but the equilibrium itself would be the strategy profile— the individuals' rules for buying and selling— that generated the outcome.

People often carelessly say "equilibrium" when they mean "equilibrium outcome," and "strategy" when they mean "action." The difference is not very important in most of the games that will appear in this chapter, but it is absolutely fundamental to thinking like a game theorist. Consider Germany's decision on whether to remilitarize the Rhineland in 1936. France adopted the strategy: *Do not fight*, and Germany responded by remilitarizing, leading to World War II a few years later. If France had adopted the strategy: *Fight if Germany remilitarizes; otherwise do not fight*, the outcome would still have been that France would not have fought. No war would have ensued,however, because Germany would not remilitarized. Perhaps it was because he thought along these lines that John von Neumann was such a hawk in the Cold War, as MacRae describes in his biography (MacRae [1992]). This difference between actions and strategies, outcomes and equilibria, is one of the hardest ideas to teach in a game theory class, even though it is trivial to state.

To find the equilibrium, it is not enough to specify the players, strategies, and payoffs, because the modeller must also decide what "best strategy" means. He does this by defining an equilibrium concept.

An **equilibrium concept** *or* **solution concept** $F : \{S_1, \ldots, S_n, \pi_1, \ldots, \pi_n\} \to s^*$ *is a rule that defines an equilibrium based on the possible strategy profiles and the payoff functions.*

We have implicitly already used an equilibrium concept in the analysis above, which picked one strategy for each of the two players as our prediction for the game (what we implicitly used is the concept of **subgame perfectness** which will reappear in Chapter 4). Only a few equilibrium concepts are generally accepted, and the remaining sections of this chapter are devoted to finding the equilibrium using the two best-known of them: dominant strategy equilibrium and Nash equilibrium.

Uniqueness

Accepted solution concepts do not guarantee uniqueness, and lack of a unique equilibrium is a major problem in game theory. Often the solution concept employed leads us to believe that the players will pick one of the two strategy profiles A or B, not C or D, but we cannot say whether A or B is more likely. Sometimes we have the opposite problem and the game has no equilibrium at all. Having no equilibrium means either that the modeller sees no good reason why one strategy profile is more likely than another, or that some player wants to pick an infinite value for one of his actions.

A model with no equilibrium or multiple equilibria is underspecified. The modeller has failed to provide a full and precise prediction for what will happen. One option is to admit that the theory is incomplete. This is not a shameful thing to do; an admission of incompleteness such as Section 5.2's Folk Theorem is a valuable negative result. Or perhaps the situation being modelled really is unpredictable, in which case to make a prediction would be wrong. Another option is to renew the attack by changing the game's description or the solution concept. Preferably it is the description that is changed, since economists look to the rules of the game for the differences between models, and not to the solution concept. If an important part of the game is concealed under the definition of equilibrium, in fact, the reader is likely to feel tricked and to charge the modeller with intellectual dishonesty.

1.2 Dominated and Dominant Strategies: The Prisoner's Dilemma

In discussing equilibrium concepts, it is useful to have shorthand for "all the other players' strategies."

For any vector $y = (y_1, \ldots, y_n)$*, denote by* y_{-i} *the vector* $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$*, which is the portion of* y *not associated with player* i*.*

Using this notation, s_{-Smith}, for instance, is the profile of strategies of every player except player *Smith*. That profile is of great interest to Smith, because he uses it to help choose his own strategy, and the new notation helps define his best response.

Player i*'s* **best response** *or* **best reply** *to the strategies* s_{-i} *chosen by the other players* *is the strategy s_i^* that yields him the greatest payoff; that is,*

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_i' \neq s_i^*. \tag{1}$$

The best response is strongly best if no other strategies are equally good, and weakly best otherwise.

The first important equilibrium concept is based on the idea of **dominance**.

The strategy s_i^d is a **dominated strategy** *if it is strictly inferior to some other strategy no matter what strategies the other players choose, in the sense that whatever strategies they pick, his payoff is lower with s_i^d. Mathematically, s_i^d is dominated if there exists a single s_i' such that*

$$\pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}) \;\; \forall s_{-i}. \tag{2}$$

Note that s_i^d is not a dominated strategy if there is no s_{-i} to which it is the best response, but sometimes the better strategy is s_i' and sometimes it is s_i''. In that case, s_i^d could have the redeeming feature of being a good compromise strategy for a player who cannot predict what the other players are going to do. A dominated strategy is unambiguously inferior to some single other strategy.

There is usually no special name for the superior strategy that beats a dominated strategy. In unusual games, however, there is some strategy that beats *every* other strategy. We call that a "dominant strategy".

The strategy s_i^* is a **dominant strategy** *if it is a player's strictly best response to* any *strategies the other players might pick, in the sense that whatever strategies they pick, his payoff is highest with s_i^*. Mathematically,*

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \;\; \forall s_{-i}, \;\; \forall s_i' \neq s_i^*. \tag{3}$$

A **dominant-strategy equilibrium** *is a strategy profile consisting of each player's dominant strategy.*

A player's dominant strategy is his strictly best response even to wildly irrational actions by the other players. Most games do not have dominant strategies, and the players must try to figure out each others' actions to choose their own.

The Dry Cleaners Game incorporated considerable complexity in the rules of the game to illustrate such things as information sets and the time sequence of actions. To illustrate equilibrium concepts, we will use simpler games, such as the Prisoner's Dilemma. In the Prisoner's Dilemma, two prisoners, Messrs Row and Column, are being interrogated separately. If each tries to blame the other, each is sentenced to eight years in prison; if both remain silent, each is sentenced to one year.[3] If just one blames the other, he is released

[3]Another way to tell the story is to say that if both are silent, then with probability 0.1 they are convicted anyway and serve ten years, for an expected payoff of $(-1, -1)$.

but the silent prisoner is sentenced to ten years. The Prisoner's Dilemma is an example of a **2-by-2 game**, because each of the two players— Row and Column— has two possible actions in his action set: *Blame* and *Silence*. Table 2 gives the payoffs.

Table 2: The Prisoner's Dilemma

		Column	
		Silence	*Blame*
	Silence	-1,-1	-10, 0
Row			
	Blame	0,-10	**- 8,-8**

Payoffs to: (Row,Column)

Each player has a dominant strategy. Consider Row. Row does not know which action Column is choosing, but if Column chooses *Silence*, Row faces a *Silence* payoff of -1 and a *Blame* payoff of 0, whereas if Column chooses *Blame*, Row faces a *Silence* payoff of -10 and a *Blame* payoff of -8. In either case Row does better with *Blame*. Since the game is symmetric, Column's incentives are the same. The dominant-strategy equilibrium is (*Blame*, *Blame*), and the equilibrium payoffs are $(-8, -8)$, which is worse for both players than $(-1, -1)$. Sixteen, in fact, is the greatest possible combined total of years in prison.

The result is even stronger than it seems, because it is robust to substantial changes in the model. Because the equilibrium is a dominant-strategy equilibrium, the information structure of the game does not matter. If Column is allowed to know Row's move before taking his own, the equilibrium is unchanged. Row still chooses *Blame*, knowing that Column will surely choose *Blame* afterwards.

The Prisoner's Dilemma crops up in many different situations, including oligopoly pricing, auction bidding, salesman effort, political bargaining, and arms races. Whenever you observe individuals in a conflict that hurts them all, your first thought should be of the Prisoner's Dilemma.

The game seems perverse and unrealistic to many people who have never encountered it before (although friends who are prosecutors assure me that it is a standard crime-fighting tool). If the outcome does not seem right to you, you should realize that very often the chief usefulness of a model is to induce discomfort. Discomfort is a sign that your model is not what you think it is— that you left out something essential to the result you expected and didn't get. Either your original thought or your model is mistaken; and finding such mistakes is a real if painful benefit of model building. To refuse to accept surprising conclusions is to reject logic.

Cooperative and Noncooperative Games

What difference would it make if the two prisoners could talk to each other before making their decisions? It depends on the strength of promises. If promises are not binding, then although the two prisoners might agree to *Silence*, they would *Blame* anyway when the time came to choose actions.

A **cooperative game** *is a game in which the players can make binding commitments, as opposed to a* **noncooperative game**, *in which they cannot.*

This definition draws the usual distinction between the two theories of games, but the real difference lies in the modelling approach. Both theories start off with the rules of the game, but they differ in the kinds of solution concepts employed. Cooperative game theory is axiomatic, frequently appealing to pareto-optimality,[4] fairness, and equity. Noncooperative game theory is economic in flavor, with solution concepts based on players maximizing their own utility functions subject to stated constraints. Or, from a different angle: cooperative game theory is a reduced-form theory, which focusses on properties of the outcome rather than on the strategies that achieve the outcome, a method which is appropriate if modelling the process is too complicated. Except for the discussion of the Nash Bargaining Solution in Chapter 12, this book is concerned exclusively with noncooperative games (For an argument that cooperative game theory is more important than I think, see Aumann [1997]).

In applied economics, the most commonly encountered use of cooperative games is to model bargaining. The Prisoner's Dilemma is a noncooperative game, but it could be modelled as cooperative by allowing the two players not only to communicate but to make binding commitments. Cooperative games often allow players to split the gains from cooperation by making **side-payments**— transfers between themselves that change the prescribed payoffs. Cooperative game theory generally incorporates commitments and side-payments via the solution concept, which can become very elaborate, while noncooperative game theory incorporates them by adding extra actions. The distinction between cooperative and noncooperative games does *not* lie in conflict or absence of conflict, as is shown by the following examples of situations commonly modelled one way or the other:

A cooperative game without conflict. Members of a workforce choose which of equally arduous tasks to undertake to best coordinate with each other.

A cooperative game with conflict. Bargaining over price between a monopolist and a monopsonist.

[4] If outcome X **strongly pareto-dominates** outcome Y, then all players have higher utility under outcome X. If outcome X **weakly pareto-dominates** outcome Y, some player has higher utility under X, and no player has lower utility. A zero-sum game does not have outcomes that even weakly pareto-dominate other outcomes. All of its equilibria are pareto-efficient, because no player gains without another player losing.

It is often said that strategy profile x "pareto dominates" or "dominates" strategy profile y. Taken literally, this is meaningless, since strategies do not necessarily have any ordering at all— one could define *Silence* as being bigger than *Blame*, but that would be arbitrary. The statement is really shorthand for "The payoff profile resulting from strategy profile x pareto-dominates the payoff profile resulting from strategy y."

A noncooperative game with conflict. The Prisoner's Dilemma.

A noncooperative game without conflict. Two companies set a product standard without communication.

1.3 Iterated Dominance: The Battle of the Bismarck Sea

Very few games have a dominant-strategy equilibrium, but sometimes dominance can still be useful even when it does not resolve things quite so neatly as in the Prisoner's Dilemma. The Battle of the Bismarck Sea, a game I found in Haywood (1954), is set in the South Pacific in 1943. General Imamura has been ordered to transport Japanese troops across the Bismarck Sea to New Guinea, and General Kenney wants to bomb the troop transports. Imamura must choose between a shorter northern route or a longer southern route to New Guinea, and Kenney must decide where to send his planes to look for the Japanese. If Kenney sends his planes to the wrong route he can recall them, but the number of days of bombing is reduced.

The players are Kenney and Imamura, and they each have the same action set, $\{North, South\}$, but their payoffs, given by Table 3, are never the same. Imamura loses exactly what Kenney gains. Because of this special feature, the payoffs could be represented using just four numbers instead of eight, but listing all eight payoffs in Table 3 saves the reader a little thinking. The 2- by-2 form with just four entries is a **matrix game**, while the equivalent table with eight entries is a **bimatrix game.** Games can be represented as matrix or bimatrix games even if they have more than two moves, as long as the number of moves is finite.

Table 3: The Battle of the Bismarck Sea

		Imamura	
		North	South
Kenney	North	**2,-2**	$2, -2$
	South	$1, -1$	$3, -3$

Payoffs to: (Kenney, Imamura)

Strictly speaking, neither player has a dominant strategy. Kenney would choose *North* if he thought Imamura would choose *North*, but *South* if he thought Imamura would choose *South.* Imamura would choose *North* if he thought Kenney would choose *South*, and he would be indifferent between actions if he thought Kenney would choose *North.* This is what the arrows are showing. But we can still find a plausible equilibrium, using the concept of "weak dominance".

Strategy s_i' is **weakly dominated** *if there exists some other strategy s_i'' for player i which is possibly better and never worse, yielding a higher payoff in some strategy profile and never* *yielding a lower payoff. Mathematically, s_i' is weakly dominated if there exists s_i'' such that*

$$\begin{aligned} &\pi_i(s_i'', s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_{-i}, \text{ and} \\ &\pi_i(s_i'', s_{-i}) > \pi_i(s_i', s_{-i}) \quad \text{for some } s_{-i}. \end{aligned} \tag{4}$$

Similarly, we call a strategy that is always at least as good as every other strategy and better than some a **weakly dominant strategy**.

One might define a **weak-dominance equilibrium** as the strategy profile found by deleting all the weakly dominated strategies of each player. Eliminating weakly dominated strategies does not help much in The Battle of the Bismarck Sea, however. Imamura's strategy of *South* is weakly dominated by the strategy *North* because his payoff from *North* is never smaller than his payoff from *South*, and it is greater if Kenney picks *South*. For Kenney, however, neither strategy is even weakly dominated. The modeller must therefore go a step further, to the idea of the iterated dominance equilibrium.

An **iterated-dominance equilibrium** *is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until only one strategy remains for each player.*

Applied to The Battle of the Bismarck Sea, this equilibrium concept implies that Kenney decides that Imamura will pick *North* because it is weakly dominant, so Kenney eliminates "Imamura chooses *South*" from consideration. Having deleted one column of Table 3, Kenney has a strongly dominant strategy: he chooses *North*, which achieves payoffs strictly greater than *South*. The strategy profile (*North, North*) is an iterated dominance equilibrium, and indeed (*North*, *North*) was the outcome in 1943.

It is interesting to consider modifying the order of play or the information structure in The Battle of the Bismarck Sea. If Kenney moved first, rather than simultaneously with Imamura, (*North*, *North*) would remain an equilibrium, but (*North*, *South*) would also become one. The payoffs would be the same for both equilibria, but the outcomes would be different.

If Imamura moved first, (*North*, *North*) would be the only equilibrium. What is important about a player moving first is that it gives the other player more information before he acts, not the literal timing of the moves. If Kenney has cracked the Japanese code and knows Imamura's plan, then it does not matter that the two players move literally simultaneously; it is better modelled as a sequential game. Whether Imamura literally moves first or whether his code is cracked, Kenney's information set becomes either {Imamura moved *North*} or {Imamura moved *South*} after Imamura's decision, so Kenney's equilibrium strategy is specified as (*North* if Imamura moved *North*, *South* if Imamura moved *South*).

Game theorists often differ in their terminology, and the terminology applied to the idea of eliminating dominated strategies is particularly diverse. The equilibrium concept used in The Battle of the Bismarck Sea might be called **iterated-dominance equilibrium** or **iterated-dominant- strategy equilibrium**, or one might say that the game is **dominance solvable**, that it can be **solved by iterated dominance**, or that the equilibrium strategy profile is **serially undominated**. Often the terms are used to mean deletion of strictly dominated strategies and sometimes to mean deletion of weakly dominated strategies. Iteration of strictly dominated strategies is, of course, a more appealing idea, but one which more rarely is applicable. For a 3-by-3 example in which iterated elimination of strictly dominated strategies does reach a unique equilibrium despite no strategy being dominant for the game as a whole see Ratliff (1997a, p. 7).

The significant difference is between strong and weak dominance. Everyone agrees that no rational player would use a strictly dominated strategy, but it is harder to argue against weakly dominated strategies. In economic models, firms and individuals are often indifferent about their behavior in equilibrium. In standard models of perfect competition, firms earn zero profits but it is crucial that some firms be active in the market and some stay out and produce nothing. If a monopolist knows that customer Smith is willing to pay up to ten dollars for a widget, the monopolist will charge exactly ten dollars to Smith in equilibrium, which makes Smith indifferent about buying and not buying, yet there is no equilibrium unless Smith buys. It is impractical, therefore, to rule out equilibria in which a player is indifferent about his actions. This should be kept in mind later when we discuss the "open-set problem" in Section 4.3.

Another difficulty is multiple equilibria. The dominant-strategy equilibrium of any game is unique if it exists. Each player has at most one strategy whose payoff in any strategy profile is strictly higher than the payoff from any other strategy, so only one strategy profile can be formed out of dominant strategies. A strong iterated-dominance equilibrium is unique if it exists. A weak iterated-dominance equilibrium may not be, because the order in which strategies are deleted can matter to the final solution. If all the weakly dominated strategies are eliminated simultaneously at each round of elimination, the resulting equilibrium is unique, if it exists, but possibly no strategy profile will remain.

Consider Table 4's Iteration Path Game. The strategy profiles (r_1, c_1) and (r_1, c_3) are both iterated dominance equilibria, because each of those strategy profiles can be found by iterated deletion. The deletion can proceed in the order (r_3, c_3, c_2, r_2), or in the order (r_2, c_2, c_1, r_3).

Table 4: The Iteration Path Game

		Column		
		c_1	c_2	c_3
	r_1	**2,12**	1,10	**1,12**
Row	r_2	0,12	0,10	0,11
	r_3	0,12	1,10	0,13

Payoffs to: (Row, Column)

Despite these problems, deletion of weakly dominated strategies is a useful tool, and it is part of more complicated equilibrium concepts such as Section 4.1's "subgame perfectness".

Zero-Sum Games

The Iteration Path Game is like the typical game in economics in that if one player gains, the other player does not necessarily lose. The outcome (2,12) is better for both players than the outcome (0,10), for example. Since economics is largely about the gains from trade, it is not surprising that win-win outcomes are possible, even if the players are each trying to maximize only their own payoffs. Some games, however, such as The Battle of Bismarck Sea, are different, because the payoffs of the players always sum to zero. This feature is important enough to have acquired a name early in the history of game theory.

A **zero-sum game** *is a game in which the sum of the payoffs of all the players is zero whatever strategies they choose. A game which is not zero-sum is* **nonzero-sum game** *or* **variable- sum.**

In a zero-sum game, what one player gains, another player must lose. The Battle of the Bismarck Sea is thus a zero-sum game, but the Prisoner's Dilemma and the Dry Cleaners Game are not. There is no way that the payoffs in those two games can be rescaled to make them zero-sum without changing the essential character of the games.

If a game is zero-sum the utilities of the players can be represented so as to sum to zero under any outcome. Since utility functions are to some extent arbitrary, the sum can also be represented to be non-zero even if the game is zero-sum. Often modellers will refer to a game as zero-sum even when the payoffs do not add up to zero, so long as the payoffs add up to some constant amount. The difference is a trivial normalization.

Although zero-sum games have fascinated game theorists for many years, they are uncommon in economics. One of the few examples is the bargaining game between two players who divide a surplus, but even this is often modelled nowadays as a nonzero-sum game in which the surplus shrinks as the players spend more time deciding how to divide it. In reality, even simple division of property can result in loss— just think of how much the lawyers take out when a divorcing couple bargain over dividing their possessions.

Although the 2-by-2 games in this chapter may seem facetious, they are simple enough for use in modelling economic situations. The Battle of the Bismarck Sea, for example, can be turned into a game of corporate strategy. Two firms, Kenney Company and Imamura Incorporated, are trying to maximize their shares of a market of constant size by choosing between the two product designs *North* and *South*. Kenney has a marketing advantage, and would like to compete head-to-head, while Imamura would rather carve out its own niche. The equilibrium is (*North*, *North*).

1.4 Nash Equilibrium: Boxed Pigs, The Battle of the Sexes, and Ranked Coordination

For the vast majority of games, which lack even iterated dominance equilibria, modellers use Nash equilibrium, the most important and widespread equilibrium concept. To introduce Nash equilibrium we will use the game Boxed Pigs from Baldwin & Meese (1979).

Two pigs are put in a box with a special control panel at one end and a food dispenser at the other end. When a pig presses the panel, at a utility cost of 2 units, 10 units of food are dispensed at the dispenser. One pig is "dominant" (let us assume he is bigger), and if he gets to the dispenser first, the other pig will only get his leavings, worth 1 unit. If, instead, the small pig is at the dispenser first, he eats 4 units, and even if they arrive at the same time the small pig gets 3 units. Thus, for example, the strategy profile ($Press$, $Press$) would yield a payoff of 5 for the big pig (10 units of food, minus 3 that the small pig eats, minus an effort cost of 2) and of 1 for the little pig (3 units of food, minus an effort cost of 2). Table 5 summarizes the payoffs for the strategies $Press$ the panel and $Wait$ by the dispenser at the other end.

Table 5: Boxed Pigs

		Small Pig		
		Press		Wait
	Press	$5, 1$	$\rightarrow$	$\boxed{\mathbf{4}}, \boxed{\mathbf{4}}$
Big Pig		$\downarrow$		$\uparrow$
	Wait	$\boxed{9}, -1$	$\rightarrow$	$0, \boxed{0}$

Payoffs to: (Big Pig, Small Pig). Arrows show how a player can increase his payoff. Best-response payoffs are boxed.

Boxed Pigs has no dominant-strategy equilibrium, because what the big pig chooses depends on what he thinks the small pig will choose. If he believed that the small pig would press the panel, the big pig would wait by the dispenser, but if he believed that the small pig would wait, the big pig would press the panel. There does exist an iterated-dominance equilibrium, $(Press, Wait)$, but we will use a different line of reasoning to justify that outcome: Nash equilibrium.

Nash equilibrium is the standard equilibrium concept in economics. It is less obviously correct than dominant-strategy equilibrium but more often applicable. Nash equilibrium is so widely accepted that the reader can assume that if a model does not specify which equilibrium concept is being used it is Nash or some refinement of Nash.

The strategy profile s^* is a **Nash equilibrium** *if no player has incentive to deviate from his strategy given that the other players do not deviate. Formally,*

$$\forall i, \;\; \pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \;\; \forall s_i'. \tag{5}$$

The strategy profile $(Press, Wait)$ is a Nash equilibrium. The way to approach Nash equilibrium is to propose a strategy profile and test whether each player's strategy is a best response to the others' strategies. If the big pig picks $Press$, the small pig, who faces a choice between a payoff of 1 from pressing and 4 from waiting, is willing to wait. If the small pig picks $Wait$, the big pig, who has a choice between a payoff of 4 from pressing and 0 from waiting, is willing to press. This confirms that $(Press, Wait)$ is a Nash equilibrium, and in fact it is the unique Nash equilibrium.[5]

It is useful to draw arrows in the tables when trying to solve for the equilibrium, since the number of calculations is great enough to soak up quite a bit of mental RAM. Another solution tip, illustrated in Boxed Pigs, is to circle payoffs that dominate other payoffs (or box, them, as is especially suitable here). Double arrows or dotted circles indicate weakly dominant payoffs. Any payoff profile in which every payoff is circled, or which has arrows pointing towards it from every direction, is a Nash equilibrium. I like using arrows better in 2-by-2 games, but circles are better for bigger games, since arrows become confusing when payoffs are not lined up in order of magnitude in the table (see Chapter 2's Table 2).

The pigs in this game have to be smarter than the players in the Prisoner's Dilemma. They have to realize that the only set of strategies supported by self-consistent beliefs is $(Press, Wait)$. The definition of Nash equilibrium lacks the "$\forall s_{-i}$" of dominant-strategy equilibrium, so a Nash strategy need only be a best response to the other Nash strategies, not to all possible strategies. And although we talk of "best responses," the moves are actually simultaneous, so the players are predicting each others' moves. If the game were repeated or the players communicated, Nash equilibrium would be especially attractive, because it is even more compelling that beliefs should be consistent.

Like a dominant-strategy equilibrium, a Nash equilibrium can be either weak or strong.

[5]This game, too, has its economic analog. If Bigpig, Inc. introduces granola bars, at considerable marketing expense in educating the public, then Smallpig Ltd. can imitate profitably without ruining Bigpig's sales completely. If Smallpig introduces them at the same expense, however, an imitating Bigpig would hog the market.

The definition above is for a weak Nash equilibrium. To define strong Nash equilibrium, make the inequality strict; that is, require that no player be indifferent between his equilibrium strategy and some other strategy.

Every dominant-strategy equilibrium is a Nash equilibrium, but not every Nash equilibrium is a dominant-strategy equilibrium. If a strategy is dominant it is a best response to *any* strategies the other players pick, including their equilibrium strategies. If a strategy is part of a Nash equilibrium, it need only be a best response to the other players' *equilibrium* strategies.

The Modeller's Dilemma of Table 6 illustrates this feature of Nash equilibrium. The situation it models is the same as the Prisoner's Dilemma, with one major exception: although the police have enough evidence to arrest the prisoners as the "probable cause" of the crime, they will not have enough evidence to convict them of even a minor offense if neither prisoner confesses. The northwest payoff profile becomes (0,0) instead of $(-1, -1)$.

Table 6: The Modeller's Dilemma

		Column		
		Silence		*Blame*
Row	*Silence*	**0**, **0**	↔	−10, 0
		↕		↓
	Blame	0, -10	→	**-8**, **-8**

Payoffs to: (Row, Column) . Arrows show how a player can increase his payoff.

The Modeller's Dilemma does not have a dominant-strategy equilibrium. It does have what might be called a weak dominant-strategy equilibrium, because *Blame* is still a weakly dominant strategy for each player. Moreover, using this fact, it can be seen that *(Blame, Blame)* is an iterated dominance equilibrium, and it is a strong Nash equilibrium as well. So the case for *(Blame, Blame)* still being the equilibrium outcome seems very strong.

There is, however, another Nash equilibrium in the Modeller's Dilemma: *(Silence, Silence)*, which is a weak Nash equilibrium. This equilibrium is weak and the other Nash equilibrium is strong, but *(Silence, Silence)* has the advantage that its outcome is pareto-superior: (0, 0) is uniformly greater than $(-8, -8)$. This makes it difficult to know which behavior to predict.

The Modeller's Dilemma illustrates a common difficulty for modellers: what to predict when two Nash equilibria exist. The modeller could add more details to the rules of the game, or he could use an **equilibrium refinement**, adding conditions to the basic equilibrium concept until only one strategy profile satisfies the refined equilibrium concept. There is no single way to refine Nash equilibrium. The modeller might insist on a strong equilibrium, or rule out weakly dominated strategies, or use iterated dominance. All of these lead to *(Blame, Blame)* in the Modeller's Dilemma. Or he might rule out Nash equilibria that are pareto-dominated by other Nash equilibria, and end up with *(Silence,*

Silence). Neither approach is completely satisfactory. In particular, do not be misled into thinking that weak Nash equilibria are to be despised. Often, no Nash equilibrium at all will exist unless the players have the expectation that player B chooses X when he is indifferent between X and Y. It is not that we are picking the equilibrium in which it is assumed B does X when he is indifferent. Rather, we are finding the *only* set of consistent expectations about behavior. (You will read more about this in connection with the "open-set problem" of Section 4.2.)

The Battle of the Sexes

The third game we will use to illustrate Nash equilibrium is The Battle of the Sexes, a conflict between a man who wants to go to a prize fight and a woman who wants to go to a ballet. While selfish, they are deeply in love, and would, if necessary, sacrifice their preferences to be with each other. Less romantically, their payoffs are given by Table 7.

Table 7: The Battle of the Sexes [6]

		Woman		
		Prize Fight		*Ballet*
	Prize Fight	**2,1**	←	0, 0
Man		↑		↓
	Ballet	0, 0	→	**1,2**

Payoffs to: (Man, Woman). Arrows show how a player can increase his payoff.

The Battle of the Sexes does not have an iterated dominance equilibrium. It has two Nash equilibria, one of which is the strategy profile (*Prize Fight, Prize Fight*). Given that the man chooses *Prize Fight*, so does the woman; given that the woman chooses *Prize Fight*, so does the man. The strategy profile (*Ballet*, *Ballet*) is another Nash equilibrium by the same line of reasoning.

How do the players know which Nash equilibrium to choose? Going to the fight and going to the ballet are both Nash strategies, but for different equilibria. Nash equilibrium assumes correct and consistent beliefs. If they do not talk beforehand, the man might go to the ballet and the woman to the fight, each mistaken about the other's beliefs. But even if the players do not communicate, Nash equilibrium is sometimes justified by repetition of the game. If the couple do not talk, but repeat the game night after night, one may suppose that eventually they settle on one of the Nash equilibria.

Each of the Nash equilibria in The Battle of the Sexes is pareto- efficient; no other strategy profile increases the payoff of one player without decreasing that of the other. In many games the Nash equilibrium is not pareto-efficient: (*Blame*, *Blame*), for example, is

[6]Political correctness has led to bowdlerized versions of this game being presented in many game theory books. This is the original, unexpurgated game.

the unique Nash equilibrium of the Prisoner's Dilemma, although its payoffs of $(-8, -8)$ are pareto- inferior to the $(-1, -1)$ generated by (*Silence*, *Silence*).

Who moves first is important in The Battle of the Sexes, unlike any of the three previous games we have looked at. If the man could buy the fight ticket in advance, his commitment would induce the woman to go to the fight. In many games, but not all, the player who moves first (which is equivalent to commitment) has a **first-mover advantage.**

The Battle of the Sexes has many economic applications. One is the choice of an industrywide standard when two firms have different preferences but both want a common standard to encourage consumers to buy the product. A second is to the choice of language used in a contract when two firms want to formalize a sales agreement but they prefer different terms. Both sides might, for example, want to add a "liquidated damages" clause which specifies damages for breach rather than trust the courts to estimate a number later, but one firm might want a value of $10,000 and the other firm, $12,000.

Coordination Games

Sometimes one can use the size of the payoffs to choose between Nash equilibria. In the following game, players Smith and Jones are trying to decide whether to design the computers they sell to use large or small floppy disks. Both players will sell more computers if their disk drives are compatible, as shown in Table 8.

Table 8: Ranked Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	$\leftarrow$	$-1, -1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1, -1$	$\rightarrow$	**1,1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

The strategy profiles (*Large*, *Large*) and (*Small*, *Small*) are both Nash equilibria, but (*Large*, *Large*) pareto- dominates (*Small*, *Small*). Both players prefer (*Large*, *Large*), and most modellers would use the pareto- efficient equilibrium to predict the actual outcome. We could imagine that it arises from pre-game communication between Smith and Jones taking place outside of the specification of the model, but the interesting question is what happens if communication is impossible. Is the pareto-efficient equilibrium still more plausible? The question is really one of psychology rather than economics.

Ranked Coordination is one of a large class of games called **coordination games**, which share the common feature that the players need to coordinate on one of multiple Nash equilibria. Ranked Coordination has the additional feature that the equilibria can be pareto ranked. Section 3.2 will return to problems of coordination to discuss the concepts of "correlated strategies" and "cheap talk." These games are of obvious relevance to analyzing the setting of standards; see, e.g., Michael Katz & Carl Shapiro (1985) and Joseph Farrell & Garth Saloner (1985). They can be of great importance to the wealth of economies– just think of the advantages of standard weights and measures (or read Charles Kindleberger (1983) on their history). Note, however, that not all apparent situations of coordination on pareto-inferior equilibria turn out to be so. One oft-cited coordination problem is that of the QWERTY typewriter keyboard, developed in the 1870s when typing had to proceed slowly to avoid jamming. QWERTY became the standard, although it has been claimed that the faster speed possible with the Dvorak keyboard would amortize the cost of retraining full-time typists within ten days (David [1985]). Why large companies would not retrain their typists is difficult to explain under this story, and Liebowitz & Margolis (1990) show that economists have been too quick to accept claims that QWERTY is inefficient. English language spelling is a better example.

Table 9 shows another coordination game, Dangerous Coordination, which has the same equilibria as Ranked Coordination, but differs in the out-of- equilibrium payoffs. If an experiment were conducted in which students played Dangerous Coordination against each other, I would not be surprised if *(Small,Small)*, the pareto-dominated equilibrium, were the one that was played out. This is true even though *(Large, Large)* is still a Nash equilibrium; if Smith thinks that Jones will pick *Large*, Smith is quite willing to pick *Large* himself. The problem is that if the assumptions of the model are weakened, and Smith cannot trust Jones to be rational, well- informed about the payoffs of the game, and unconfused, then Smith will be reluctant to pick *Large* because his payoff if Jones picks *Small* is then -1,000. He would play it safe instead, picking *Small* and ensuring a payoff of at least -1. In reality, people do make mistakes, and with such an extreme difference in payoffs, even a small probability of a mistake is important, so *(Large, Large)* would be a bad prediction.

Table 9: Dangerous Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	$\leftarrow$	$-1000, -1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1, -1$	$\rightarrow$	**1,1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Games like Dangerous Coordination are a major concern in the 1988 book by Harsanyi and Selten, two of the giants in the field of game theory. I will not try to describe their approach here, except to say that it is different from my own. I do not consider the fact that one of the Nash equilibria of Dangerous Coordination is a bad prediction as a heavy blow against Nash equilibrium. The bad prediction is based on two things: using the Nash equilibrium concept, and using the game Dangerous Coordination. If Jones might be confused about the payoffs of the game, then the game actually being played out is not Dangerous Coordination, so it is not surprising that it gives poor predictions. The rules of the game ought to describe the probabilities that the players are confused, as well as the payoffs if they take particular actions. If confusion is an important feature of the situation, then the two-by-two game of Table 9 is the wrong model to use, and a more complicated game of incomplete information of the kind described in Chapter 2 is more appropriate. Again, as with the Prisoner's Dilemma, the modeller's first thought on finding that the model predicts an odd result should not be "Game theory is bunk," but the more modest "Maybe I'm not describing the situation correctly" (or even "Maybe I should not trust my 'common sense' about what will happen").

Nash equilibrium is more complicated but also more useful than it looks. Jumping ahead a bit, consider a game slightly more complex than the ones we have seen so far. Two firms are choosing outputs Q_1 and Q_2 simultaneously. The Nash equilibrium is a pair of numbers (Q_1^*, Q_2^*) such that neither firm would deviate unilaterally. This troubles the beginner, who says to himself,

> "Sure, Firm 1 will pick Q_1^* if it thinks Firm 2 will pick Q_2^*. But Firm 1 will realize that if it makes Q_1 bigger, then Firm 2 will react by making Q_2 smaller. So the situation is much more complicated, and (Q_1^*, Q_2^*) is not a Nash equilibrium. Or, if it is, Nash equilibrium is a bad equilibrium concept."

If there is a problem in this model, it is not Nash equilibrium but the model itself. Nash equilibrium makes perfect sense as a stable outcome in this model. The beginner's hypothetical is false because if Firm 1 chooses something other than Q_1^*, Firm 2 would not observe the deviation till it was too late to change Q_2– remember, this is a simultaneous move game. The beginner's worry is really about the rules of the game, not the equilibrium concept. He seems to prefer a game in which the firms move sequentially, or maybe a repeated version of the game. If Firm 1 moved first, and then Firm 2, then Firm 1's strategy would still be a single number, Q_1, but Firm 2's strategy– its action rule– would have to be a function, $Q_2(Q_1)$. A Nash equilibrium would then consist of an equilibrium number, Q_1^{**}, and an equilibrium function, $Q_2^{**}(Q_1)$. The two outputs actually chosen, Q_1^{**} and $Q_2^{**}(Q_1^{**})$, will be different from the Q_1^* and Q_2^* in the original game. And they should be different– the new model represents a very different real-world situation. Look ahead, and you will see that these are the Cournot and Stackelberg models of Chapter 3.

One lesson to draw from this is that it is essential to figure out the mathematical form the strategies take before trying to figure out the equilibrium. In the simultaneous move game, the strategy profile is a pair of non-negative numbers. In the sequential game, the strategy profile is one nonnegative number and one function defined over the nonnegative numbers. Students invariably make the mistake of specifying Firm 2's strategy as a number, not a function. This is a far more important point than any beginner realizes. Trust me– you're going to make this mistake sooner or later, so it's worth worrying about.

1.5 Focal Points

Thomas Schelling's 1960 book, *The Strategy of Conflict*, is a classic in game theory, even though it contains no equations or Greek letters. Although it was published more than 40 years ago, it is surprisingly modern in spirit. Schelling is not a mathematician but a strategist, and he examines such things as threats, commitments, hostages, and delegation that we will examine in a more formal way in the remainder of this book. He is perhaps best known for his coordination games. Take a moment to decide on a strategy in each of the following games, adapted from Schelling, which you win by matching your response to those of as many of the other players as possible.

1 Circle one of the following numbers: 100, 14, 15, 16, 17, 18.

2 Circle one of the following numbers 7, 100, 13, 261, 99, 666.

3 Name Heads or Tails.

4 Name Tails or Heads.

5 You are to split a pie, and get nothing if your proportions add to more than 100 percent.

6 You are to meet somebody in New York City. When? Where?

Each of the games above has many Nash equilibria. In example (1), if each player thinks every other player will pick 14, he will too, and this is self- confirming; but the same is true if each player thinks every other player will pick 15. But to a greater or lesser extent they also have Nash equilibria that seem more likely. Certain of the strategy profiles are **focal points:** Nash equilibria which for psychological reasons are particularly compelling.

Formalizing what makes a strategy profile a focal point is hard and depends on the context. In example (1), 100 is a focal point, because it is a number clearly different from all the others, it is biggest, and it is first in the listing. In example (2), Schelling found 7 to be the most common strategy, but in a group of Satanists, 666 might be the focal point. In repeated games, focal points are often provided by past history. Examples (3) and (4) are identical except for the ordering of the choices, but that ordering might make a difference. In (5), if we split a pie once, we are likely to agree on 50:50. But if last year we split a pie in the ratio 60:40, that provides a focal point for this year. Example (6) is the most interesting of all. Schelling found surprising agreement in independent choices, but the place chosen depended on whether the players knew New York well or were unfamiliar with the city.

The **boundary** is a particular kind of focal point. If player Russia chooses the action of putting his troops anywhere from one inch to 100 miles away from the Chinese border, player China does not react. If he chooses to put troops from one inch to 100 miles *beyond* the border, China declares war. There is an arbitrary discontinuity in behavior at the boundary. Another example, quite vivid in its arbitrariness, is the rallying cry, "Fifty-Four

Forty or Fight!," which refers to the geographic parallel claimed as the boundary by jingoist Americans in the Oregon dispute between Britain and the United States in the 1840s.[7]

Once the boundary is established it takes on additional significance because behavior with respect to the boundary conveys information. When Russia crosses an established boundary, that tells China that Russia intends to make a serious incursion further into China. Boundaries must be sharp and well known if they are not to be violated, and a large part of both law and diplomacy is devoted to clarifying them. Boundaries can also arise in business: two companies producing an unhealthful product might agree not to mention relative healthfulness in their advertising, but a boundary rule like "Mention unhealthfulness if you like, but don't stress it," would not work.

Mediation and **communication** are both important in the absence of a clear focal point. If players can communicate, they can tell each other what actions they will take, and sometimes, as in Ranked Coordination, this works, because they have no motive to lie. If the players cannot communicate, a mediator may be able to help by suggesting an equilibrium to all of them. They have no reason not to take the suggestion, and they would use the mediator even if his services were costly. Mediation in cases like this is as effective as arbitration, in which an outside party imposes a solution.

One disadvantage of focal points is that they lead to inflexibility. Suppose the pareto-superior equilibrium (*Large*, *Large*) were chosen as a focal point in Ranked Coordination, but the game was repeated over a long interval of time. The numbers in the payoff matrix might slowly change until (*Small, Small*) and (*Large, Large*) both had payoffs of, say, 1.6, and (*Small, Small*) started to dominate. When, if ever, would the equilibrium switch?

In Ranked Coordination, we would expect that after some time one firm would switch and the other would follow. If there were communication, the switch point would be at the payoff of 1.6. But what if the first firm to switch is penalized more? Such is the problem in oligopoly pricing. If costs rise, so should the monopoly price, but whichever firm raises its price first suffers a loss of market share.

[7]The threat was not credible: that parallel is now deep in British Columbia.

NOTES

N1.2 Dominant Strategies: The Prisoner's Dilemma

- Many economists are reluctant to use the concept of cardinal utility (see Starmer [2000]), and even more reluctant to compare utility across individuals (see Cooter & Rappoport [1984]). Noncooperative game theory never requires interpersonal utility comparisons, and only ordinal utility is needed to find the equilibrium in the Prisoner's Dilemma. So long as each player's rank ordering of payoffs in different outcomes is preserved, the payoffs can be altered without changing the equilibrium. In general, the dominant strategy and pure strategy Nash equilibria of games depend only on the ordinal ranking of the payoffs, but the mixed strategy equilibria depend on the cardinal values. Compare Section 3.2's Chicken game with Section 5.6's Hawk-Dove.

- If we consider only the ordinal ranking of the payoffs in 2-by-2 games, there are 78 distinct games in which each player has strict preference ordering over the four outcomes and 726 distinct games if we allow ties in the payoffs. Rapoport, Guyer & Gordon's 1976 book, *The 2x2 Game*, contains an exhaustive description of the possible games.

- If we allow players to randomize their action choices (the "mixed strategies" of Chapter 3), it can happen that some action is strictly dominated by a randomized strategy, even though it is not dominated by any nonrandom strategy. An example is in Chapter 3. Jim Ratliff's web notes are good on this topic; see Ratliff (1997a, 1997b). If random strategies are allowed, it becomes much more difficult to check for dominance and to use the iterative dominance ideas of Section 1.3.

- The Prisoner's Dilemma was so named by Albert Tucker in an unpublished paper, although the particular 2-by-2 matrix, discovered by Dresher and Flood, was already well known. Tucker was asked to give a talk on game theory to the psychology department at Stanford, and invented a story to go with the matrix, as recounted in Straffin (1980), Poundstone (1992, pp. 101-118), and Raiffa (1992, pp. 171-173).

- In the Prisoner's Dilemma the notation *cooperate* and *defect* is often used for the moves. This is bad terminology, because it is easy to confuse with *cooperative* games and with *deviations*. It is also often called the Prisoners' Dilemma (rs', not r's). Whether one looks at from the point of the individual or the group, the prisoners have a problem.

- The Prisoner's Dilemma is not always defined the same way. If we consider just ordinal payoffs, then the game in Table 10 is a prisoner's dilemma if $T(temptation) > R(revolt) > P(punishment) > S(Sucker)$, where the terms in parentheses are mnemonics. This is standard notation; see, for example, Rapoport, Guyer & Gordon (1976, p. 400). If the game is repeated, the cardinal values of the payoffs can be important. The requirement $2R > T + S > 2P$ should be added if the game is to be a standard Prisoner's Dilemma, in which $(Silence, Silence)$ and $(Blame, Blame)$ are the best and worst possible outcomes in terms of the sum of payoffs. Section 5.3 will show that an asymmetric game called the One-Sided Prisoner's Dilemma has properties similar to the standard Prisoner's Dilemma, but does not fit this definition.

  Sometimes the game in which $2R < T + S$ is also called a "Prisoner's Dilemma", but in it the sum of the players' payoffs is maximized when one blames the other and the other is silent. If the game were repeated or the prisoners could use the correlated equilibria defined in Section 3.2, they would prefer taking turns being silent, which would make the game a coordination game similar to The Battle of the Sexes. David Shimko has suggested the name "Battle of the Prisoners" for this (or, perhaps, "The Sex Prisoners' Dilemma").

Table 10: A General Prisoner's Dilemma

		Column		
		Silence		*Blame*
	Silence	R, R	$\rightarrow$	S, T
Row		$\downarrow$		$\downarrow$
	Blame	T, S	$\rightarrow$	**P,P**

Payoffs to: (Row, Column). Arrows show how a player can increase his payoff.

- Herodotus (429 B.C., III-71) describes an early example of the reasoning in the Prisoner's Dilemma in a conspiracy against the Persian emperor. A group of nobles met and decided to overthrow the emperor, and it was proposed to adjourn till another meeting. One of them named Darius then spoke up and said that if they adjourned, he knew that one of them would go straight to the emperor and reveal the conspiracy, because if nobody else did, he would himself. Darius also suggested a solution— that they immediately go to the palace and kill the emperor.

  The conspiracy also illustrates a way out of coordination games. After killing the emperor, the nobles wished to select one of themselves as the new emperor. Rather than fight, they agreed to go to a certain hill at dawn, and whoever's horse neighed first would become emperor. Herodotus tells how Darius's groom manipulated this randomization scheme to make him the new emperor.

- Philosophers are intrigued by the Prisoner's Dilemma: see Campbell & Sowden (1985), a collection of articles on the Prisoner's Dilemma and the related Newcombe's paradox. Game theory has even been applied to theology: if one player is omniscient or omnipotent, what kind of equilibrium behavior can we expect? See Brams's 1983 book, *Superior Beings*, and his 1980 book, *Biblical Games: A Strategic Analysis of Stories from the Old Testament*.

N1.4 Nash Equilibrium: Boxed Pigs, the Battle of the Sexes, and Ranked Coordination

For a history of the idea of Nash equilibrium, see Roger Myerson's 1999 article, "Nash Equilibrium and the History of Game Theory." E. Roy Weintraub's 1992 collection of essays, *Toward a History of Game Theory*, Norman Macrae's 1992 *John von Neumann*, William Poundstone's 1992 *Prisoner's Dilemma: John von Neumann, Game Theory, and the Puzzle of the Bomb*, Sylvia Nasar's 1998 *A Beautiful Mind*, and Mary and Robert Dimand's 1996 *A History of Game Theory* are other places to learn about the history of game theory. Good profiles of economists can be found in Michael Szenberg's 1992 *Eminent Economists: Their Life Philosophies* and 1998 *Passion and Craft: Economists at Work* and Sergiu Hart's 2005 "An Interview with Robert Aumann, " http://www.ma.huji.ac.il/ hart/abs/aumann.html, is also illuminating. Leonard (1995) discusses the "pre-history" from 1928 to 1944.

- I invented the payoffs for Boxed Pigs from the description of one of the experiments in Baldwin & Meese (1979). They do *not* think of this as an experiment in game theory, and they describe the result in terms of "reinforcement." The Battle of the Sexes is taken from p. 90 of Luce & Raiffa (1957). I have changed their payoffs of $(-1, -1)$ to $(-5, -5)$ to fit the story.

- Some people prefer the term "equilibrium point" to "Nash equilibrium," but the latter is more euphonious, since the discoverer's name is "Nash" and not "Mazurkiewicz."

- Bernheim (1984a) and Pearce (1984) use the idea of mutually consistent beliefs to arrive at a different equilibrium concept than Nash. They define a **rationalizable strategy** to be a strategy which is a best response for some set of rational beliefs in which a player believes that the other players choose their best responses. The difference from Nash is that not all players need have the same beliefs concerning which strategies will be chosen, nor need their beliefs be consistent. Every Nash equilibrium is rationalizable, but not every rationalizable equilibrium is Nash. Thus, the idea provides an argument for why Nash equilibria might be played, but not for why *just* Nash equilibria would be played. In a two-player game, the set of rationalizable strategies is the set which survive iterated deletion of strictly dominated strategies, but in a game with three or more players the set might be smaller. Ratliff (1997a) has an excellent discussion with numerical examples.

- Jack Hirshleifer (1982) uses the name "The Tender Trap" for a game essentially the same as Ranked Coordination. It has also been called the " Assurance Game".

- O. Henry's story,"The Gift of the Magi" is about a coordination game noteworthy for the reason communication is ruled out. A husband sells his watch to buy his wife combs for Christmas, while she sells her hair to buy him a watch fob. Communication would spoil the surprise, a worse outcome than discoordination.

- Macroeconomics has more game theory in it than is readily apparent. The macroeconomic concept of *rational expectations* faces the same problems of multiple equilibria and consistency of expectations as Nash equilibrium. Game theory is now often explicitly used in macroeconomics; see the books by Canzoneri & Henderson (1991) and Cooper (1999).

N1.5 Focal Points

- Besides his 1960 book, Schelling has written books on diplomacy (1966) and the oddities of aggregation (1978). Political scientists are now looking at the same issues more technically; see Brams & Kilgour (1988) and Ordeshook (1986). Douglas Muzzio's 1982 *Watergate Games*, Thomas Flanagan's 1998 *Game Theory and Canadian Politics*, and especially William Riker's 1986 *The Art of Political Manipulation* are absorbing examples of how game theory can be used to analyze specific historical episodes.

- In Chapter 12 of *The General Theory*, Keynes (1936) suggests that the stock market is a game with multiple equilibria, like a contest in which a newspaper publishes the faces of 20 girls, and contestants submit the name of the one they think most people would submit as the prettiest. When the focal point changes, big swings in predictions about beauty and value result.

- Not all of what we call boundaries have an arbitrary basis. If the Chinese cannot defend themselves as easily once the Russians cross the boundary at the Amur River, they have a clear reason to fight there.

- Crawford & Haller (1990) take a careful look at focalness in repeated coordination games by asking which equilibria are objectively different from other equilibria, and how a player can learn through repetition which equilibrium the other players intend to play. If on the first repetition the players choose strategies that are Nash with respect to each other, it seems focal for them to continue playing those strategies, but what happens if they begin in disagreement?

Problems

1.1. Nash and Iterated Dominance (medium)

(a) Show that every iterated dominance equilibrium s^* is Nash.

(b) Show by counterexample that not every Nash equilibrium can be generated by iterated dominance.

(c) Is every iterated dominance equilibrium made up of strategies that are not weakly dominated?

1.2. 2-by-2 Games (easy)
Find or create examples of 2-by-2 games with the following properties:

(a) No Nash equilibrium (you can ignore mixed strategies).

(b) No weakly pareto-dominant strategy profile.

(c) At least two Nash equilibria, including one equilibrium that pareto- dominates all other strategy profiles.

(d) At least three Nash equilibria.

1.3. Pareto Dominance (medium) (from notes by Jong-Shin Wei)

(a) If a strategy profile s^* is a dominant-strategy equilibrium, does that mean it weakly pareto-dominates all other strategy profiles?

(b If a strategy profile s strongly pareto-dominates all other strategy profiles, does that mean it is a dominant-strategy equilibrium?

(c) If s weakly pareto-dominates all other strategy profiles, then must it be a Nash equilibrium?

1.4. Discoordination (easy)
Suppose that a man and a woman each choose whether to go to a prize fight or a ballet. The man would rather go to the prize fight, and the woman to the ballet. What is more important to them, however, is that the man wants to show up to the same event as the woman, but the woman wants to avoid him.

(a) Construct a game matrix to illustrate this game, choosing numbers to fit the preferences described verbally.

(b) If the woman moves first, what will happen?

(c) Does the game have a first-mover advantage?

(d) Show that there is no Nash equilibrium if the players move simultaneously.

1.5. Drawing Outcome Matrices (easy)
It can be surprisingly difficult to look at a game using new notation. In this exercise, redraw the outcome matrix in a different form than in the main text. In each case, read the description of the game and draw the outcome matrix as instructed. You will learn more if you do this from the description, without looking at the conventional outcome matrix.

(a) The Battle of the Sexes (Table 7). Put (*Prize Fight, Prize Fight*) in the northwest corner, but make the woman the row player.

(b) The Prisoner's Dilemma (Table 2). Put (*Blame, Blame*) in the northwest corner.

(c) The Battle of the Sexes (Table 7). Make the man the row player, but put (*Ballet, Prize Fight*) in the northwest corner.

1.6. Finding Nash Equilibria (medium)
Find the Nash equilibria of the game illustrated in Table 11. Can any of them be reached by iterated dominance?

Table 11: An Abstract Game

		Column		
		Left	*Middle*	*Right*
	Up	10,10	$0,0$	$-1,15$
Row:	*Sideways*	$-12,1$	8, 8	$-1,-1$
	Down	15,1	8,-1	$0,0$

Payoffs to: (Row, Column).

1.7. Finding More Nash Equilibria (medium)
Find the Nash equilibria of the game illustrated in Table 12. Can any of them be reached by iterated dominance?

Table 12: Flavor and Texture

		Brydox	
		Flavor	*Texture*
Apex:	*Flavor*	-2,0	0,1
	Texture	-1,-1	0,-2

Payoffs to: (Apex, Brydox).

1.8. Which Game? (medium)
Table 13 is like the payoff matrix for what game that we have seen? (a) a version of the Battle of the Sexes. (b) a version of the Prisoner's Dilemma. (c) a version of Pure Coordination. (d) a version of the Legal Settlement Game. (e) none of the above.

Table 13: Which Game?

		COLUMN	
		A	B
ROW	A	3,3	0,1
	B	5,0	-1,-1

1.9. Choosing Computers (easy)
The problem of deciding whether to adopt IBM or HP computers by two offices in a company is most like which game that we have seen?

1.10. Campaign Contributions (easy)
The large Wall Street investment banks have recently agreed not to make campaign contributions to state treasurers, which up till now has been a common practice. What was the game in the past, and why can the banks expect this agreement to hold fast?

1.11. A Sequential Prisoner's Dilemma (hard)
Suppose Row moves first, then Column, in the Prisoner's Dilemma. What are the possible actions? What are the possible strategies? Construct a normal form, showing the relationship between strategy profiles and payoffs.

Hint: The normal form is *not* a two-by-two matrix here.

1.12. Three-by-Three Equilibria (medium)
Identify any dominated strategies and any Nash equilibria in pure strategies in the game of Table 14.

Table 14: A Three-By-Three Game

			Column	
		Left	**Middle**	*Right*
	Up	**1,4**	$5,-1$	$0,1$
Row:	**Sideways**	$-1,0$	-2,-2	$-3,4$
	Down	$0,3$	$9,-1$	$5,0$

Payoffs to: (Row, Column).

Fisheries: A Classroom Game for Chapter 1

Each of eight countries in a fishery decides how many fish to catch each decade. Each country picks an integer number X_t as its fishing catch for decade t. The country's profit for decade t is

$$20X_t - X_t^2. \tag{6}$$

Thus, diminishing returns set in after a certain point and the marginal cost is too high for further fishing to be profitable.

The fish population starts at 112 (14 per country) and the game continues for 5 decades. Let Q_1 denote the fish population at the start of Decade 1. In Decade 2, the population is

$$1.5 * (Q_1 - (X_{1t} + X_{2t} + X_{3t} + ...)), \text{rounded up}, \tag{7}$$

where X_{it} is Country i's catch in Decade t.

If $X_{11} = 30$ and $X_{21} = X_{31} = ... = X_{81} = 3$, then the first country's profit is $20 * 30 - 30^2 = 600 - 900 = -300$, and each other country earns $20 * 3 - 3^2 = 60 - 6 = 54$. The second-year fish population would be $Q_2 = 1.5 * (112 - 30 - 7[3]) = 1.5(82 - 21) = 1.5(61) = 92$.

1. In the first scenario, one fishing authority chooses the catch for all eight countries to try to maximize the catch over all 5 decades. Each country will propose quotas for all 8 countries for the first year. The class will discuss the proposals and the authority will deliberate and make its choice. Once the catch is finalized, the instructor calculates the next year's fish population, and the process repeats to pick the next year's catch.

2. The countries choose independently. Each country writes down its catch on a piece of paper, which it hands in to the instructor. The instructor opens them up as he receives them. He does not announce each country's catch until the end of this scenario's 5 decades, but he does announce the total catch. If the attempted catch exceeds the total fish population, those countries which handed in their catches first get priority, and a country's payoff is $20Z_t - X_t^2$, where Z_t is its actual catch and X_t is its attempted catch, what it wrote down on its paper. Do this for 5 decades.

3. Repeat Scenario 2, but with each country's actual (not attempted) catch announced at the end of each decade.

4. Repeat Scenario 3, but this time any countries that so wish can form a binding treaty and submit their catches jointly, on one piece of paper.