or might only exist in mixed strategies.

(1) An unbounded strategy space
Suppose in a stock market game that Smith can borrow money and buy as many shares x of stock as he likes, so his strategy set, the amount of stock he can buy, is $[0, \infty)$, a set which is unbounded above. (Note, by the way, that we thus assume that he can buy fractional shares, e.g., $x = 13.4$, but cannot sell short, e.g., $x = -100$.)

If Smith knows that the price is lower today than it will be tomorrow, his payoff function will be $\pi(x) = x$ and he will want to buy an infinite number of shares, which is not an equilibrium purchase. If the amount he buys is restricted to be less than or equal to 1,000, however, then the strategy set is bounded (by 1,000), and an equilibrium exists, $-x = 1{,}000$.

x=1000

Sometimes, as in the Cournot Game discussed earlier in this chapter, the unboundedness of the strategy sets does not matter because the optimum is an interior solution. In other games, though, it is important, not just to get a determinate solution but because the real world is a rather bounded place. The solar system is finite in size, as is the amount of human time past and future.

(2) An open strategy space
Again consider Smith. Let his strategy be $x \in [0, 1,000)$, which is the same as saying that $0 \leq x < 1{,}000$, and his payoff function be $\pi(x) = x$. Smith's strategy set is bounded (by 0 and 1,000), but it is open rather than closed, because he can choose any number less than 1,000, but not 1,000 itself. This means no equilibrium will exist, because he wants to buy 999.999… shares. This is just a technical problem; we ought to have specified Smith's strategy space to be $[0, 1,000]$, and then an equilibrium would exist, at $x = 1,000$.

(3) A discrete strategy space (or, more generally, a nonconvex strategy space)
Suppose we start with an arbitrary pair of strategies s_1 and s_2 for two players. If the players' strategies are strategic complements, then if Player 1 increases his strategy in response to s_2, Player 2 will increase his strategy in response to that. An equilibrium will occur where the players run into diminishing returns or increasing costs, or where they hit the upper bounds of their strategy sets. If, on the other hand, the strategies are strategic substitutes, then if Player 1 increases his strategy in response to s_2, Player 2 will in turn want to reduce his strategy. If the strategy spaces are continuous, this can lead to an equilibrium, but if they are discrete, Player 2 cannot reduce his strategy just a little bit – he has to jump down a discrete level. That could then induce Player 1 to increase his strategy by a discrete amount. This jumping of responses can be never-ending – there is no equilibrium.

That is what is happening in the Welfare Game of table 3.1 in this chapter. No compromise is possible between a little aid and no aid, or between working and not working – until we introduce mixed strategies. That allows for each player to choose a continuous amount of his strategy.

This problem is not limited to games such as 2-by-2 games that have discrete strategy spaces. Rather, it is a problem of "gaps" in the strategy space. Suppose we had a game in which the government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non convex – there is gap in it. (For a space $\{x\}$ to be convex, it must be true that if x_1 and x_2 are in the space, so is $\theta x_1 + (1 - \theta)x_2$ for any $\theta \in [0, 1]$.) Without mixed strategies, an equilibrium to the game might well not exist.