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Answers to Odd-Numbered Problems, 4th Edition of Games and Information, Rasmusen

PROBLEMS FOR CHAPTER 3: Mixed and Continuous Strategies

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This appendix contains answers to the odd-numbered problems in the gourth edition of *Games and Information* by Eric Rasmusen, which I am working on now and perhaps will come out in 2006. The answers to the even- numbered problems are available to instructors or self-studiers on request to me at Erasmuse@indiana.edu.

Other books which contain exercises with answers include Bierman & Fernandez (1993), Binmore (1992), Fudenberg & Tirole (1991a), J. Hirshleifer & Riley (1992), Moulin (1986), and Gintis (2000). I must ask pardon of any authors from whom I have borrowed without attribution in the problems below; these are the descendants of problems that I wrote for teaching without careful attention to my sources.

PROBLEMS FOR CHAPTER 3: Mixed and Continuous Strategies

3.1. Presidential Primaries

Smith and Jones are fighting it out for the Democratic nomination for President of the United States. The more months they keep fighting, the more money they spend, because a candidate must spend one million dollars a month in order to stay in the race. If one of them drops out, the other one wins the nomination, which is worth 11 million dollars. The discount rate is r per month. To simplify the problem, you may assume that this battle could go on forever if neither of them drops out. Let θ denote the probability that an individual player will drop out each month in the mixed- strategy equilibrium.

(a) In the mixed-strategy equilibrium, what is the probability θ each month that Smith will drop out? What happens if r changes from 0.1 to 0.15?

Answer. The value of exiting is zero. The value of staying in is $V = \theta(10) + (1-\theta)(-1+\frac{V}{1+r})$. Thus, $V - (1-\theta)\frac{V}{1+r} = 10\theta - 1 + \theta$, and $V = \frac{(11\theta-1)(1+r)}{(r+\theta)}$. As a result, $\theta = 1/11$ in equilibrium.

The discount rate does not affect the equilibrium outcome, so a change in r produces no observable effect.

(b) What are the two pure-strategy equilibria?

Answer. (Smith drops out, Jones stays in no matter what) and (Jones drops out, Smith stays in no matter what).

(c) If the game only lasts one period, and the Republican wins the general election if both Democrats refuse to give up (resulting in Democrat payoffs of zero), what is the probability γ with which each Democrat drops out in a symmetric equilibrium?

Answer. The payoff matrix is shown in Table A3.1.

Table A3.1: Fighting Democrats

		Jones	
		Exit (γ)	*Stay* ($1-\gamma$)
Smith	*Exit* (γ)	0,0	0, 10
	Stay ($1-\gamma$)	10,0	-1,-1

The value of exiting is $V(exit) = 0$. The value of staying in is $V(Stay) = 10\gamma + (-1)(1-\gamma) = 11\gamma - 1$. Hence, each player stays in with probability $\gamma = 1/11$ — the same as in the war of attrition of part (a).

3.3. Uniqueness in Matching Pennies

In the game Matching Pennies, Smith and Jones each show a penny with either heads or tails up. If they choose the same side of the penny, Smith gets both pennies; otherwise, Jones gets them.

(a) Draw the outcome matrix for Matching Pennies.

Table A3.2: "Matching Pennies"

		Jones	
		Heads (θ)	*Tails* ($1-\theta$)
Smith:	*Heads* (γ)	$1, -1$	$-1, 1$
	Tails ($1-\gamma$)	$-1, 1$	$1, -1$

Payoffs to: (Smith, Jones).

(b) Show that there is no Nash equilibrium in pure strategies.

Answer. ($Heads, Heads$) is not Nash, because Jones would deviate to $Tails$. *Heads, Tails* is not Nash, because Smith would deviate to $Tails$. *(Tails, Tails)* is not Nash, because Jones would deviate to $Heads$. *(Tails, Heads)* is not Nash, because Smith would deviate to $Heads$.

(c) Find the mixed-strategy equilibrium, denoting Smith's probability of $Heads$ by γ and Jones' by θ.

Answer. Equate the pure strategy payoffs. Then for Smith, $\pi(Heads) = \pi(Tails)$, and

$$\theta(1) + (1-\theta)(-1) = \theta(-1) + (1-\theta)(1), \qquad (1)$$

which tells us that $2\theta - 1 = -2\theta + 1$, and $\theta = 0.5$. For Jones, $\pi(Heads) = \pi(Tails)$, so

$$\gamma(-1) + (1-\gamma)(1) = \gamma(1) + (1-\gamma)(-1), \tag{2}$$

which tells us that $1 - 2\gamma = 2\gamma - 1$ and $\gamma = 0.5$.

(d) Prove that there is only one mixed-strategy equilibrium.

Answer. Suppose $\theta > 0.5$. Then Smith will choose *Heads* as a pure strategy. Suppose $\theta < 0.5$. Then Smith will choose *Tails* as a pure strategy. Similarly, if $\gamma > 0.5$, Jones will choose *Tails* as a pure strategy, and if $\gamma < 0.5$, Jones will choose *Heads* as a pure strategy. This leaves (0.5, 0.5) as the only possible mixed- strategy equilibrium.

Compare this with the multiple equilibria in problem 3.5. In that problem, there are three players, not two. Should that make a difference?

3.5. A Voting Paradox

Adam, Karl, and Vladimir are the only three voters in Podunk. Only Adam owns property. There is a proposition on the ballot to tax property-holders 120 dollars and distribute the proceeds equally among all citizens who do not own property. Each citizen dislikes having to go to the polling place and vote (despite the short lines), and would pay 20 dollars to avoid voting. They all must decide whether to vote before going to work. The proposition fails if the vote is tied. Assume that in equilibrium Adam votes with probability θ and Karl and Vladimir each vote with the same probability γ, but they decide to vote independently of each other.

(a) What is the probability that the proposition will pass, as a function of θ and γ?

Answer. The probability that Adam loses can be decomposed into three probabilities— that all three vote, that Adam does not vote but one other does, and that Adam does not vote but both others do. These sum to $\theta\gamma^2 + (1-\theta)2\gamma(1-\gamma) + (1-\theta)\gamma^2$, which is, rearranged, $\gamma^2 + 2\gamma(1-\gamma)(1-\theta)$ or $\gamma(2\gamma\theta - 2\theta + 2 - \gamma)$.

(b) What are the two possible equilibrium probabilities γ_1 and γ_2 with which Karl might vote? Why, intuitively, are there two symmetric equilibria?

<u>*Answer.*</u> The equilibrium is in mixed strategies, so each player must have equal payoffs from his pure strategies. Let us start with Adam's payoffs. If he votes, he loses 20 immediately, and 120 more if both Karl and Vladimir have voted.

$$\pi_a(Vote) = -20 + \gamma^2(-120). \tag{3}$$

If Adam does not vote, then he loses 120 if either Karl or Vladimir vote, or if both vote:

$$\pi_a(Not\ Vote) = (2\gamma(1-\gamma) + \gamma^2)(-120) \tag{4}$$

Equating $\pi_a(Vote)$ and $\pi_a(Not\ Vote)$ gives

$$0 = 20 - 240\gamma + 240\gamma^2. \tag{5}$$

The quadratic formula solves for γ:

$$\gamma = \frac{12 \pm \sqrt{144 - 4 \cdot 1 \cdot 12}}{24}. \tag{6}$$

This equations has two solutions, $\gamma_1 = 0.09$ (rounded) and $\gamma_2 = 0.91$(rounded).

Why are there two solutions? If Karl and Vladimir are sure not to vote, Adam will not vote, because if he does not vote he will win, 0-0. If Karl and Vladimir are sure to vote, Adam will not vote, because if he does not vote he will lose, 2-0, but if he does vote, he will lose anyway, 2-1. Adam only wants to vote if Karl and Vladimir vote with moderate probabilities. Thus, for him to be indifferent between voting and not voting, it suffices either for γ to be low or to be high– it just cannot be moderate.

(c) What is the probability θ that Adam will vote in each of the two symmetric equilibria?

<u>*Answer.*</u> Now use the payoffs for Karl, which depend on whether Adam and Vladimir vote.

$$\pi_c(Vote) = -20 + 60[\gamma + (1-\gamma)(1-\theta)] \tag{7}$$

$$\pi_c(Not\ Vote) = 60\gamma(1-\theta). \tag{8}$$

Equating these and using $\gamma^* = 0.09$ gives $\theta = 0.70$ (rounded). Equating these and using $\gamma^* = 0.91$ gives $\theta = 0.30$ (rounded).

(d) What is the probability that the proposition will pass?

<u>*Answer.*</u> The probability that Adam will lose his property is, using the equation in part (a) and the values already discovered, either 0.06 (rounded) $(= (0.7)(0.09)^2 + (0.3)(2(0.09)(0.91) + (0.09)^2))$ or 0.94 (rounded $(= (0.3)(0.91)^2 + (0.7)(2(0.91)(0.09) + (0.91)^2))$.

3.7. Nash Equilibrium

Find the unique Nash equilibrium of the game in Table 9.

Table 9: A Meaningless Game

		Column		
		Left	*Middle*	*Right*
	Up	1,0	$10, -1$	$0, 1$
Row:	*Sideways*	$-1, 0$	-2,-2	$-12, 4$
	Down	$0, 2$	$823, -1$	$2, 0$

Payoffs to: (Row, Column).

<u>*Answer.*</u> The equilibrium is in mixed strategies. Denote Row's probability of Up by γ and Column's probability of $Left$ by θ. Strategies *Sideways* and $Middle$ are strongly dominated strategies, so we can forget about them. Row has no reason ever to choose *Sideways*, and Column has no reason ever to choose $Middle$.

In equilibrium, Row must be indifferent between Up and $Down$, so

$$\pi_R(Up) = \theta(1) + (1-\theta)(0) = \pi_R(Down) = \theta(0) + (1-\theta)(2)$$

This yields $\theta^* = 2/3$. Column must be indifferent between $Left$ and $Right$, so

$$\pi_R(Left) = \gamma(0) + (1-\gamma)(2) = \pi_R(Right) = \gamma(1) + (1-\gamma)(0)$$

This yields $\gamma^* = 2/3$.

3.9 (hard). Cournot with Heterogeneous Costs
On a seminar visit, Professor Schaffer of Michigan told me that in a Cournot model with a linear demand curve $P = \alpha - \beta Q$ and constant marginal cost C_i for firm i, the equilibrium industry output Q depends on $\Sigma_i C_i$, but not on the individual levels of C_i. I may have misremembered. Prove or disprove this assertion. Would your conclusion be altered if we made some other assumption on demand? Discuss.

Answer. A good approach when stymied is to start with a simple case. Here, the two-firm problem is the obvious simple case. Prove the proposition for the simple case, and then use that as a pattern to extend it. (Also, you can disprove a general proposition using a simple counterexample, though you cannot prove one using a simple example.)

Note that you cannot assume symmetry of strategies in this game. It is plausible, though not always correct (remember Chicken), when players are identical, but they are not here— firms have different costs. So we would expect their equilibrium outputs to differ.

The proposition is true.

$$\pi_j = (\alpha - \beta \Sigma_i Q_i - C_j) Q_j,$$

so

$$\frac{d\pi_j}{dQ_j} = \alpha - \beta \Sigma_{i \neq j} Q_i - 2\beta Q_j - C_j = 0,$$

and

$$Q_j = \frac{\alpha - C_j - \beta \Sigma_{i \neq j} Q_i}{2\beta}.$$

Industry output is

$$\Sigma_j Q_j = \Sigma_j \frac{\alpha - C_j - \beta \Sigma_{i \neq j} Q_i}{2\beta} = \Sigma_j \frac{\alpha - C_j}{2\beta} - \Sigma_j \frac{\Sigma_{i \neq j} Q_i}{2}.$$

The first term of this last expression depends on the sum of the firms' cost parameters, but not on their individual levels. The second term adds up the outputs of all but one firm N times, and so equals $(N-1)$ times the sum of the output, $(N-1)\Sigma_j Q_j$. Thus,

$$\begin{aligned}\Sigma_j Q_j &= \Sigma_j \tfrac{\alpha - C_j}{2\beta} - (N-1)\left(\tfrac{\Sigma_{i \neq j} Q_i}{2}\right) \\ &= \Sigma_j \tfrac{\alpha - C_j}{\beta(N+1)}.\end{aligned}$$

This does not depend on the cost parameters except through their sum. Q.E.D.

A caveat: This proof implicitly assumed that every firm had low enough costs that it would produce positive output. If it produces zero output, it is at a corner solution, and the first order condition does not hold, so the proof fails. Thus, the validity of the proposition depends on the following being true for every j:

$$Q_j = \frac{C_j - \alpha - \beta\Sigma_{i \neq j} Q_i}{2\beta} > 0.$$

This condition is not stated in terms of the primitive parameters (it depends on $\Sigma_{i \neq j} Q_i$), so to be quite proper I ought to solve it out further, but I will not do that here.

The result does depend on linear demand. This can be shown by counterexample. Suppose $P = \alpha - \beta Q^2$. Then, attempting the construction above,

$$\pi_j = (\alpha - \beta(\Sigma_i Q_i)^2 - C_j)Q_j,$$

so

$$\frac{d\pi_j}{dQ_j} = \alpha - 3\beta Q_j^2 - 2\beta\Sigma_{i \neq j} Q_i Q_j - C_j = 0.$$

Solving this for Q_j will involve taking a square root of C_j. But if Q_j is a function of the square root of C_j, then increasing C_j by a given amount and decreasing C_l by the same amount will *not* keep the sum of Q_j and Q_l the same, unlike before, where Q_j was a linear function of C_j. So the proposition fails for quadratic demand, and, more generally, whenever demand is nonlinear.

3.11. Finding Nash Equilibria
Find all of the Nash equilibria for the game of Table 10.

Table 10: A Takeover Game

		Target		
		Hard	*Medium*	*Soft*
	Hard	-3, -3	$-1, 0$	$4, 0$
Raider:	*Medium*	$0, 0$	2,2	$3, 1$
	Soft	0,0	$2, 4$	$3, 3$

Payoffs to: (Raider, Target).

Answer. The three equilibria are in pure strategies: (Hard, Soft), (Medium, Medium), and (Soft, Medium).

There are two mixed strategy equilibria.

(1) (Raider: Hard. Target: Mix between Medium and Soft.) Notice that for Target, Hard is a dominated strategy. That means it will not be part of any mixed strategy equilibrium. Next, notice that Soft is weakly dominated. Thus, if Raider ever plays anything but Hard, Target will want strongly to play Medium. What if Raider plays Hard? Then Target would be willing to mix between Medium and Soft. If Target plays Medium with a probability of γ, Raider's payoff from Hard is $(-1)\gamma + 4(1-\gamma)$, whereas his payoff from Medium or Soft is $(2)\gamma + 3(1-\gamma)$. Equating these yields $\gamma^* =.25$. If γ is no bigger than .25, we have a Nash equilibrium.

(2) (Raider: Mix between Medium and Soft. Target: Medium.) How about if Target plays Medium and Raider mixes? Raider would only want to mix between Medium and Soft. But that would generate a Nash equilibrium, for any mixing probability, since Raider gets 2 no matter what, and Target prefers Medium no matter what the mixing probability may be.

3.13. The Kosovo War
Senator Robert Smith of New Hampshire said of the US policy in Serbia

of bombing but promising not to use ground forces, "It's like saying we'll pass on you but we won't run the football." (*Human Events*, p. 1, April 16, 1999.) Explain what he meant, and why this is a strong criticism of U.S. policy, using the concept of a mixed strategy equilibrium. (Foreign students: in American football, a team can choose to throw the football (to pass it) or to hold it and run with it to move towards the goal.) Construct a numerical example to compare the U.S. expected payoff in (a) a mixed strategy equilibrium in which it ends up not using ground forces, and (b) a pure strategy equilibrium in which the U.S. has committed not to use ground forces.

Answer. Senator Smith meant that by declaring our action, we have allowed the Yugoslavs to choose a better response (for them) than if we left them uncertain. Thus, the declaration reduces the expected U.S. payoff. Rather than mixing– which means to be unpredictable– we chose a pure strategy.

Ane example can show this. Suppose that the US has the two alternatives of Air and Ground, and the Yugoslavs have the two alternatives of Air Defense and Ground Defense. Air and Air Defense represent policies of just positioning forces for an air war; Ground and Ground Defense represent policies that also prepare for ground war.

Let the payoffs be as in Table A3.3.

Table A3.3: The Kosovo War

		Yugoslavia	
		Air Defense (γ)	Ground Defense
US	Air (θ)	0,0	1,-1
	Ground	2,-5	-2,-2

Payoffs to: (U.S., Yugoslavia).

(a) In the mixed strategy equilibrium, Yugoslavia chooses its probability of Air Defense to equate the US payoffs from Air and Ground. Thus,

$$\pi_{US}(Air) = \gamma(0) + (1-\gamma)(1) = 2\gamma + (1-\gamma)(-2) = \pi_{US}(Ground). \qquad (9)$$

This reduces to $1-\gamma = 2\gamma - 2 + 2\gamma$, so $3 = 5\gamma$, and $\gamma = 3/5$. The U.S. expected payoff from choosing Air is then $\pi_{US}(Air) = \gamma(0) + (1-\gamma)(1) = 1 - 3/5 = .4$.

(b) If the U.S. instead moves first and chooses Air, Yugoslavia will respond with Air Defense, and the U.S. expected payoff is 0.

Thus, by volunteering to move first, the U.S. reduces its payoff.

3.15. Coupon Competition

Two marketing executives are arguing. Smith says that reducing our use of coupons will make us a less aggressive competitor, and that will hurt our sales. Jones says that reducing our use of coupons will make us a less aggressive competitor, but that will end up helping our sales.

Discuss, using the effect of reduced coupon use on your firm's reaction curve, under what circumstance each executive could be correct.

Answer. There are a couple of ways to look at this problem.

(1) One way is that the important strategy is coupon use directly. Smith thinks that coupons are strategic substitutes, so when we reduce our use of coupons, our rival will increase their use, and we will be hurt. Jones thinks that coupons are strategic complements, so when we reduce our use of coupons, our rival will reduce their use too, to the benefit of both of us.

(2) A second way is in terms of how coupon use affects how the two companies play a game in the consumer market.

Smith thinks that our firm is in a market with downward sloping reaction curves in the important strategy– strategic substitutes, as with Cournot competition. If we use fewer coupons, that will shift in our reaction curve, and we will end up with lower sales. We need to be "lean and hungry", because if we use coupons to make us softer in the product market, our rival will react by being tougher.

The important strategy might be, for example, output, and if we use more coupons, that will make us less willing to produce high output in reaction to what our rival does, because each sale will be profitable. In the end, we will contract our output and our rival will increase his.

Jones thinks that our firm is in a market with upward sloping reaction curves in the important strategy– strategic complements, as with Bertrand competition. If the important variable is price, and we use fewer coupons, that will shift out our reaction curve, and we will increase our price. So will our rival, and we will both end up with higher profits.

We thus adopt a "fat cat" strategy– we use more coupons to make us softer in the product market, and our rival becomes softer in response.