Zermelo and the Early History of Game Theory[1]

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Abstract

In the modern literature on game theory there are several versions of what is known as Zermelo's theorem. It is shown that most of these modern statements of Zermelo's theorem bear only a partial relationship to what Zermelo really did. We also give a short survey and discussion of the closely related but almost unknown work by König and Kálmar. Their papers extend and considerably generalize Zermelo's approach. A translation of Zermelo's paper is included in the appendix.

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1 Introduction

It is generally agreed that the first formal theorem in the theory of games was proved by E. Zermelo[2] in an article on chess appearing in German in 1913 (Zermelo (1913)). In the modern literature on game theory there are many variations in the statements of this theorem. Some writers claim that Zermelo showed that chess is determinate, e.g. Aumann (1989b, p.1), Eichberger (1993, p.9) or Hart (1992, p.30): "In chess, either white can force a win, or black can force a win, or both sides can force a draw." Others state more general propositions under the heading of Zermelo's theorem, e.g. Mas Colell et al. (1995, p.272): "Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived by backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner." Dimand and Dimand (1996, p.107) claim that Zermelo showed that white can not lose: "[I]n a finite game, there exists a strategy whereby a first mover (...) cannot lose, but it is not clear whether there is a strategy whereby the first mover can win." In addition many authors claim that Zermelo's method of proof was by backward induction, e.g. Binmore (1992, p.32): "Zermelo used this method way back in 1912 to analyze Chess. It requires starting from the end of the game and then working backwards to its beginning. For this reason, the technique is sometimes called 'backward induction'."

Despite a growing interest in the history of game theory, see for example Aumann (1989a), Dimand and Dimand (1996, 1997), Kuhn (1997), Leonard (1995) and Weintraub (1992), confusion, at least in the English language literature, as to the contribution made by Zermelo and some of the other early game theorists seems to prevail. This problem may be due, in part, to a language barrier. Many of the early papers in game theory were not written in English and have not been translated. For example, to the best of our knowledge, there is no English version of the Zermelo article on chess. The same holds for the lesser known but related work by König (1927)[3]. A second paper related to that of Zermelo, Kalmár (1928/29)[4], has recently been translated, see Dimand and Dimand (1997)[5]. The lack of an English translation may help to explain the apparent confusion in the modern game theory literature as to what Zermelo's theorem states and the method of proof employed. It appears that there is only one accurate summary of Zermelo's paper. This was published in a book on the

[2] Ernst Friedrich Ferdinand Zermelo (1871-1951), was a German mathematician. He studied mathematics, physics and philosophy at Halle, Freiburg and Berlin where he received his doctorate in 1894. He taught at Göttingen, Zürich and Freiburg and is best known for his work on the axiom of choice and axiomatic set theory.

[3] Dénes König (1884-1944), was a Hungarian mathematician, the son of the mathematician Julius König. He studied mathematics in Budapest and Göttingen and received his doctorate in 1907. He spent his whole career in Budapest, first as an assistant and later as a professor. Most of König's work was in the field of combinatorics and he wrote the first comprehensive treatise on graph theory, *Theorie der endlichen und unendlichen Graphen* (Theory of Finite and Infinite Graphs).

[4] László Kalmár (1905-1976) was also a Hungarian mathematician. He studied mathematics and physics in Budapest. From 1930 until his death he worked at Szeged University, first as an assistant, later as a professor. His main research was in mathematical logic, computer science and cybernetics.

[5] However, the translation of Kalmár's paper contains so many severe mistakes that it is almost impossible to understand what Kalmár really did.

history of game theory by Vorob'ev (1975) which is unfortunately only available in the original Russian version or in a German translation.

In this note, we attempt to shed some light on the original statement and proof of Zermelo's theorem and the closely related work of König and Kalmár. This will clarify the relationship between Zermelo's result and the modern statements of it. It is shown that most of the modern statements of Zermelo's theorem are to some degree incorrect - only the statement on the determinateness of chess comes close to what Zermelo did, but even this covers only a minor part of his paper. A translation of Zermelo's paper is included in the appendix.[6]

2 Zermelo's two theorems on chess

In his paper, Zermelo concentrates on the analysis of two person games without chance moves where the players have strictly opposing interests. He also assumes that in the game only finitely many positions are possible. However, he allows for infinite sequences of moves since he does not consider stopping rules. Thus, he allows for the possibility of infinite games. This is in contrast to what is normally assumed in the modern literature.[7] He remarks that there are many games of this type but he uses the game of chess as an example since it is the best known of them.

The Zermelo paper addresses two problems: First, what does it mean for a player to be in a 'winning' position and is it possible to define this in an objective mathematical manner; secondly, if he is in a winning position, can the number of moves needed to enforce the win be determined? To answer the first question, he states that a necessary and sufficient condition is the nonemptyness of a certain set, containing all possible sequences of moves such that a player (say white) wins independently of how the other player (black) plays. However, should this set be empty, the best a player could achieve would be a draw. So he defines another set containing all possible sequences of moves such that a player can postpone his loss for an infinite number of moves, which implies a draw.[8] If this set is also empty, the other player can enforce a win. This is the basis for all modern versions of Zermelo's theorem. The possibility of both sets being empty means that white can not guarantee that he will not lose. This contradicts the 'first mover has an advantage' version of Zermelo's theorem given by Dimand and Dimand (1996).

However, this problem was only of minor interest for Zermelo. He was much more interested in the following question: Given that a player (say white) is in 'a winning position', how long does it take for white to enforce a win? Zermelo claimed that it will never take more moves than there are positions in the game. His proof of this is by contradiction: Assume that white can win in a number of moves which is larger than the number of positions. Of course, at least one winning position must have appeared twice. So white could have played at the first occurrence in the same way as he does at the second and thus could have won in less moves than there are positions.

[6] A Russian translation was published in 1961.

[7] All of Aumann (1989b), Binmore (1992), Dimand and Dimand (1996), Eichberger (1993), Hart (1992) and Mas Colell et al. (1995), for example, assume a finite game.

[8] Zermelo does not consider the stalemate position, which ends the game in finitely many moves without any party winning the game.

Notice that in Zermelo's paper, in contrast to what is often claimed, no use is made of backward induction. The first time a proof by backward induction is used seems to be in von Neumann and Morgenstern (1944). The first mentioning of Zermelo in connection with induction was made by Kuhn (1953).

3 König's paper and Zermelo's proof

Thirteen years after Zermelo, König published a paper 'Über eine Schlußweise aus dem Endlichen ins Unendliche' (1927) (On a Method of Conclusion From the Finite to the Infinite). In this paper, König states a general lemma from the theory of sets which he formulates in a graph-theoretic framework. This theorem states that: *If the countably infinite set of points (= vertices) of an infinite graph G can be partitioned into countably many finite non-empty sets $E_1, E_2, E_3, \ldots$ such that each point of E_{n+1} $(n = 1, 2, 3 \ldots)$ is connected with a point of E_n by an edge, there exists in the graph an 'infinite path' $a_1, a_2, a_3, \ldots$, containing in each of the sets E_n a point a_n.* (König (1928, p.121))

He applies this theorem to a number of different topics including the colouring of maps, relationships between relatives, and to the theory of games. The latter application was suggested to him by John von Neumann. Von Neumann conjectured, and König proved the proposition that "if q is such a winning position (...) there exists a number N which depends on q such that starting from this position q, white can enforce a win in less than N moves."

This is a generalization of Zermelo's second problem to games with an infinite number of positions. However, from each position there are only finitely many new positions that can be reached. An example would be chess played on an infinite board, where the pieces have to move as on a normal chessboard. König shows that if one of the players can win at all, there is only a finite number of moves necessary to do so.

In addition, he argued that Zermelo's proof was incomplete for two reasons: First, he remarks that Zermelo failed to prove that a player, say white, who is in a winning position is *always* able to enforce a win in a number of moves that is less than the number of positions in the game. Zermelo had argued that white could do so by changing his behaviour at the first occurrence of any repeated winning position and thus win without repetition. However Zermelo implicitly assumes that black would never change his behaviour at any occurrence of such a position. He just considered the special case of unchanging behaviour on black's part. What he needed to show was that his claim is true for *all* possible moves by black.

The second argument of König was, that the strategy 'do the same at the first occurrence of a position as at the second and thus win in less moves' cannot be carried out if it is black's turn to move in this position. However, the second argument is incorrect since Zermelo considers two positions as different if black or white has to move.

In an appendix to König's paper Zermelo provided a new proof of his theorem without referring to white winning without any repeated positions. Instead, he uses the lemma of König. Zermelo also supplies a proof of von Neumann's conjecture without referring to the general lemma. However, as König points out, Zermelo implicitly proves the lemma itself.

4 Kalmár's generalization of the work of Zermelo and König

One year after the publication of König's work, Kalmár published a paper 'Zur Theorie der abstrakten Spiele' (1928/29) (On the Theory of Abstract Games). Starting from the work of Zermelo and König, he generalizes both models by allowing not only infinitely many positions in a game, but also infinitely many new positions being able to be reached from any given position. The major question he considers is that of Zermelo and König: If a player is in a winning position, is there an upper bound to the number of moves that it takes him to enforce a win?

As König pointed out, in the original formulation of Zermelo's proof there is a gap since Zermelo claimed, but did not show, that a player who is in a winning position can always win 'without repetition'. However, König did not try to bridge this gap but used a different method of proof instead. In contrast, Kalmár's approach returns to Zermelo's original idea. Without making any assumption on the finiteness of the number of positions etc., he is able to show that even in this much more general class of games Zermelo's claim holds: If a win is possible, it can be enforced without any position appearing twice.

In the first section of his paper Kalmár defines the concepts of a game, which is given by a set of positions q_i and a set of ordered pairs (q_i, q_j), where q_i is a position at which player i has the move and q_j is a position at which player j has the move, such that $q_i \rightarrow q_j$ is a feasible move. In other words, this set implies the rules of the game. Further, winning and losing positions are defined as well as the idea of a 'subgame'. However, his concept of a subgame is different from the concept used in the modern literature. In Kalmár's terminology, a subgame is given by any subset of the positions and the corresponding subset of feasible moves. He also introduces the concept of a strategy which he calls a 'tactic'. A tactic 'in the strict sense' (i.t.s.s.) for player A is a subgame such that each move which is feasible for player B in the original game is also feasible in the subgame, i.e. does not restrict player B.

Using the concept of a tactic in the strict sense he defines winning, non-losing or losing positions in the strict sense. Of course, a position is only called a winning position if a player can win in a finite number of moves. He then shows that a winning position i.t.s.s. for player A is always a losing position i.t.s.s. for player B.

To introduce these concepts 'in a weak sense', Kalmár uses the notion of a 'script game' $\mathcal{S}$ of a given game S. A position in the script game is defined as a position q_n in the game S *including* the history of this position, i.e. the sequence $q_0, q_1, q_2, \ldots, q_n$. Of course, moves in the script game have to be consistent with the rules of the game S.

Using the script game, he defines a tactic 'in the weak sense' (i.t.w.s.) which is a tactic in the strict sense in the script game. In other words: a tactic i.t.s.s. depends only on the current position while a tactic i.t.w.s. takes into account the whole history of the game. Analogously, he defines winning and losing positions etc. in the weak sense and proves that a winning position i.t.w.s. for one player is a losing position i.t.w.s. for the other. In a footnote, Kalmár mentions, that König informed him, that this theorem was known to von Neumann. This comment suggests that the three men were aware, at least indirectly, of each other's work. Of course, if a player can enforce a win without taking into account the history of the game, he can also enforce a win if he does so, i.e. a winning position i.t.s.s. is always a winning position i.t.w.s. He also proves that a losing position in the strict sense is the same as in the weak sense.

In section II he uses the concepts and theorems developed to formulate and prove the first of his two main theorems: *If player A is in a winning position q_0, then q_0 is also a winning position without repetition for A.* (Kálmar (1928/29, p.79)) Here, a winning position is without repetition if there exists a winning strategy such that during the play of the game no position is repeated.

To prove his claim, Kalmár characterizes the set of winning positions for player A as follows: The set of winning positions i.t.w.s. is the smallest set $\mathcal{M}$ of positions in the game S with the following closure property: If it is A's turn to move and if A can make a move to a position in $\mathcal{M}$, then A has already started from a position in $\mathcal{M}$. If every move of B leads to a position in $\mathcal{M}$, then B has started from a position in $\mathcal{M}$.

He shows that every set $\mathcal{M}$ with this property contains the set of winning positions for A and that the winning positions without repetition have this closure property. Since the set of winning positions is the smallest set with this property, the set of winning positions without repetition contains the set of all winning positions. Or stated otherwise, if player A is in a winning position, he is also in a winning position without repetition.

This result shows that the gap in Zermelo's proof can be bridged using Zermelo's original idea of non-repetition of positions. This is in contrast to König's conjecture which suggests that for the proof of Zermelo's theorem the boundedness of the number of moves has to be shown first.

In the last part of his paper, Kalmár proves that if a player is in a winning position, there exists a - possibly transfinite - ordinal number of moves in which this player can win independently of the behaviour of his opponent.

If in addition, the cardinality of the set of possible moves is smaller than a transfinite cardinal number μ, then a player in a winning position can win in $\alpha < \mu$ moves. The possibly transfinite ordinal number α is the generalization of the natural number N in König's theorem.

In the summary of his paper, Kalmár gives a clear and concise formulation of what is referred to today as Zermelo's theorem, as stated in the first interpretation above.

"Each position of the game S belongs either to the set of the winning positions of A, $\mathcal{G}_A$ or to the set of winning positions of B, $\mathcal{G}_B$ or it belongs to the set $\mathcal{R}$ of draw positions, i.e. positions where A as well as B can avoid a loss by using an appropriate non losing tactic. For each position which belongs to $\mathcal{G}_A$ ($\mathcal{G}_B$), there is a winning tactic (also in the strict sense) G_A (G_B) which depends only on the game S by which players A (B) can enforce a win. For each position which belongs to $\mathcal{R}$ there is a non-losing tactic (also in the strict sense) R_A (R_B) which depends only on the game S by which A (B) can avoid a loss." (Kalmár (1928/9, p.84))

Kalmár's generalization of both Zermelo's and König's frameworks is the last contribution in a line of research which was mainly concerned with the following question: Given a winning position, how quickly can a win be enforced? His paper proves the claim made by Zermelo, but doubted by König, that winning without repetition is possible if winning is possible at all.

In a recent book by Dimand and Dimand (1996) some comments on the work of Kalmár are included, which are however mostly incorrect. They claim that "... Kalmár attempted to show that a game of perfect information has a solution by giving a more general proof of non-repetition which, unlike König's, did not depend on any finiteness assumption. The original thought process followed by Kalmár was, in fact, backwards induction. Kalmár's proof of non-repetition by backward induction (a concept which in itself makes non-repetition intuitive) rested on defining the types of positions which could be reached in play as winning, non-losing or losing. Unfortunately, Kalmár did not show that the types of positions he defined must appear on every branch of the potentially infinitely and thus infinitely branched game tree. Without this sort of spanning argument for the types of nodes defined, Kalmár's proof was not valid. Interesting features of Kalmár's approach were his definition of the 'script game' (what we call a subgame) and his definition of strategy."(p.124-5)

First, it was not Kalmár's intention to show that a solution for this class of games exists, but that if a player can win, he can do so without repetition and that there is an upper bound to the number of moves necessary. His proof is *not* an existence proof. Secondly, Zermelo's original thought process was not backward induction but the idea of non-repetition. Thirdly Kalmár's proof of non-repetition is not by backward induction, but by characterizing the set of winning positions and by showing that the set of winning positions without repetition is equal to this set. Fourthly, his proof does not rest on defining the types of positions as winning, non-losing or losing. The characterization of a winning position is sufficient for the proof of non-repetition. He does not need any spanning argument and his proof is perfectly valid. Finally, the concept of a 'script game' is not the same as a subgame in the modern sense but a position in the game with its history. A subgame looks 'forward' from a given position while a script game looks 'back'.

5 Conclusion

This short survey on the work of Zermelo, König and Kalmár shows, that these early game theorists were dealing with what today would be called two-person zero-sum games with perfect information. The common starting point for their analysis was the concept of a winning position, defined in a precise mathematical way: If a player is in a winning position, then he can always enforce a win no matter what strategy the other player may employ.

Zermelo, König, and Kalmár's main interest was to find an answer to the question: Given that a player is in a winning position is there an upper bound on the number of moves in which he can enforce a win? Or, for the case of being in a losing position, how long can a loss be postponed?

Thus, the problem of strategic interaction and the problem of an equilibrium were not concerns for Zermelo, König, and Kalmár. They did not ask the question: How should a player behave to achieve a good result? This was the main question von Neumann asked in his paper 'Zur Theorie der Gesellschaftsspiele' (1928) (On the Theory of Strategic Games). In contrast to the work of Zermelo, König, and Kalmár, von Neumann's main concerns were the strategic interaction between players and the concept of an equilibrium. These two ideas have become the building blocks of modern noncooperative game theory. The concerns of Zermelo, König, and Kalmár have been answered at a very high level of generality in the paper by Kalmár and thus have not generated an ongoing research agenda.

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Appendix

Ernst Zermelo: On an Application of Set Theory to the Theory of the Game of Chess[9]

The following considerations are independent of the special rules of the game of chess and are valid in principle just as well for all similar games of reason, in which two opponents play against each other; for the sake of determinateness they shall be exemplified by chess as the best known of all games of this kind. Also they do not deal with any method of practical play, but only with the answer to the question: can the value of an arbitrary position, which could possibly occur during the play of a game as well as the best possible move for one of the playing parties be determined or at least defined in a mathematically-objective manner, without having to make reference to more subjective-psychological notions such as the 'perfect player' and similar ideas? That this is possible at least in singular special cases is shown by the so called 'chess problems', i.e. examples of positions in which it can be proved that the player whose turn it is to move can enforce checkmate in a prescribed number of moves. However, it seems to me worth considering whether such an evaluation of a position is at least theoretically conceivable and does make any sense at all in other cases as well, where the exact execution of the analysis finds a practically insurmountable obstacle in the enormous complication of possible continuations, and only this validation would give the secure basis for the practical theory of the 'endgames' and the 'openings' as we find them in textbooks on chess. The method used in the following for the solution of the problem is taken from the 'theory of sets' and the 'logical calculus' and shows the fertility of these mathematical disciplines in a case, where almost exclusively *finite* totalities are concerned.

Since the number of squares and of the moving pieces is finite, so also is the set P of possible positions $p_0, p_1, p_2, \ldots, p_t$, where positions always have to be considered as different, depending on whether white or black has to move, whether one of the parties already has castled, a given pawn has been 'promoted' etc.

Now let q be one of these positions, then starting from q, 'endgames' $\mathbf{q} = (q, q_1, q_2, \ldots)$ are possible, that is sequences of positions, which begin with q and follow each other in accordance with the rules of the game, so that each position q_λ emerges from the previous one $q_{\lambda-1}$ by an admissible move of either white or black in an alternating way. Such a possible endgame $\mathbf{q}$ can find its natural end either in a 'checkmate' or in a 'stalemate' position but could also - at least theoretically - go on forever in which case the game would without doubt has to be called a draw or 'remis'. The totality Q of all these 'endgames' $\mathbf{q}$ associated with q is always a well defined, finite or infinite subset of the set P^a, which comprises all possible countable sequences formed by elements p of P.[10]

Among these $\mathbf{q}$ endgames some can lead to a win for white in r or less 'moves' (i.e. simple changes of position $p_{\lambda-1} \to p_\lambda$, but not double moves) however this also depends in general on the play of the opponent. What properties does a

[9] Translation by Ulrich Schwalbe and Paul Walker. In our translation we tried to stay as close as possible to the German original.

[10] In modern terminology, P^a would be called the game tree and Q a subgame.

position q have to have so that white, independently of how black plays, can *enforce* a win in at most r moves? I claim, the necessary and sufficient condition for that is the existence of a non-vanishing subset $U_r(q)$ of the set Q with the following properties:

1. All elements $\mathbf{q}$ of $U_r(q)$ end in at most r moves with a win for white, such that no sequence contains more than $r+1$ elements and $U_r(q)$ is definitely finite.

2. If $\mathbf{q} = (q, q_1, q_2, \ldots)$ is an arbitrary element of $U_r(q)$, q_λ an arbitrary element of this sequence which corresponds to a move carried out by black, i.e. always one of even or odd order, depending on whether at q it is white's or black's turn to move, and finally q'_λ a possible variant, such that black could have moved from $q_{\lambda-1}$ to q'_λ as well as to q_λ, then $U_r(q)$ contains in addition at least an element of the form $\mathbf{q}'_\lambda = (q, q_1, \ldots, q_{\lambda-1}, q'_\lambda, \ldots)$, which shares with $\mathbf{q}$ the first λ elements. Indeed in this and only in this case white can start with an arbitrary element q of $U_r(q)$ and in every case, where black plays q'_λ instead of q_λ white can carry on playing with a corresponding $\mathbf{q}'_\lambda$, i.e. win under all contingencies in at most r moves.

Of course there can be several such subsets $U_r(q)$, but the sum of any two always has the same properties and also the union $\bar{U}_r(q)$ of all such $U_r(q)$, which is uniquely determined by q and r and definitely has to be different from 0[11], i.e. has to contain at least one element if such $U_r(q)$ exist at all.

Thus, $\bar{U}_r(q) \neq 0$ is the necessary and sufficient condition such that white can enforce a win in at most r moves. If $r < r'$ then $\bar{U}_r(q)$ is always a subset of $\bar{U}_{r'}(q)$ since every set $U_r(q)$ definitely satisfies the conditions imposed on $U_{r'}(q)$, i.e. has to be contained in $\bar{U}_{r'}(q)$, and to the smallest $r = \rho$, for which $\bar{U}_r(q) \neq 0$ corresponds the common component $U^*(q) = \bar{U}_\rho(q)$ of all such $\bar{U}_r(q)$; this contains all continuations such that white must win in the shortest time. Now all these minimum values $\rho = \rho_q$ have on their part a maximum $\tau \leq t$ which is independent of q, where $t+1$ denotes the number of possible positions, thus $U(q) = \bar{U}_\tau(q) \neq 0$ is the necessary and sufficient condition that in position q some $\bar{U}_r(q)$ does not vanish and white is 'in a winning position' at all. Namely if in a position q the win can be enforced at all, then it can be enforced in at most t moves as we want to show. Indeed every endgame $\mathbf{q} = (q, q_1, q_2, \ldots, q_n)$ with $n > t$ would have to contain at least one position $q_\alpha = q_\beta$ a second time and white could have played at the first appearance of it in the same way as at the second and thus could have won earlier than by move n, i.e. $\rho \leq t$.

If on the other hand $U(q) = 0$, so that white can only achieve a draw, if the opponent plays correctly, but white can also be 'in a losing position' and will try in this case to postpone a checkmate as long as possible. If he should hold out until the sth move there must exist a subset $V_s(q)$ with the following properties:

1. There is no endgame contained in $V_s(q)$ where white loses before the sth move.

2. If $\mathbf{q}$ is an arbitrary element of $V_s(q)$ and if in $\mathbf{q}$ the element q_λ can be replaced with q'_λ by black using an allowed move, then $V_s(q)$ contains at

[11] To denote an empty set, Zermelo uses the symbol 0 instead of $\emptyset$.

least one element of the form

$$\mathbf{q}'_\lambda = (q, q_1, \ldots, q_{\lambda-1}, q'_\lambda, \ldots)$$

that coincides with $\mathbf{q}$ up to the λth member and then continues with q'_λ.

Also these sets $V_s(q)$ are all subsets of their union $\bar{V}_s(q)$ which is uniquely determined by q and s and which has the same property as $V_s(q)$ itself, and for $s > s'$ now $\bar{V}_s(q)$ becomes a subset of $\bar{V}_{s'}(q)$. The numbers s for which $\bar{V}_s(q)$ differs from 0 are either infinite or $\leq \sigma \leq \tau \leq t$, since the opponent, if he can win at all, must be able to enforce a win in at most τ moves[12]. Thus if and only if $V(q) = \bar{V}_{\tau+1}(q) \neq 0$ white can obtain a draw and in the other case, by virtue of $V^*(q) = \bar{V}_\sigma(q)$ he can postpone the loss for at least $\sigma \leq \tau$ moves. Since every $U_r(q)$ certainly satisfies the conditions imposed on $V_s(q)$, each $\bar{U}_r(q)$ is a subset of each set $\bar{V}_s(q)$, and $U(q)$ is a subset of $V(q)$. The result of our examination is thus the following:

To each of the positions q that are possible during play, there correspond two well-defined subsets $U(q)$ and $V(q)$ of the totality of the endgames beginning with q where the second contains the first. If $U(q)$ is different from 0, then white can enforce a win, independently of how black might play and can do so in at most ρ moves by virtue of a certain subset $U^*(q)$ of $U(q)$, but not for certain in fewer moves. If $U(q) = 0$ but $V(q) \neq 0$, then white can at least enforce a draw by virtue of the endgames contained in $V(q)$. However, if $V(q)$ vanishes also and the opponent plays correctly, white can postpone the loss up until the σth move at best by virtue of a well defined subset $V^*(q)$ of continuations. In any case only the games contained in U^* respectively V^* have to be considered as 'correct' for white, with any other continuation he would, if in a winning position, forfeit or delay the certain win or otherwise make possible or accelerate the loss of the game given that the opponent plays correctly. Of course an exact analogy exists for black and only those games that satisfy both conditions *simultaneously* could be considered as played 'correctly' until the end, in any case they form a well defined subset $W(q)$ of Q.

The numbers t and r are independent of the position and only determined by the rules of the game. To each possible position there corresponds a number $\rho = \rho_q$ or $\sigma = \sigma_q$ smaller than τ, depending on whether white or black can enforce a win in ρ respectively σ moves but not less. The special theory of the game would have, as far as possible, to determine these numbers or at least include them within certain boundaries, which hitherto has only been possible for special cases such as the 'problems' or the real 'endgames'. The question as to whether the starting position p_0 is already a 'winning position' for one of the parties is still open. Would it be answered exactly, chess would of course lose the character of a game at all.

[12] Zermelo doesn't define the number σ; it denotes the smallest number of moves for which white can postpone his loss.