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3 Mixed and Continuous Strategies

A pure strategy maps each of a player's possible information sets to one action. $s_i : \omega_i \rightarrow a_i$.

A mixed strategy maps each of a player's possible information sets to a probability distribution over actions.

$$s_i : \omega_i \rightarrow m(a_i), \;\; where \;\; m \geq 0 \;\text{ and }\; \int_{A_i} m(a_i) da_i = 1.$$

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	**3,2**	$\rightarrow$	$-1, 3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1, 1$	$\leftarrow$	**0,0**

Payoffs to: (Government, Pauper). Arrows show how a player can increase his payoff.

If the government plays *Aid* with probability θ_a and the pauper plays *Work* with probability γ_w, the government's expected payoff is

$$\begin{aligned} \pi_{Government} &= \theta_a[3\gamma_w + (-1)(1-\gamma_w)] + [1-\theta_a][-1\gamma_w + 0(1-\gamma_w)] \\ &= \theta_a[3\gamma_w - 1 + \gamma_w] - \gamma_w + \theta_a\gamma_w \\ &= \theta_a[5\gamma_w - 1] - \gamma_w. \end{aligned} \quad (1)$$

Differentiate the payoff function with respect to the choice variable to obtain the first-order condition.

$$\begin{aligned} 0 = \frac{d\pi_{Government}}{d\theta_a} = 5\gamma_w - 1 \\ \Rightarrow \gamma_w = 0.2. \end{aligned} \quad (2)$$

We obtained the pauper's strategy by differentiating the government's payoff!

THE LOGIC

1 I assert that an optimal mixed strategy exists for the government.

2 If the pauper selects *Work* **more than 20 percent of the time, the government always selects** *Aid*. **If the pauper selects** *Work* **less than 20 percent of the time, the government never selects** *Aid*.

3 If a mixed strategy is to be optimal for the government, the pauper must therefore select *Work* **with probability exactly 20 percent.**

To obtain the probability of the government choosing Aid:

$$
\begin{aligned}
\pi_{Pauper} &= \gamma_w(2\theta_a + 1[1-\theta_a]) + (1-\gamma_w)(3\theta_a + [0][1-\theta_a]) \\
&= 2\gamma_w\theta_a + \gamma_w - \gamma_w\theta_a + 3\theta_a - 3\gamma_w\theta_a \\
&= -\gamma_w(2\theta_a - 1) + 3\theta_a.
\end{aligned}
\tag{3}
$$

The first- order condition is

$$
\begin{aligned}
&\frac{d\pi_{Pauper}}{d\gamma_w} = -(2\theta_a - 1) = 0, \\
&\Rightarrow \theta_a = 1/2.
\end{aligned}
\tag{4}
$$

The Payoff-Equating Method

In equilibrium, each player is willing to mix only because he is indifferent between the pure strategies he is mixing over. This gives us a better way to find mixed strategies.

First, guess which strategies are being mixed between.

Then, see what mixing probability for the other player makes a given player indifferent.

The Welfare Game

		Pauper		
		Work (γ_w)		$Loaf$ ($1-\gamma_w$)
Government	Aid (θ_a)	**3,2**	$\rightarrow$	$-1, 3$
		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1, 1$	$\leftarrow$	**0,0**

Here,

$$\pi_g(Aid) = \gamma_w(3) + (1-\gamma_w)(-1) = \pi_g(No\ aid) = \gamma_w(-1) + (1-\gamma_w)(0)$$

So $\gamma_w(3+1+1) = 1$**, so** $\gamma_w = .2$**.**

$$\pi_p(Work) = \theta_a(2) + (1-\theta_a)(1) = \pi_p(Loaf) = \theta_a(3) + (1-\theta_a)(0)$$

so $\theta_a(2-1-3) = -1$ **and** $\theta_a = .5$**.**

Interpreting Mixed Strategies

A player who selects a mixed strategy is always indifferent between two pure strategies and an entire continuum of mixed strategies.

What matters is that a player's strategy appear random to other players, not that it really be random.

It could be based on time of day, temperature, etc.

It could be there is a population of identical players, each of whom picks a pure strategy. But each would still be indifferent about his strategy.

Harsanyi based an interpretation on this: model it as an incomlpete info game,a nd let the incomplete info shrink to zero.

Here do the Gintis example of mixing.

Then do the soccer example, which is true randomization.

Pure Strategies Dominated by a Mixed Strategy

		Column	
		North	*South*
	North	**0,0**	**4,-9**
Row	*South*	**4,-6**	**0,0**
	Defense	**1,-1**	**1,-1**

Payoffs to: (Row, Column)

For Row, *Defense* is strictly dominated by (0.5 *North*, 0.5 *South*), though that is not the Nash equilibrium.

Row's expected payoff from (.5,.5) if Column plays *North* is .5(0) + .5(4) = 2. Row's expected payoff from it if Column plays *South* is .5(4) + .5(0)= 2.

Row's expected payoff from this mixed strategy if Column plays *North* with probability N is

$$0.5(N)(0) + 0.5(1-N)(4) + 0.5(N)(4) + 0.5(1-N)(0) = 2, \tag{5}$$

so whatever response Column picks, Row's expected payoff is higher from the mixed strategy than his payoff of 1 from *Defense*.

Column's strategy must make Row willing to randomize, for a Nash equilibrium. Thus, if c is Column's probability of North, we need $\pi_r(North) = c(0) + (1-c)4 = \pi_r(South) = c(4) + (1-c)(0)$, so $4 - 4c = 4c$ so $c = 1/2$.

Row's strategy must make Column willing to randomize, for a Nash equilibrium. Thus, if r is Row's probability of North, we need $\pi_c(North) = r(0) + (1-c)(-6) = \pi_c(South) = c(-9) + (1-c)(0)$, so $-6 - 6c = -9c$ so $c = 3/5$.

Note that $c = 3/5$ is even better than $c = .5$ for Row in equilibrium. He gets a payoff of $\pi_r(South) = c(4) + (1-c)(0) = 2.4$

Chicken

		Jones		
		Continue **(θ)**		*Swerve* **($1-\theta$)**
Smith:	*Continue* **(θ)**	$-3, -3$	$\rightarrow$	**2, 0**
		$\downarrow$		$\uparrow$
	Swerve **($1-\theta$)**	**0, 2**	$\leftarrow$	**1, 1**

$$\pi_{Jones}(Swerve) = (\theta_{Smith}) \cdot (0) + (1 - \theta_{Smith}) \cdot (1)$$

$$= (\theta_{Smith}) \cdot (-3) + (1 - \theta_{Smith}) \cdot (2) = \pi_{Jones}(Cont \quad (6)$$

From equation (6) we can conclude that $1 - \theta_{Smith} = 2 - 5\theta_{Smith}$**, so** $\theta_{Smith} = 0.25$.

In the symmetric equilibrium, both players choose the same probability, so we can replace θ_{Smith} **with simply** θ**.**

The two teenagers will survive with probability $1 - (\theta \cdot \theta) = 0.9375$.

How can we prove there is no asymmetric mixed-strategy equilibrium, with unequal mixing probabilities?

		Jones		
		$Continue$ **(θ)**		$Swerve$ **($1-\theta$)**
Smith:	$Continue$ **(θ)**	x, x	$\rightarrow$	**2, 0**
		$\downarrow$		$\uparrow$
	$Swerve$ **($1-\theta$)**	**0, 2**	$\leftarrow$	**1, 1**

$$\theta = \frac{1}{1-x}. \tag{7}$$

If $x = -3$, this yields $\theta = 0.25$, as was just calculated.

If $x = -9$, it yields $\theta = 0.10$.

If $x = 0.5$, the equilibrium probability of continuing appears to be $\theta = 2$. What is going on?

In the mixed-strategy equilibrium, the expected payoff is $\pi(swerve) = \theta(0) + (1-\theta)(1)$.

Note that this is decreasing in θ.

The War of Attrition

The possible actions are $Exit$ and $Continue$. In each period that both $Continue$, each earns -1. If a firm exits, its losses cease and the remaining firm obtains the value of the market's monopoly profit, which we set equal to 3. We will set the discount rate equal to $r > 0$.

(1) Continue in each period, Exit in each period

(2) Each exits with probability θ if it hasn't yet.

Let Smith's payoffs be V_{stay} if he stays and V_{exit} if he exits.

$V_{exit} = 0$.

$$V_{stay} = \theta \cdot (3) + (1-\theta)\left(-1 + \left[\frac{V_{stay}}{1+r}\right]\right), \quad (8)$$

which, after a little manipulation, becomes

$$V_{stay} = \left(\frac{1+r}{r+\theta}\right)(4\theta - 1). \quad (9)$$

Thus, $\theta = 0.25$.

This does not have to solved with the dynamic programming/Bellman equation method.

Timing games

Pre-emption games: the reward goes to the player who chooses the move which ends the game, and a cost is paid if both players choose that move, but no cost is incurred in a period when neither player chooses it.

Grab the Dollar. A dollar is placed on the table between Smith and Jones, who each must decide whether to grab for it or not. If both grab, each is fined one dollar. This could be set up as a one-period game, a T period game, or an infinite-period game, but the game definitely ends when someone grabs the dollar.

Grab the Dollar

		Jones		
		Grab		*Don't Grab*
	Grab	$-1,-1$	$\rightarrow$	**1,0**
Smith:		$\downarrow$		$\uparrow$
	Don't Grab	**0,1**	$\leftarrow$	$0,0$

		Jones	
		Grab	*Don't Grab*
	Grab	$-1, -1 \rightarrow$	**1,0**
Smith:		$\downarrow$	$\uparrow$
	Don't Grab	**0,1** $\leftarrow$	$0, 0$

Let s be Smith's probability of grabbing and j be Jones's. If Smith grabs, that ends the game:

$$\pi_s(grab) = j(-1) + (1 - j)(1)$$

If he chooses not to grab, then the game continues, and if Jones does not grab either, he remains in the same position as at the start:

$$\pi_s(not\ grab) = j(0) + (1 - j)(\frac{1}{1 + r}\pi_s(not\ grab))$$

The only value that solves this second equation is $\pi_s(not\ grab) = 0$. Equating that to $\pi_s(grab)$ gives us

$0 = j(-1) + (1 - j)(1)$**, so** $j = .5$

		Jones	
		Grab	*Don't Grab*
Smith:	*Grab*	$-1,-1 \rightarrow$	**1,0**
		$\downarrow$	$\uparrow$
	Don't Grab	**0,1** $\leftarrow$	$0,0$

Suppose we had an equilibrium where if the second period is reached, Smith grabs with probability one. What will happen to the mixed strategy in the first period?

Smith would have to equate his first period payoffs thus:

$$\pi_s(grab) = j(-1)+(1-j)(1) = \pi(don't) = j(0)+(1-j)(\frac{1}{1+r}(1))$$

If $r = 0$**, these are equal only if** $j = 0$ **or** $j = 1$**. So there can't be an equilibrium with mixed strategies in the first period and pure strategies in the second.**

If $r \neq 0$ **then some algebra shows that** $j = \frac{r}{1+2r}$**.**

As for Jones:

$$\pi_j(grab) = s(-1) + (1-s)(1) = \pi(don't) = 0,$$

so $s = 1/2$**.**

Smith probably wins in the first period because of the forecast that he would otherwise win in the second period.

Asymmetric Grab the Dollar

		Jones		
		Grab		*Don't Grab*
	Grab	$-2, -3$	$\rightarrow$	**1,0**
Smith:		$\downarrow$		$\uparrow$
	Don't Grab	**0,1**	$\leftarrow$	$0, 0$

Let s be Smith's probability of grabbing, and j be Jones's. Then Smith equates his payoffs thus (remember: the continuation payoff is zero):

$$\pi(grab) = j(-2)+(1-j)(1) = \pi(don't) = j(0)+(1-j)(0)$$

so $-2j + 1 - j = 0$ and $j = 1/3$.

Jones equates his payoffs thus:

$$\pi(grab) = s(-3)+(1-s)(1) = \pi(don't) = s(0)+(1-s)(0)$$

Then $-3s + 1 - s = 0$, and s =1/4.

Smith has a smaller probability of grabbing even tho his penalty from Grab, Grab is less. That is because for Smith, with his smaller penalty, not to always want to grab requires that Jones have a bigger probability of grabbing.

Patent Race for a New Market

Players
Three identical firms, Apex, Brydox, and Central.

The Order of Play
Each firm simultaneously chooses research spending $x_i \geq 0$, $(i = a, b, c)$.

Payoffs
Firms are risk neutral and the discount rate is zero. Innovation occurs at time $T(x_i)$ where $T' < 0$. The value of the patent is V, and if several players innovate simultaneously they share its value.

$$\pi_i = \begin{cases} V - x_i & \text{if } T(x_i) < Min\{T(x_j, T(x_k)\} \quad \textbf{(wins)} \\ \frac{V}{2} - x_i & \text{if } T(x_i) = Min\{T(x_j), T(x_k)\} < Max\{T(x_j), T(x_k)\} \quad \textbf{(shares with} \\ \frac{V}{3} - x_i & \text{if } T(x_i) = T(x_j = T(x_k) \quad \textbf{(shares with 2 2 other firms)} \\ -x_i & \text{if } T(x_i) > Min\{T(x_j, T(x_k)\} \quad \textbf{(loses)} \end{cases}$$

No pure strategy Nash equilibria, because the payoff functions are discontinuous.

A slight difference in research by one player can make a big difference in the payoffs, as shown in the figure for fixed values of x_b and x_c. (The research levels shown are not equilibrium values.) If Apex chose any research level x_a less than V, Brydox would respond with $x_a + \varepsilon$ and win the patent. If Apex chose $x_a = V$, then Brydox and Central would respond with $x_b = 0$ and $x_c = 0$, which would make Apex want to switch to $x_a = \varepsilon$.

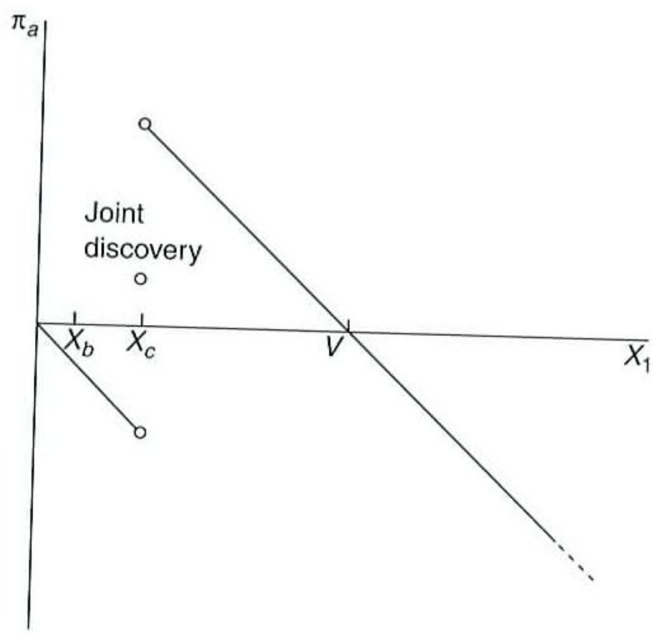


Figure 3.1 The payoffs in Patent Race for a New Market.

Figure 1: The Payoffs in Patent Race for a New Market

Denote the probability that firm i chooses a research level less than or equal to x as $M_i(x)$. This function describes the firm's mixed strategy.

Since we know that the pure strategies $x_a = 0$ and $x_a = V$ yield zero payoffs, if Apex mixes over $[0, V]$ then the expected payoff for every strategy mixed between must also equal zero.

$$\pi_a(x_a) = V \cdot Pr(x_a \geq X_b, x_a \geq X_c) - x_a = 0 = \pi_a(x_a = 0), \tag{10}$$

which can be rewritten as

$$V \cdot Pr(X_b \leq x_a) Pr(X_c \leq x_a) - x_a = 0, \tag{11}$$

or

$$V \cdot M_b(x_a) M_c(x_a) - x_a = 0. \tag{12}$$

We can rearrange equation (12) to obtain

$$M_b(x_a) M_c(x_a) = \frac{x_a}{V}. \tag{13}$$

If all three firms choose the same mixing distribution M, then

$$M(x) = \left(\frac{x}{V}\right)^{1/2} \text{ for } 0 \leq x \leq V. \tag{14}$$

This is an "all-pay auction."

Correlated Strategies

Aumann (1974, 1987) has pointed out that it is often important whether players can use the same randomizing device for their mixed strategies. If they can, these are correlated strategies.

In Chicken, the only mixed-strategy equilibrium is the symmetric one in which each player chooses $Continue$ **with probability 0.25 and the expected payoff is 0.75. A correlated equilibrium would be for the two players to flip a coin and for Smith to choose** $Continue$ **if it comes up heads and for Jones to choose** $Continue$ **otherwise. The probability of each choosing** $Continue$ **is 0.5, and the expected payoff for each is 1.0.**

Cheap talk (Crawford & Sobel [1982]). Cheap talk refers to costless communication when players can lie without penalty.

In Ranked Coordination, cheap talk instantly allows the players to make the desirable outcome a focal point, though it does not get rid of the other equilibria.

The Civic Duty Game

		Jones		
		Ignore (γ)		*Telephone* ($1-\gamma$)
	Ignore (γ)	$0, 0$	$\rightarrow$	**10,7**
Smith:		$\downarrow$		$\uparrow$
	Telephone ($1-\gamma$)	**7,10**	$\leftarrow$	$7, 7$

Payoffs to: (Row, Column). Arrows show how a player can increase his payoff.

In the N-player version of the game, the payoff to Smith is 0 if nobody calls, 7 if he himself calls, and 10 if one or more of the other $N-1$ players calls.

If all players use the same probability γ of $Ignore$, the probability that the other $N-1$ players besides Smith all choose $Ignore$ is γ^{N-1}, so the probability that one or more of them chooses $Telephone$ is $1-\gamma^{N-1}$.

$$\pi_{Smith}(Telephone) = 7 = \pi_{Smith}(Ignore) = \gamma^{N-1}(0)+(1-\gamma^{N-1})(10) \tag{15}$$

Equation (15) tells us that

$$\gamma^{N-1} = 0.3 \tag{16}$$

so

$$\gamma^* = 0.3^{\frac{1}{N-1}}. \tag{17}$$

$$\gamma^* = 0.3^{\frac{1}{N-1}}. \qquad (18)$$

If $N = 2$, Smith chooses $Ignore$ with a probability of 0.30.

As N increases, Smith's expected payoff remains equal to 7 whether $N = 2$ or $N = 38$, since his expected payoff equals his payoff from the pure strategy of $Telephone$. The value γ^*, the probability of $Ignore$ for each player, rises with N.

If $N = 38$, the value of γ^* is about 0.97. (Kitty Genovese case) (think of juries)

The probability that nobody calls is γ^{*N}. Equation (16) shows that $\gamma^{*N-1} = 0.3$, so $\gamma^{*N} = 0.3\gamma^*$, which is increasing in N because γ^* is increasing in N. If $N = 2$, the probability that neither player phones the police is $\gamma^{*2} = 0.09$.

When there are 38 players, the probability rises to γ^{*38}, about 0.29. The more people that watch a crime, the less likely it is to be reported.

Randomizing Is Not Always Mixing:

Assume that the benefit of preventing or catching cheating is 4, the cost of auditing is C, where $C < 4$, the cost to the suspects of obeying the law is 1, and the cost of being caught is the fine $F > 1$.

Auditing Game I, II

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
IRS:	*Audit* (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	**0,0**	$\leftarrow$	$4, -1$

$$\pi(Audit) = \theta(4-C) + (1-\theta)(4-C) =$$
$$\pi(Trust) = \theta(0) + (1-\theta)(4), 4 - C = 4 - 4\theta, \theta^* = C/4$$

$$\pi(Cheat) = \gamma(-F) + (1-\gamma)(0) =$$
$$\pi(Obey) = \gamma(-1) + (1-\gamma)(-1), -\gamma F = -1, \gamma^* = 1/F.$$

The payoffs are $\pi_{gov} = 4 - C$ and $\pi_{suspect} = -1$.

Auditing Game II makes this sequential: The government moves first. The payoffs are identical, but there is always auditing and never cheating.

In Auditing Game I, the equilibrium strategy was to audit all suspects with probability $1/F$ and none of them otherwise.

That is different from announcing in advance that the IRS will audit a random sample of $1/F$ of the suspects.

For Auditing Game III, suppose the IRS moves first, but let its move consist of the choice of the proportion α of tax returns to be audited.

We know that the IRS is willing to deter the suspects from cheating, since it would be willing to choose $\alpha = 1$ and replicate the result in Auditing Game II if it had to. It chooses α so that

$$\pi_{suspect}(Obey) \geq \pi_{suspect}(Cheat), \tag{19}$$

i.e.,

$$-1 \geq \alpha(-F) + (1-\alpha)(0). \tag{20}$$

In equilibrium, therefore, the IRS chooses $\alpha = 1/F$ and the suspects respond with $Obey$. The IRS payoff is $(4 - \alpha C)$, which is better than the $(4-C)$ in the other two games, and the suspect's payoff is -1, exactly the same as before.

The Cournot Game

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose quantities q_a and q_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$, and we will assume it to be linear (for generalization see Chapter 14), and, in fact, will use the following specific function:

$$p(Q) = 120 - q_a - q_b. \tag{21}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_aq_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_aq_b - q_b^2. \tag{22}$$

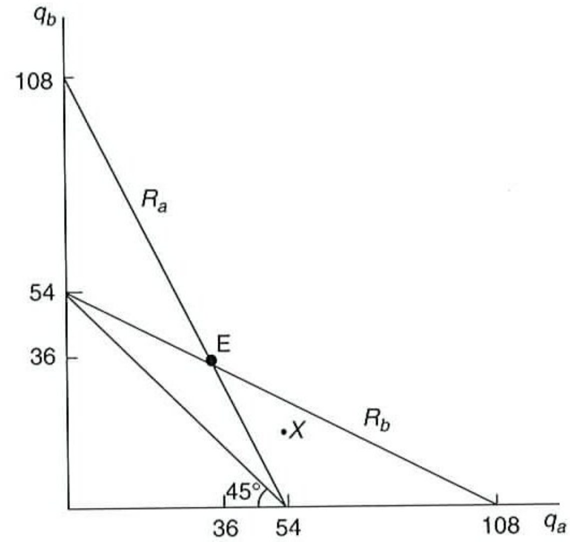


Figure 3.2 Reaction curves in the Cournot Game.

Figure 2: Reaction Curves in the Cournot Game

The monopoly output maximizes $pQ - cQ = (120 - Q - c)Q$ with respect to the total output of Q, resulting in the first-order condition

$$120 - c - 2Q = 0, \tag{23}$$

which implies a total output of $Q = 54$ and a price of 66.

To find the "Cournot-Nash" equilibrium, we need to refer to the best-response functions or reaction functions. If Brydox produced 0, Apex would produce the monopoly output of 54. If Brydox produced $q_b = 108$ or greater, the market price would fall to 12 and Apex would choose to produce zero. The best response function is found by maximizing Apex's payoff, $\pi_{Apex} = (120 - c)q_a - q_a^2 - q_a q_b$, with respect to his strategy, q_a. This generates the first-order condition $120 - c - 2q_a - q_b = 0$, or

$$q_a = 60 - \left(\frac{q_b + c}{2}\right) = 54 - \left(\frac{1}{2}\right) q_b. \qquad (24)$$

$$q_a = 54 - \left(\frac{1}{2}\right) q_b. \qquad (25)$$

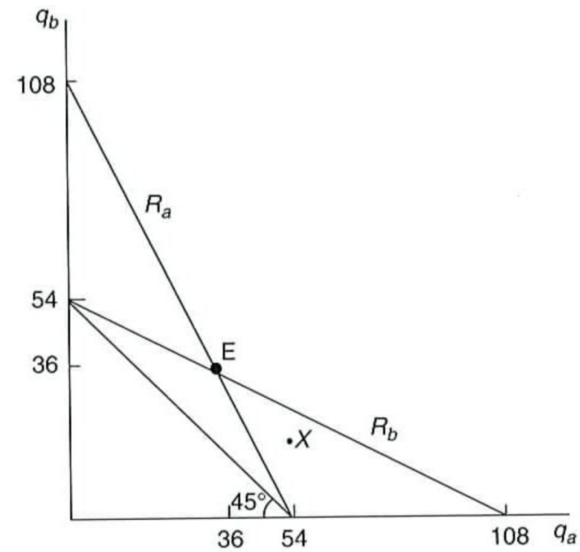


Figure 3.2 Reaction curves in the Cournot Game.

The reaction functions of the two firms are labelled R_a and R_b in Figure 2. Where they cross, point E, is the Cournot-Nash equilibrium, the Nash equilibrium when the strategies consist of quantities.

Algebraically, it is found by solving the two reaction functions for q_a and q_b, which generates the unique equilibrium, $q_a = q_b = 40 - c/3 = 36$. The equilibrium price is then 48 (= 120-36-36).

The Stackelberg Game

Players
Firms Apex and Brydox

The Order of Play
1 Apex chooses quantity q_a from the set $[0, \infty)$.
2 . Brydox chooses quantity q_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$:

$$p(Q) = 120 - q_a - q_b. \tag{26}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_a q_b - q_b^2. \tag{27}$$

The actions and payoffs are identical to the Cournot Game. All that has changed is that it is sequential now. And— that we act as if the sequence mattered. We will look for an asymmetric equilibrium in quantities now.

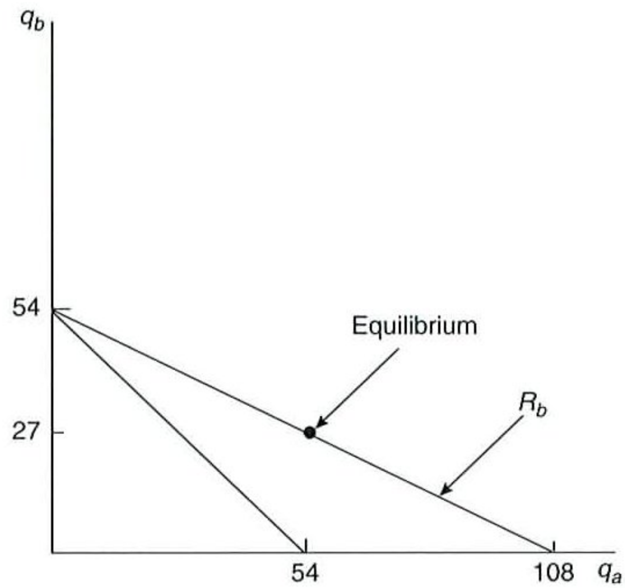


Figure 3.3 Stackelberg equilibrium.

If Apex forecasts Brydox's output to be $q_b = 60 - \frac{q_a+c}{2}$**, Apex can substitute this into his payoff function in (22) to obtain**

$$\pi_a = (120 - c)q_a - q_a^2 - q_a(60 - \frac{q_a + c}{2}). \tag{28}$$

Maximizing his payoff with respect to q_a **yields the first-order condition**

$$(120 - c) - 2q_a - 60 + q_a + \frac{c}{2} = 0, \tag{29}$$

so $q_a = 60 - c/2 = 54$**. Once Apex chooses this output, Brydox chooses his output to be** $q_b = 27$**.**

The Bertrand Game

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q(p) = 120 - p$. The payoff function for Apex (Brydox's would be analogous) is

$$\pi_a = \begin{cases} (120 - p_a)(p_a - c) & \text{if } p_a \leq p_b \\ \\ \frac{(120-p_a)(p_a-c)}{2} & \text{if } p_a = p_b \\ \\ 0 & \text{if } p_a > p_b \end{cases}$$

The Bertrand Game has a unique Nash equilibrium: $p_a = p_b = c = 12$, with $q_a = q_b = 54$. That this is a weak Nash equilibrium is clear: if either firm deviates to a higher price, it loses all its customers and so fails to increase its profits to above zero. In fact, this is an example of a Nash equilibrium in weakly dominated strategies.

That the equilibrium is unique is less clear. To see why it is, divide the possible strategy profiles

into four groups:

$p_a < c$ or $p_b < c$. In either of these cases, the firm with the lowest price will earn negative profits, and could profitably deviate to a price high enough to reduce its demand to zero.

$p_a > p_b > c$ or $p_b > p_a > c$. In either of these cases the firm with the higher price could deviate to a price below its rival and increase its profits from zero to some positive value.

$p_a = p_b > c$. In this case, Apex could deviate to a price ϵ less than Brydox and its profit would rise, because it would go from selling half the market quantity to selling all of it with an infinitesimal decline in profit per unit sale.

$p_a > p_b = c$ or $p_b > p_a = c$. In this case, the firm with the price of c could move from zero profits to positive profits by increasing its price slightly while keeping it below the other firm's price.

The Differentiated Bertrand Game

Let us now move to a different duopoly market, where the demand curves facing Apex and Brydox are

$$q_a = 24 - 2p_a + p_b \tag{30}$$

and

$$q_b = 24 - 2p_b + p_a, \tag{31}$$

and they have constant marginal costs of $c = 3$**.**

The payoffs are

$$\pi_a = (24 - 2p_a + p_b)(p_a - c) \tag{32}$$

and

$$\pi_b = (24 - 2p_b + p_a)(p_b - c). \tag{33}$$

Apex and Brydox simultaneously choose prices p_a **and** p_b **from the set** $[0, \infty)$**.**

Maximizing Apex's payoff by choice of p_a**, we obtain the first- order condition,**

$$\frac{d\pi_a}{dp_a} = 24 - 4p_a + p_b + 2c = 0, \tag{34}$$

and the reaction function,

$$p_a = 6 + \left(\frac{1}{2}\right)c + \left(\frac{1}{4}\right)p_b = 7.5 + \left(\frac{1}{4}\right)p_b. \tag{35}$$

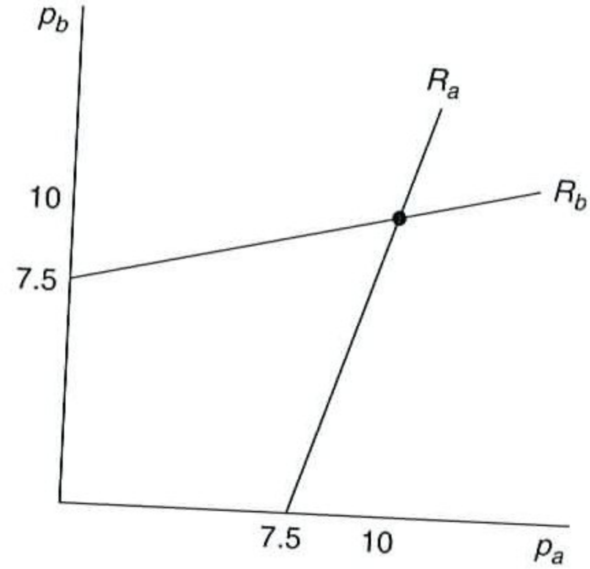
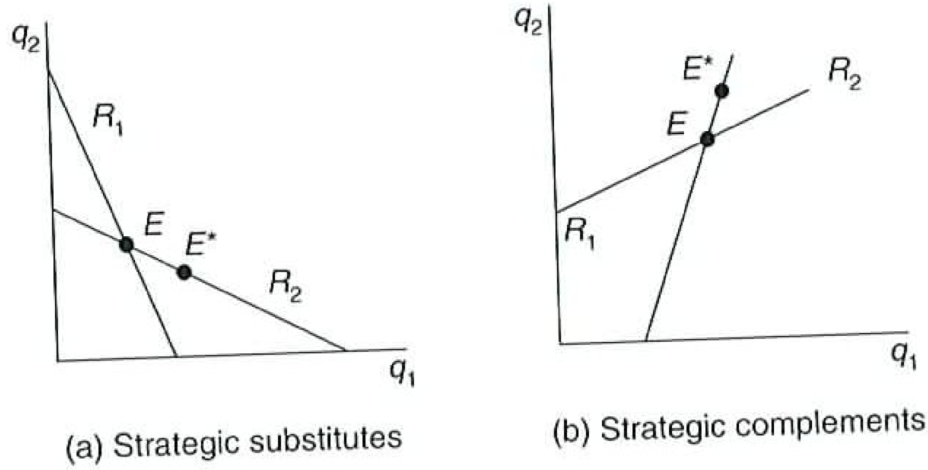


Figure 3.4 Bertrand reaction functions with differentiated products.

Brydox's reaction curve slopes upwards too. Equilibrium occurs where $p_a = p_b = 10$. The quantity each firm produces is 14, which is below the 21 each would produce at prices of $p_a = p_b = c = 3$. Figure 4 shows that the reaction functions intersect.



(a) Strategic substitutes (b) Strategic complements

Figure 3.5 Cournot versus Differentiated Bertrand reaction functions (strategic substitutes v[...] strategic complements).

Esther Gal-Or (1985) notes that if reaction curves slope down (as with strategic substitutes and Cournot) there is a first-mover advantage, whereas if they slope upwards (as with strategic complements and differentiated Bertrand) there is a second-mover advantage.

Supermodularity is a related concept. With only one choice variable, as here, it boils down to, for players i and j with strategies s_i, s_j:

$$\frac{\partial^2 \pi_i}{\partial s_i s_j} \geq 0.$$

Four common reasons why an equilibrium might not exist

(1) An unbounded strategy space

Smith can borrow money and buy as much tin as he wants for \$6/pound. He knows that the price will be \$7/pound tomorrow. What quantity x will he buy, if his borrowing is unlimited?

Choosing x in the strategy set $[0, \infty)$ when his payoff function is $\pi = (1)x$, there is no best strategy.

(2) An open strategy space

Now say that government regulations constrain him to buy less than 1,000 pounds. His strategy is $x \in [0, 1, 000)$, which is bounded by 1000.

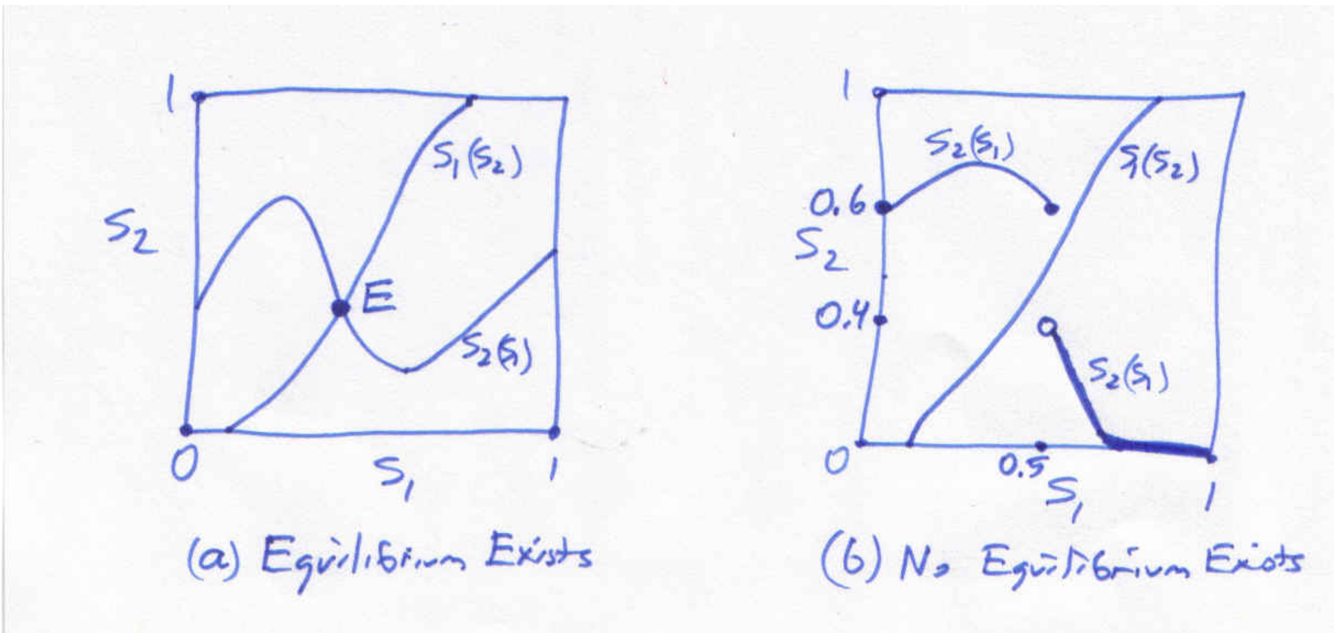
(3) A discrete strategy space (or, more generally, a nonconvex strategy space)

If the strategies are strategic substitutes, then if player 1 increases his strategy in response to s_2, player 2 will in turn want to reduce his strategy. If the strategy spaces are discrete, player 2 cannot reduce his strategy just a little bit– he has to jump down a discrete level. That could then induce Player 1 to increase his strategy by a discrete amount. This jumping of responses can be never-ending–there is no equilibrium.

This is a problem of "gaps" in the strategy space. Suppose we had a game in which the government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non-convex.

(4) A discontinuous reaction function arising from nonconcave or discontinuous payoff functions

For a Nash equilibrium to exist, we need for the reaction functions of the players to intersect. If the reaction functions are discontinuous, they might not.



In Panel (a) a Nash equilibrium exists, at the point, E, where the two reaction functions intersect. In Panel (b) no Nash equilibrium exists. Firm 2's reaction function $s_2(s_1)$ is discontinuous at the point $s_1 = 0.5$. It jumps down from $s_2(0.5) = 0.6$ to $s_2(0.50001) = 0.4$. The reaction curves never intersect, and no equilibrium exists.

If the two players can use mixed strategies, then an equilibrium will exist even for the game in Panel (b).

A first reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is discontinuous in either his own or the other players' strategies. This is what happens in Chapter 14's Hotelling Pricing Game, where if Player 1's price drops enough (or Player 2's price rises high enough), all of Player 2's customers suddenly rush to Player 1.

A second reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is not concave.

If firms are Cournot competitors with different marginal costs, they will have different market shares. The Herfindahl Index equals the weighted average of their price-cost margins multiplied by the industry elasticity of demand (and multiplied by -10,000).