6 Dynamic Games with Incomplete Information

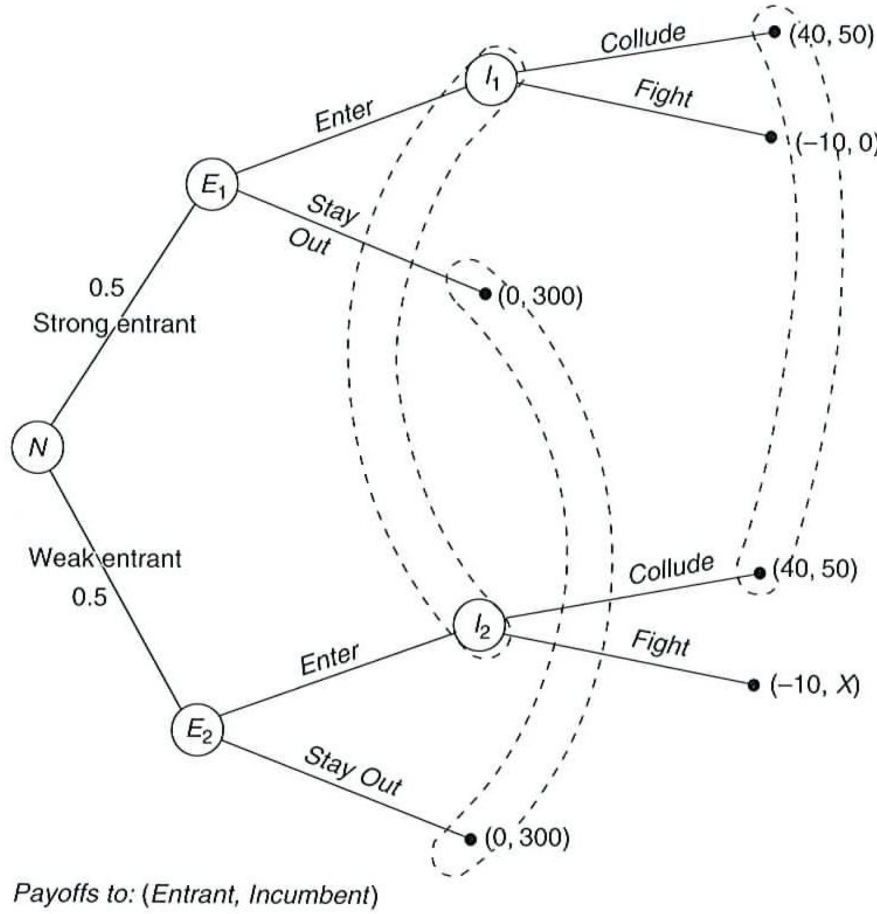

Figure 6.1 Entry Deterrence II, III, and IV.

Entry Deterrence II: Fighting Is Never Profitable: X=1

Subgame perfectness does not rule out any Nash equilibria. The only subgame is the entire game.

Trembling-Hand Perfectness

Trembling-hand perfectness — Selten (1975) says a strategy that is to be part of an equilibrium must be optimal for the player even if there is a small chance that the other player's hand will "tremble" :

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The strategy profile s^* *is a* **trembling-hand perfect** *equilibrium if for any* ϵ *there is a vector of positive numbers* $\delta_1, \ldots, \delta_n \in [0,1]$ *and a vector of completely mixed strategies* $\sigma_1, \ldots \sigma_n$ *such that the perturbed game where every strategy is replaced by* $(1-\delta_i)s_i + \delta_i\sigma_i$ *has a Nash equilibrium in which every strategy is within distance* ϵ *of* s^*.

This is hard to use, and undefined when games have continuous strategy spaces because it is hard to work with mixtures of a continuum).

Perfect Bayesian Equilibrium and Sequential Equilibrium (Kreps & Wilson (1982))

The profile of beliefs and strategies is called an assessment.

On the equilibrium path, all that the players need to update their beliefs are their priors and Bayes' s Rule. Off the equilibrium path, this is not enough. Suppose that in equilibrium, the entrant always enters. If the entrant stays out, what is the incumbent to think about the probability the entrant is weak? Bayes' s Rule does not help, because when $Prob(data) = 0$**, which is the case for data such as** $Stay\ Out$ **which is never observed in equilibrium, the posterior belief cannot be calculated using Bayes' s Rule.**

$$Prob(Weak|Stay\ Out) = \frac{Prob(Stay\ Out|Weak)Prob(Weak)}{Prob(Stay\ Out)}. \tag{1}$$

The posterior $Prob(Weak|Stay\ Out)$ **is undefined, because this requires dividing by zero.**

A perfect bayesian equilibrium is a strategy profile s and a set of beliefs μ such that at each node of the game:

(1) The strategies for the remainder of the game are Nash given the beliefs and strategies of the other players.
(2) The beliefs at each information set are rational given the evidence appearing thus far in the game (meaning that they are based, if possible, on priors updated by Bayes' s Rule, given the observed actions of the other players under the hypothesis that they are in equilibrium).

Kreps & Wilson (1982b) use this idea to form their equilibrium concept of sequential equilibrium, but they impose a third condition to restrict beliefs further:

(3) The beliefs are the limit of a sequence of rational beliefs, i.e., if (μ^*, s^*) is the equilibrium assessment, then some sequence of rational beliefs and completely mixed strategies converges to it:

$$(\mu^*, s^*) = Lim_{n\to\infty}(\mu^n, s^n)\ for\ some\ sequence\ (\mu^n, s^n)\ in\ \{\mu, s\}.$$

Back to Entry Deterrence II

A PBE for Entry Deterrence II :

Entrant: $Enter|Weak$, $Enter|Strong$
Incumbent: *Collude*
Beliefs: $Prob$($Strong|$ *Stay Out*) = 0.4

There is no perfect bayesian equilibrium in which the entrant chooses *Stay Out*.

$Fight$ **is a bad response even under the most optimistic possible belief, that the entrant is** $Weak$ **with probability 1.**

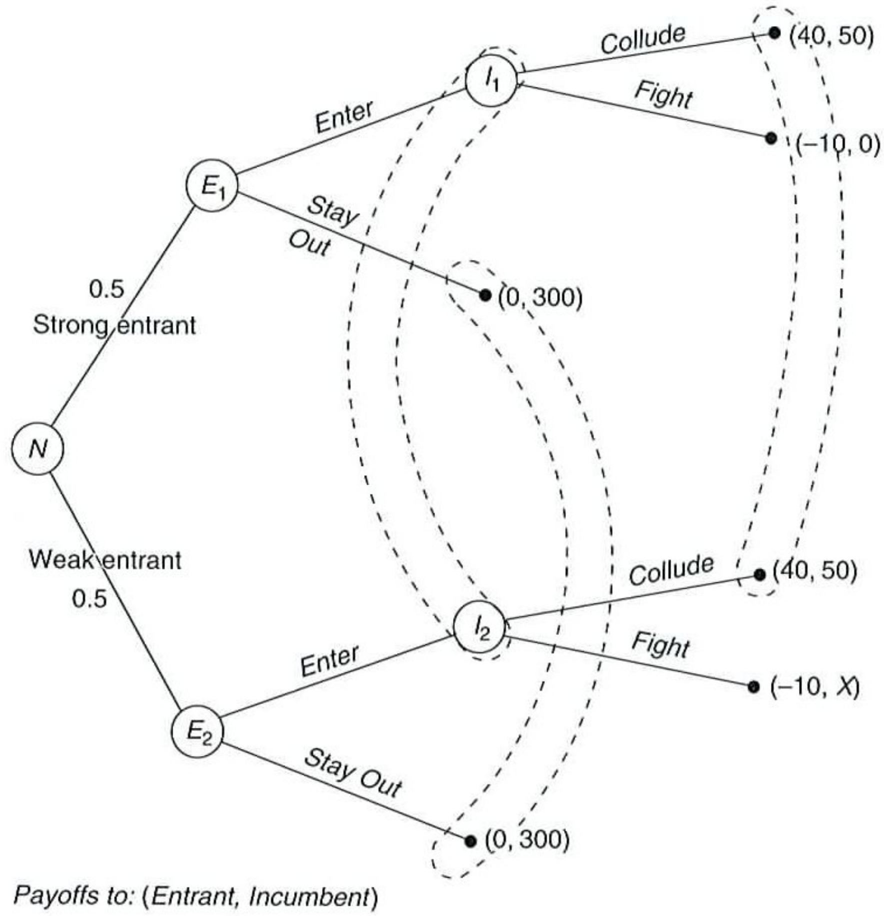

Figure 6.1 Entry Deterrence II, III, and IV.

In Entry Deterrence III, assume $X = 60$, not $X = 1$. Fighting is now more profitable for the incumbent than collusion if the entrant is $Weak$.

The first equilibrium we'll examine uses

passive conjectures— **"posterior equals prior"**

for out-of-equilibrium beliefs, but could use ANY beliefs— it is completely robust.

A plausible pooling equilibrium for Entry Deterrence III
Entrant: $Enter|Weak$, $Enter|Strong$
Incumbent: *Collude*, **Out-of-equilibrium beliefs:** $Prob(Strong|$ *Stay Out*$) = 0.5$

In choosing whether to enter, the entrant must predict the incumbent's behavior.

If the probability that the entrant is $Weak$ is 0.5, the expected payoff to the incumbent from choosing *Fight* is 30 ($= 0.5[0] + 0.5[60]$), which is less than the payoff of 50 from $Collude$.

The incumbent will collude, so the entrant enters. The entrant may know that the incumbent's payoff is actually 60, but that is irrelevant to the incumbent's behavior.

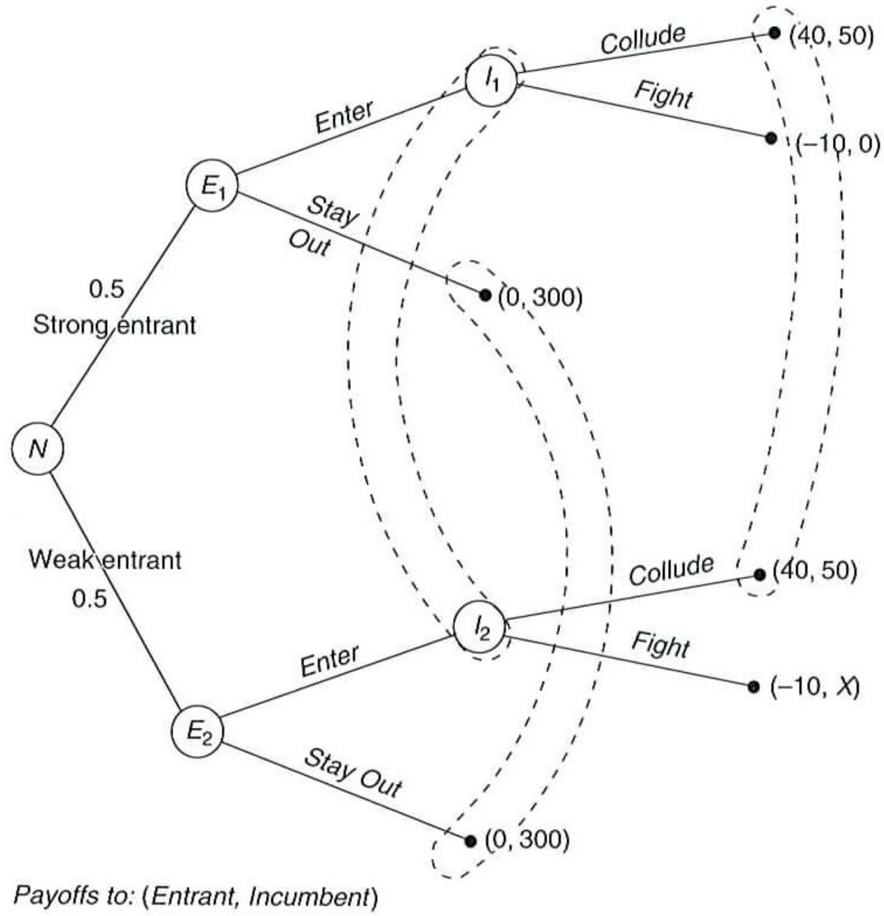

Figure 6.1 Entry Deterrence II, III, and IV.

An implausible equilibrium for Entry Deterrence III
Entrant: *Stay Out*$|Weak$, *Stay Out*$|Strong$
Incumbent: *Fight*,
Out-of-equilibrium beliefs: $Prob(Strong|Enter) = 0.1$

If the entrant were to deviate and enter, the incumbent would calculate his payoff from fighting to be 54 ($= 0.1[0] + 0.9[60]$), which is greater than the *Collude* payoff of 50. The entrant would therefore stay out.

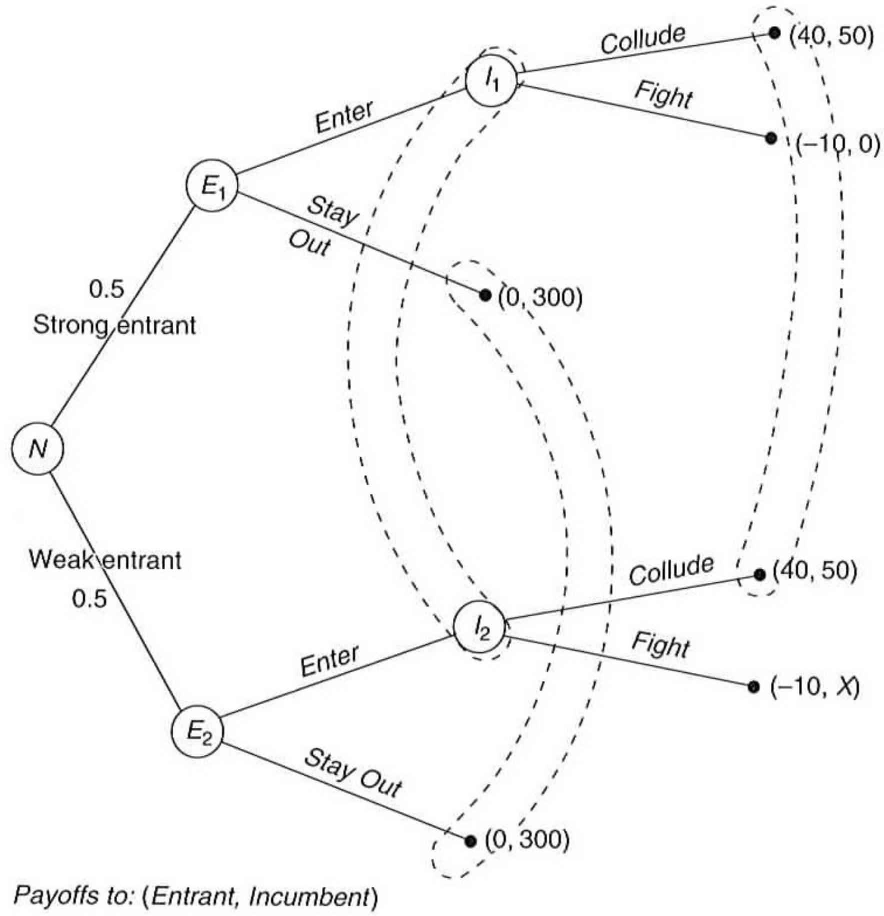

Figure 6.1 Entry Deterrence II, III, and IV.

A conjectured separating equilibrium for Entry Deterrence III
Entrant: *Stay Out*$|Weak$, *Enter*$|Strong$
Incumbent: $Collude$

This turns out not to be an equilibrium.

A Mixed-Strategy Equilibrium for Entry Deterrence III

The prior for the probability that the entrant is strong is .5. In this game, the weak and the strong entrant both get the same payoff from entering. The strong entrant is strong only in the sense that the incumbent doesn't want to fight him.

Let the probability that the incumbent colludes be α.

$$\pi_(enter) = \alpha(40) + (1-\alpha)(-10) = \pi_(stay; out) = 0$$

Thus, $\alpha = .2$.

Let θ be the posterior probability that an entrant who enters is Strong.

$$\pi_(fight) = \theta(0) + (1-\theta)(60) = \pi_(collude) = 50$$

Thus, $\theta = 1/6$.

Let β_s and β_w be the probabilities with which the strong and weak entrants enter. We need

$$\theta = \frac{1}{6} = \frac{.5 \cdot \beta_s}{.5 \cdot \beta_w}$$

There are lots of values which satisfy this condition, e.g. $\beta_s = 1/6, \beta_w = 1$ or $\beta_s = 1/12, \beta_w = 1/2$ or $\beta_s = 1/10, \beta_w = 6/10$.

The weak entrant is more likely to enter! The reason is that if the strong entrant were to enter with greater probability, the incumbent would want to $Collude$.

The PhD Admissions Game: A Separating Equilibrium

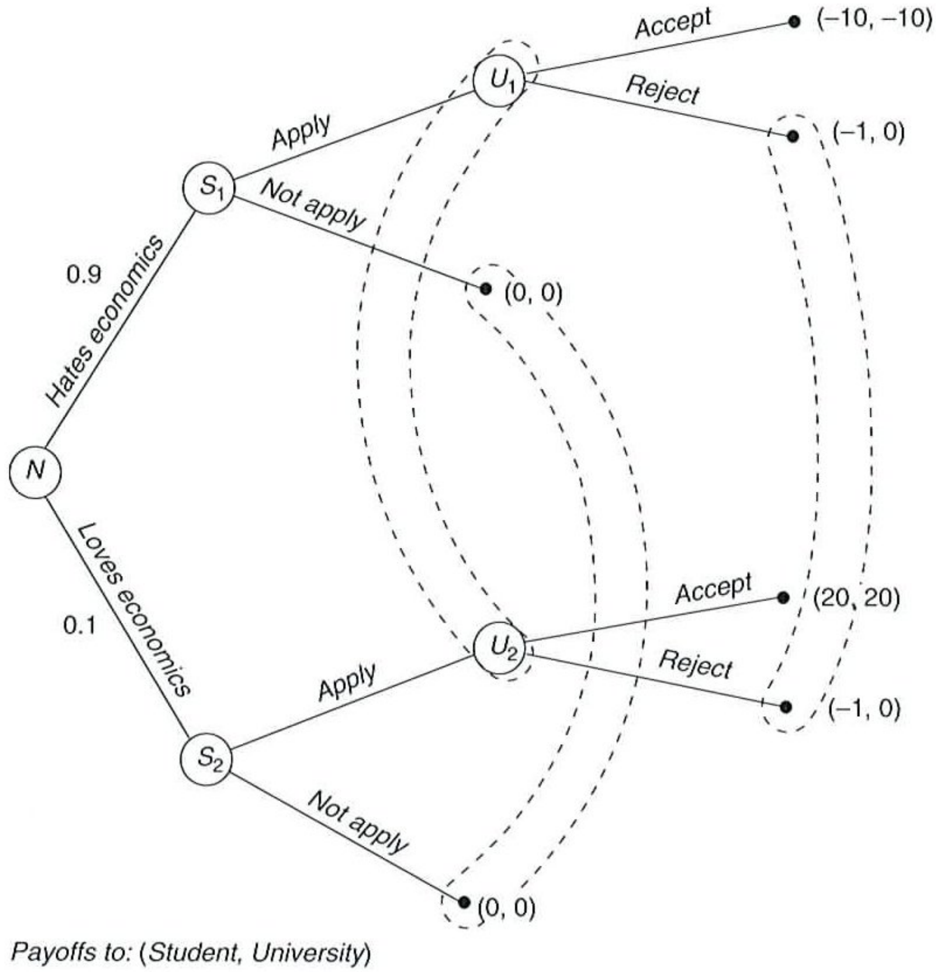

Figure 6.2 The PhD Admissions Game.

A separating equilibrium for the PhD Admissions Game

Student: *Apply* |*Lover*, *Do Not Apply* | *Hater*

University: $Admit$

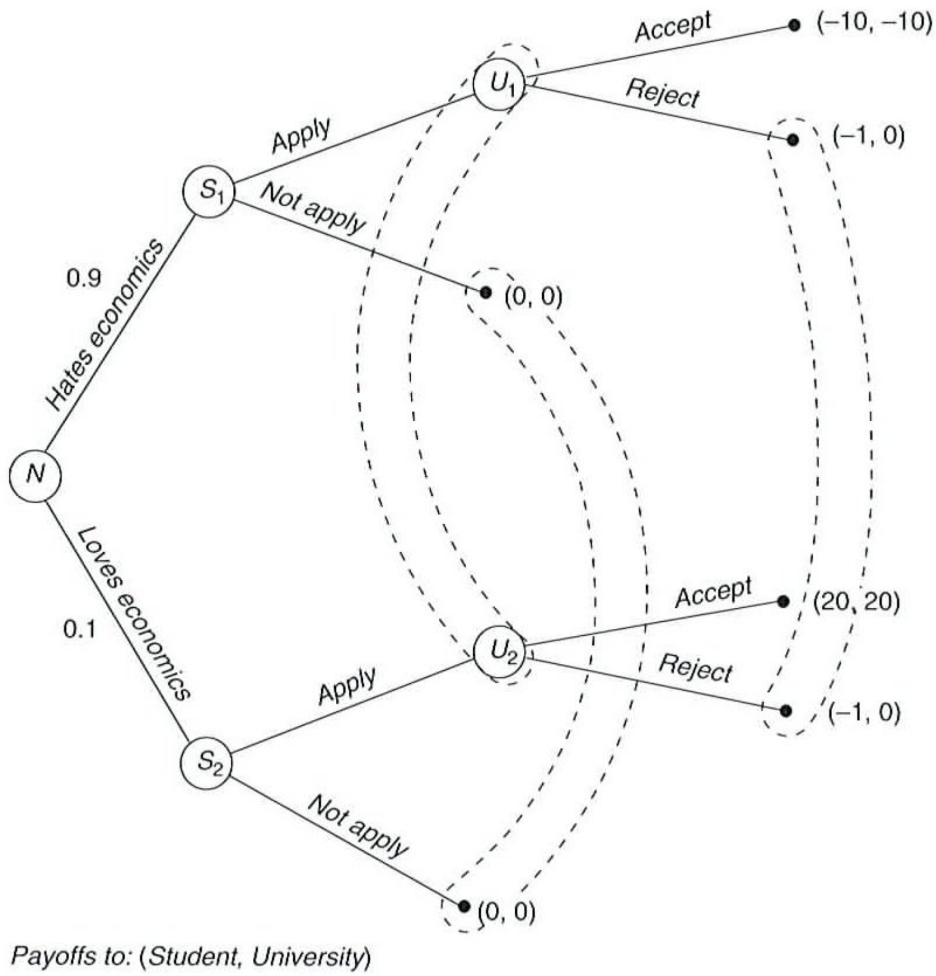

Figure 6.2 The PhD Admissions Game.

A pooling equilibrium for the PhD Admissions Game

Student: *Do Not Apply* $|$*Lover*, *Do Not Apply* $|$*Hater*

University: $Reject$, **Out-of-equilibrium beliefs:** $Prob$(*Hater* $|$*Apply*) = 0.9 **(passive conjectures)**

Passive Conjectures. $Prob$(*Hater*|*Apply*) = 0.9

This supports the pooling equilibrium.

Complete Robustness. $Prob$(*Hater*|*Apply*) $= m$, $0 \leq m \leq 1$

Under this approach, the equilibrium strategy profile must consist of responses that are best, given any and all out-of-equilibrium beliefs. Our equilibrium for Entry Deterrence II satisfied this requirement. Complete robustness rules out a pooling equilibrium in the PhD Admissions Game, because a belief like $m = 0$ makes accepting applicants a best response, in which case only the *Lover* will apply.

The Intuitive Criterion. $Prob(\textit{Hater}|\textit{Apply}) = 0$

Under the Intuitive Criterion of Cho & Kreps (1987), if there is a type of informed player who could not benefit from the out-of-equilibrium action no matter what beliefs were held by the uninformed player, the uninformed player's belief must put zero probability on that type.

Here, the *Hater* **could not benefit from applying under any possible beliefs of the university, so the university puts zero probability on an applicant being a** *Hater*. **This argument will not support the pooling equilibrium.**

An Ad Hoc Specification. $Prob(\textit{Hater}|\textit{Apply}) = 1$

Sometimes the modeller can justify beliefs by the circumstances of the particular game. Here, one could argue that anyone so foolish as to apply knowing that the university would reject them could not possibly have the good taste to love economics. This supports the pooling equilibrium also.

The Beer-Quiche Game of Cho & Kreps (1987). Player I is weak or strong and doesn't want to duel. Player II wants to duel only if player I is weak. Player II does not know player I's type, but he observes what player I has for breakfast. Weak players prefer quiche for breakast, strong players prefer beer.

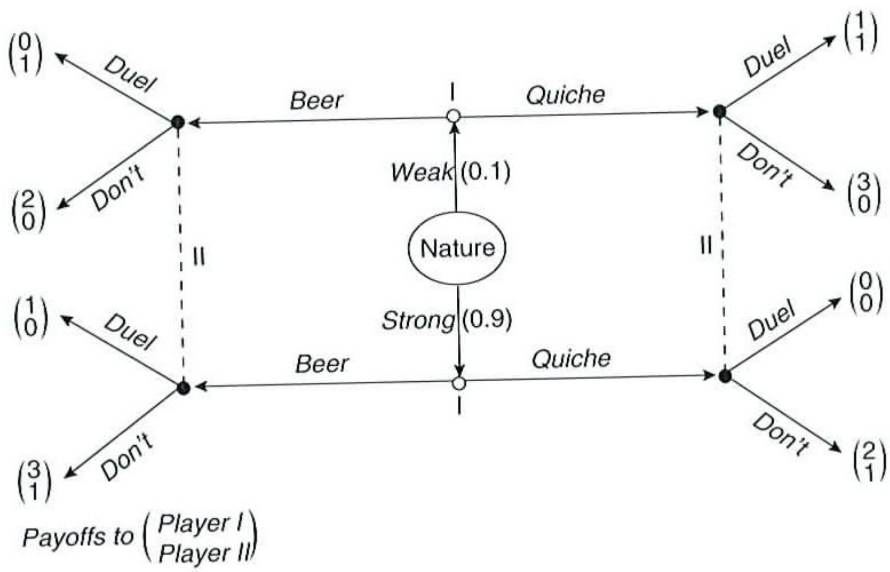

Figure 6.5 The Beer–Quiche Game.

E_1: **Player I has beer. Player II doesn't duel if beer, does duel if quiche. Out-of-equilibrium belief: a quiche-eating player I is weak with probability over 0.5.**

E_2: **Player I has quiche. Player II duel if beer doesn't duel if quiche. Out-of-equilibrium belief: a beer-drinking player I is weak with probability over 0.5.**

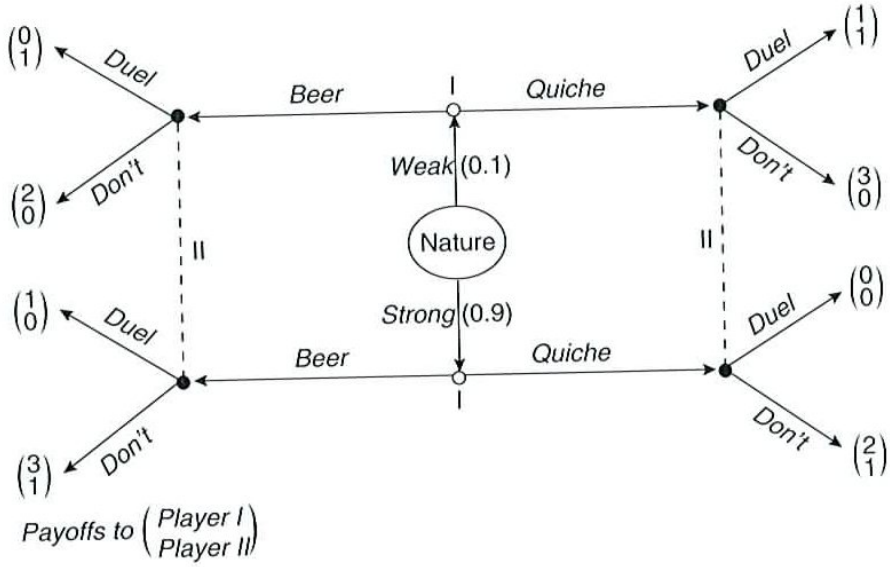

Figure 6.5 The Beer–Quiche Game.

E_2: Player I has quiche. Player II duel if beer doesn't duel if quiche. Out-of-equilibrium belief: a beer-drinking player I is weak with probability over 0.5.

Intuitive Criterion: player I could deviate to BEER by giving the following convincing speech,

> **I am having beer for breakfast, which ought to convince you I am strong. The only conceivable benefit to me of breakfasting on beer comes if I am strong. I would never wish to have beer for breakfast if I were weak, but if I am strong and this message is convincing, then I benefit from having beer for breakfast.**

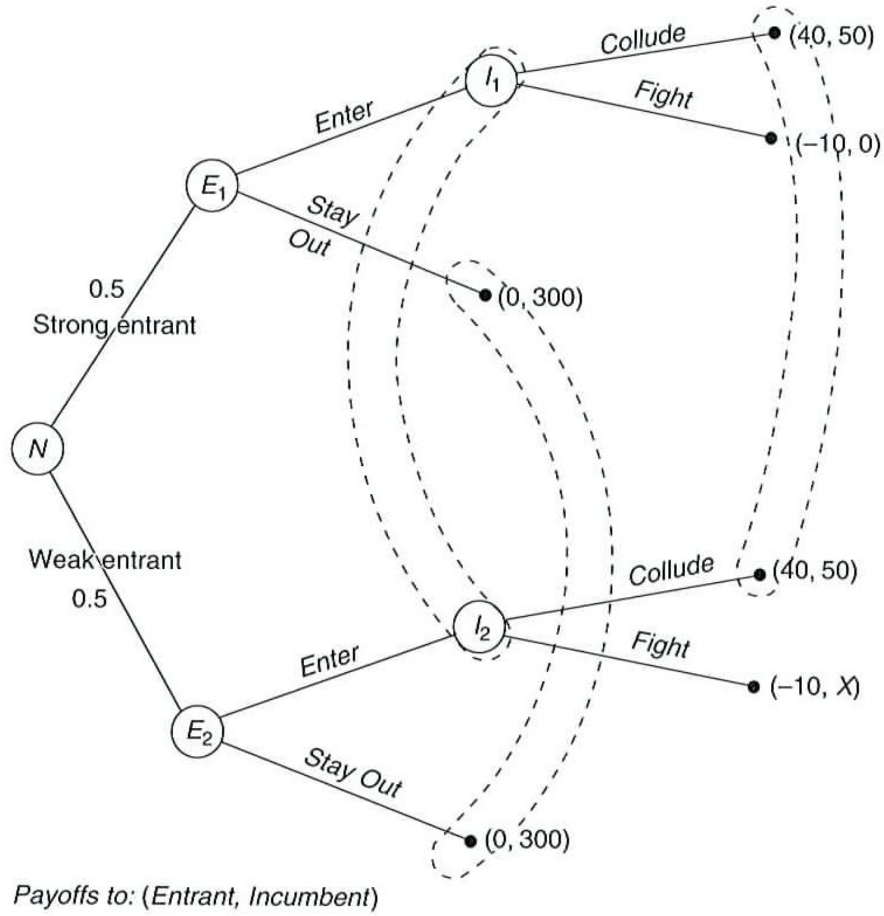

Figure 6.1 Entry Deterrence II, III, and IV.

Entry Deterrence IV: The Incumbent Benefits from His Own Ignorance

Let $X = 300$. The entrant knows his type, but the incumbent does not.

Equilibrium for Entry Deterrence IV
Entrant: *Stay Out* $|Weak$, *Stay Out* $|Strong$
Incumbent: *Fight*,
Out-of-equilibrium beliefs: $Prob$(*Strong*|*Enter*) = 0.5 (passive conjectures)

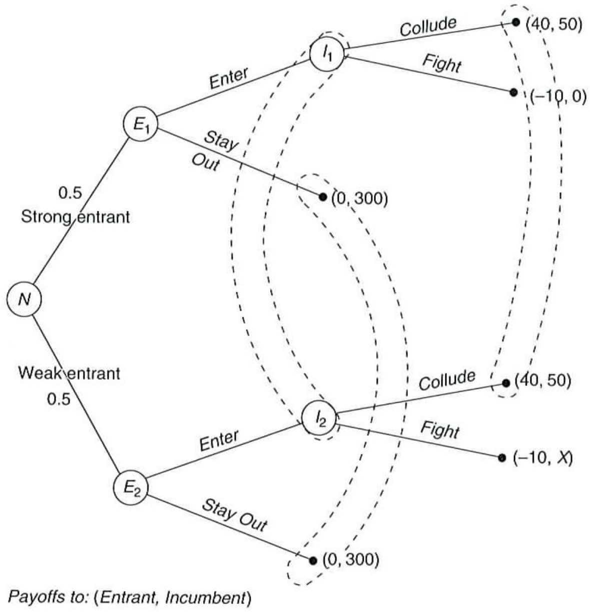

Figure 6.1 Entry Deterrence II, III, and IV.

There is no pure-strategy pooling equilibrium in which both types of entrant enter, because then the incumbent's expected payoff from *Fight* would be 150 ($= 0.5[0] + 0.5[300]$), which is greater than the *Collude* payoff of 50. Nor is there a pure-strategy separating equilibrium.

There exists a mixed-strategy equilibrium too.

Entry Deterrence V: Lack of Common Knowledge of Ignorance: Both the entrant and the incumbent know the payoff from (*Enter, Fight*), but the entrant does not know whether the incumbent knows.

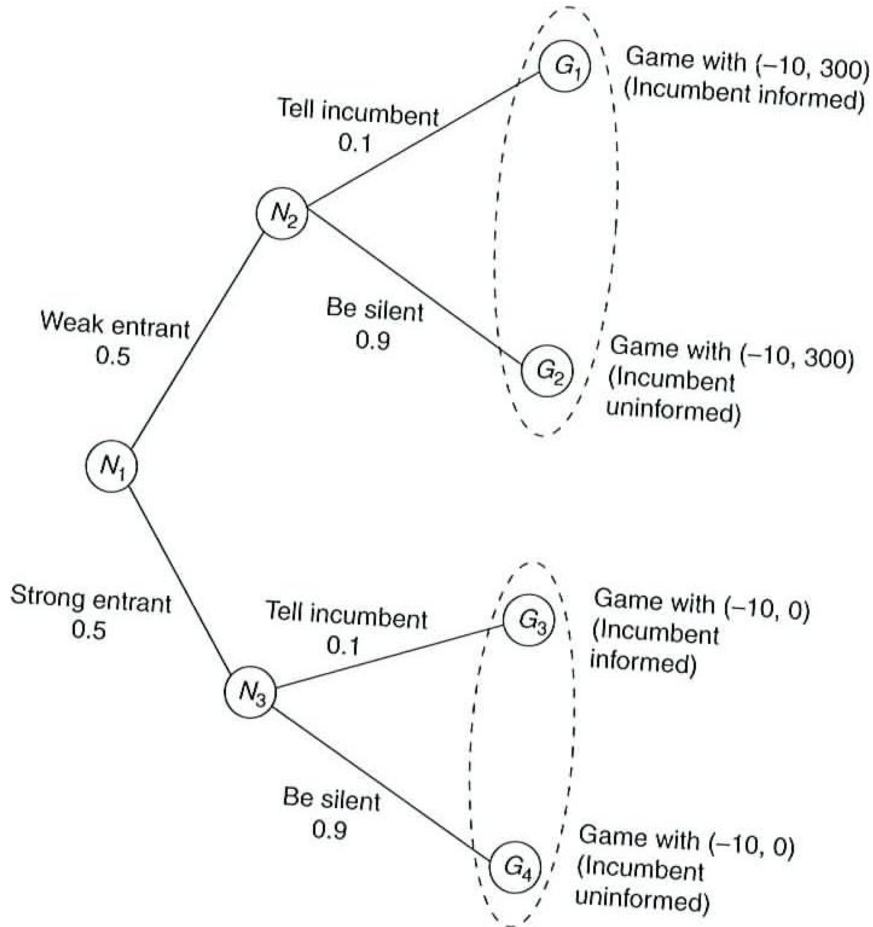

Figure 6.3 Entry Deterrence V.

Entrant: *Stay Out*$|Weak$, *Stay Out*$|Strong$

Incumbent: *Fight|Nature said "Weak"*, *Collude|Nature said "Strong"*, *Fight |Nature said nothing*,

Out-of-equilibrium beliefs:

$Prob$($Strong|$*Enter, Nature said nothing*) **= 0.5 (passive conjectures)**

Equilibrium for Entry Deterrence V
Entrant: *Stay Out*$|Weak$**,** *Stay Out*$|Strong$
Incumbent: *Fight*$|$*Nature said "Weak"***,** *Collude*$|$*Nature said "Strong"***,** *Fight* $|$*Nature said nothing***, Out-of-equilibrium beliefs:** $Prob$**(** $Strong|$*Enter, Nature said nothing)* **= 0.5 (passive conjectures)**

With probability 0.9, Nature has said nothing and the incumbent calculates his expected payoff from *Fight* **to be 150, and with probability 0.05** $(= 0.1[0.5])$**) Nature has told the incumbent that the entrant is weak and the payoff from** *Fight* **is 300. Even if the entrant is strong and Nature tells this to the incumbent, the entrant would choose** *Stay Out***, because he does not know that the incumbent knows, and his expected payoff from** *Enter* **would be** -5 **(**$= [0.9][-10] + 0.1[40]$**).**

Kreps, Milgrom, Roberts, Wilson (1982) : The Gang of Four Model

One way to incorporate incomplete information would be to assume that with 30% probability Row is a player who blindly follows the strategy of Tit-for-Tat.

If Column thinks he is playing against a Tit-for-Tat player, his optimal strategy is *Silence* **until near the last period (how near depending on the parameters), and then** *Blame***.**

If he were not certain of this, but the probability were high that he faced a Tit-for-Tat player, Row would choose that same strategy.

But it turns out that even a small probability of a Tit-for-Tat player can make a big difference.

Theorem 6.1: The Gang of Four Theorem
Consider a T-stage, repeated Prisoner's Dilemma, without discounting but with a probability γ of a Tit-for-Tat player. In any perfect bayesian equilibrium, the number of stages in which either player chooses Blame is less than some number M that depends on γ but not on T.

In equilibrium, $Blame$ is played in the periods near T. Before that there is a period of mixing, and before that they play $Silence$.

The significance of the Gang of Four theorem is that while the players do resort to $Blame$ as the last period approaches, the number of periods during which they $Blame$ is independent of the total number of periods. Suppose $M = 2,500$. If $T = 2,500$, there might be $Blame$ every period. But if $T = 10,000$, there are 7,500 periods without a $Blame$ move. For reasonable probabilities of the unusual type, the number of periods of cooperation can be much larger.

Wilson has set up an entry deterrence model in which the incumbent fights entry (the equivalent of *Silence* above) up to seven periods from the end, although the probability the entrant is of the unusual type is only 0.008.

Gang of Four Intuition

		Column	
		Silence	*Blame*
	Silence	**5,5**	**-5,10**
Row:			
	Blame	**10,-5**	**0,0**

Payoffs to: (Row,Column)

Consider what would happen in a 10,001-period PD with a probability of 0.01 that Row is playing the Grim Strategy.

A best response for Column to a known Grim player is ($Blame$ **only in the last period, unless Row chooses** $Blame$ **first, in which case respond with** $Blame$**).**

Column's payoff will be 50,010 (= (10,000)(5) + 10).

Blame Always **would just yield 10 as a payoff.**

Suppose instead that if Row is not Grim, he will choose $Blame$ **every period. The outcome will be (**$Blame, Silence$**) in the first period and** *(Blame, Blame*) **thereafter, for a payoff to Column of** $-5(= -5 + (10,000)(0))$**. If the probabilities of the two outcomes are 0.01 and 0.99, Column's expected payoff is 495.15.**

If instead Row follows a strategy of ($Blame$ **every period), his expected payoff is just 0.1 (**$= 0.01(10) + 0.99(0)$**).**

		Column	
		Silence	*Blame*
Row:	*Silence*	**5,5**	**-5,10**
	Blame	**10,-5**	**0,0**

Payoffs to: (Row,Column)

The aggressive strategy is not Row's best response to Column's strategy. A better response is for Row to choose $Silence$ until the second-to-last period, and then $Blame$. Row's payoff would rise from 10 to (9,999)(5) + 10.

Given that Column is cooperating in the early periods, Row will cooperate also. Still not Nash, but we see why Column chooses $Silence$ in the first period.

Theorem 6.2: The Incomplete Information Folk Theorem(Fudenberg & Maskin [1986] p. 547)

For any two-person repeated game without discounting, the modeller can choose a form of irrationality so that for any probability $\epsilon > 0$ there is some finite number of repetitions such that with probability $(1 - \epsilon)$ a player is rational and the average payoffs in some sequential equilibrium are closer than ϵ to any desired payoffs greater than the minimax payoffs.

THE AXELROD TOURNAMENT: Contestants submitted strategies for a 200-repetition Prisoner's Dilemm
.

Since the strategies could not be updated during play, players could precommit, but the strategies could be as complicated as they wished.

Strategies were submitted in the form of computer programs. In Axelrod's first tournament, 14 programs were submitted as entries. Every program played every other program, and the winner was the one with the greatest sum of payoffs over all the plays. The winner was Anatol Rapoport, whose strategy was Tit-for-Tat.

What strategy could have beat Rapoport and all the others?

After the results of the first tournament were announced, Axelrod ran a second tournament, adding a probability $\theta = 0.00346$ that the game would end each round so as to avoid the Chainstore Paradox. The winner among the 62 entrants was again Anatol Rapoport with Tit-for-Tat.

Before choosing his tournament strategy, Rapoport had written an entire book on The Prisoner's Dilemma in analysis, experiment, and simulation.

Why did he choose such a simple strategy as Tit-for-Tat?

Tit-for-Tat has three strong points.

1. **It never initiates blaming (niceness);**
2. **It retaliates instantly against blaming (provocability);**
3. **It forgives someone who plays** $Blame$ **but then goes back to cooperating (it is forgiving).**

Tit-for-Tat never beats any other strategy in a one-on-one contest. In an elimination tournament, Tit-for- Tat would be eliminated early, because it scores *high* payoffs but never the *highest* payoff.

In a game in which players occasionally blamed because of trembles, two Tit-for-Tat players facing each other would do very badly.

“Reputation Acquisition in Debt Markets” JPE, 1989

Douglas Diamond (1989) explains why old firms are less likely than young firms to default on debt. The three types of risk-neutral firms, R, S, and RS, are “born” at time zero and borrow to finance projects at the start of each of T periods.

Type RS firms can choose independently risky projects with negative expected values or safe projects with low but positive expected values.

Although the risky projects are worse in expectation, if they are successful the return is much higher than from safe projects. Type R firms can only choose risky projects, and type S firms only safe projects.

At the end of each period the projects bring in their profits and loans are repaid, after which new loans and projects are chosen for the next period. Lenders cannot see which project is chosen or a firm’s current profits, but they can seize the firm’s assets if a loan is not repaid, which always happens if the risky project was chosen and turned out unsuccessfully.

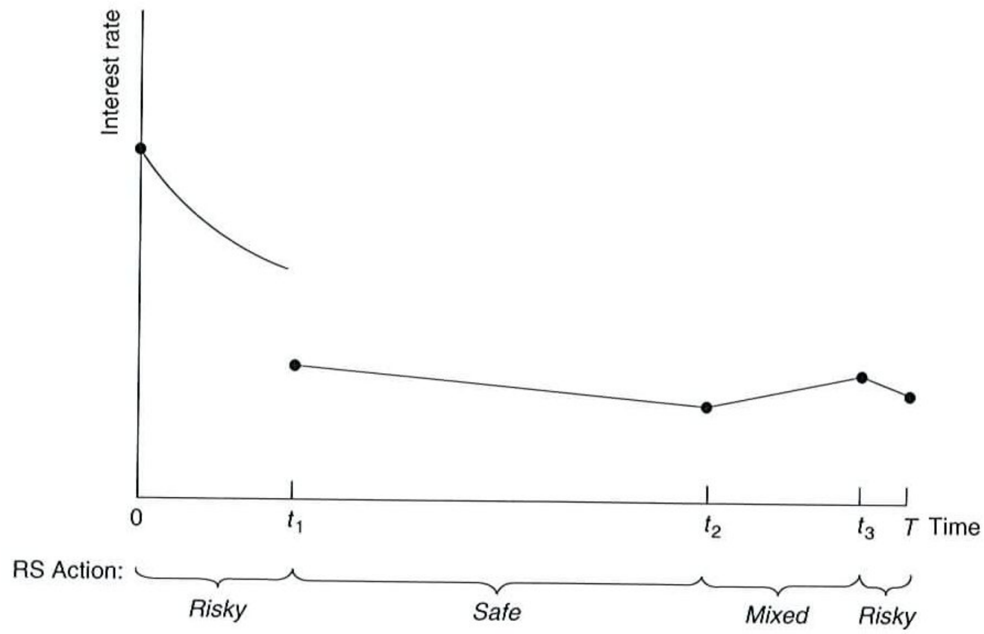

Figure 6.4 The interest rate over time.

The equilibrium path has three parts. The RS firms start by choosing risky projects. Their downside risk is limited by bankruptcy, but if the project is successful the firm keeps large profits after repaying the loan. Over time, the number of firms with access to the risky project (the RS's and R's) diminishes through bankruptcy, while the number of S's remains unchanged.

Lenders can therefore maintain zero profits while lowering their interest rates. When the interest rate falls, the value of a stream of safe investment profits minus interest payments rises relative to the expected value of the few periods of risky returns minus interest payments before bankruptcy.

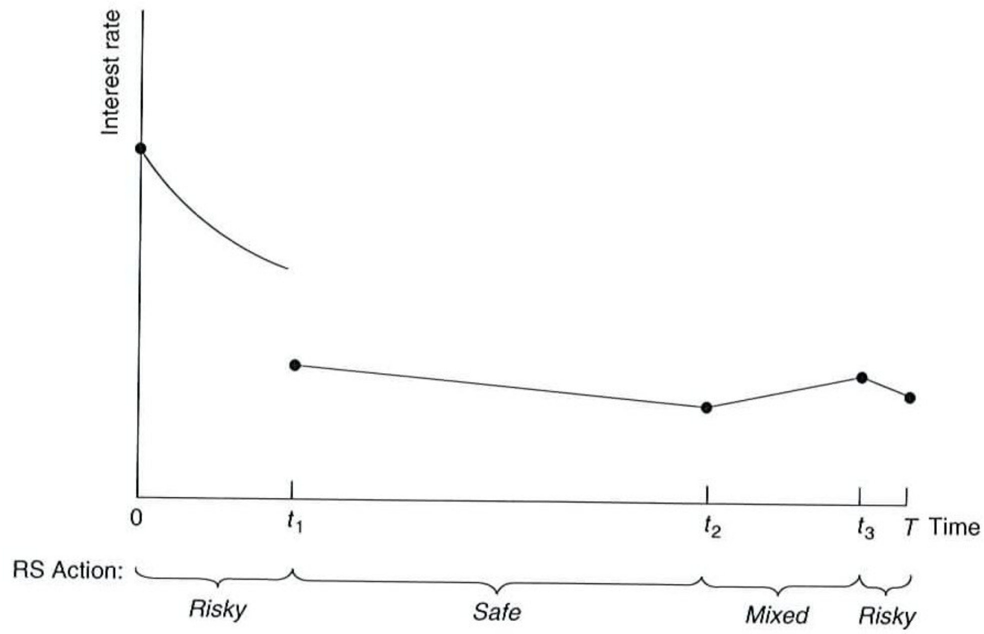

Figure 6.4 The interest rate over time.

After the interest rate has fallen enough, the second phase of the game begins when the RS firms switch to safe projects, at t_1. Only the tiny and diminishing group of type R firms continue to choose risky projects. Since the lenders know that the RS firms switch, the interest rate falls sharply at t_1. A firm that is older is less likely to be a type R, so it is charged a lower interest rate.

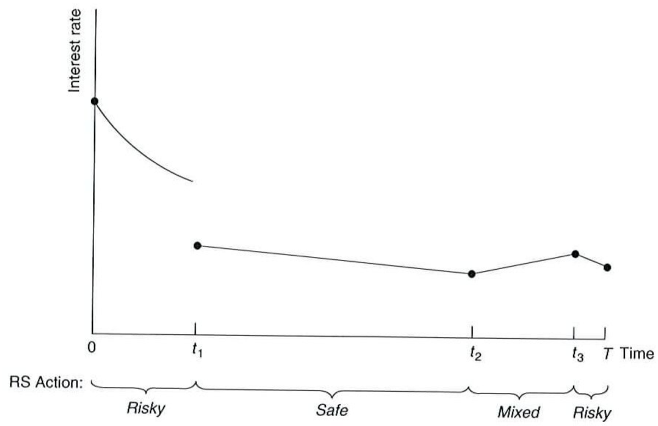

Figure 6.4 The interest rate over time.

Towards T, the value of future profits from safe projects declines and the RS's are again tempted to choose risky projects.

Between t_2 and t_3, the RS's follow a mixed strategy, an increasing number choosing risky projects. The interest rate rises as a result.

At t_3, the interest rate is high enough and the end of the game is close enough that the RS's revert to the pure strategy of choosing risky projects. The interest rate then falls as the number of RS's diminishes because of failed risky projects.

Why three types of firms, not two?

Types S and RS are clearly needed, but why type R?

The little extra detail in the game description allows simplification of the equilibrium, because with three types bankruptcy is never out-of-equilibrium behaviour, since the failing firm might be a type R.

Bayes's Rule can therefore always be applied, elminating the problem of ruling out peculiar beliefs and absurd perfect bayesian equilibria.

This is a Gang of Four model but differs from previous examples in an important respect: the Diamond model is not stationary, and as time progresses, some firms of types R and RS go bankrupt, which changes the lenders' payoff functions. Thus, it is not a repeated game.