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Eric Rasmusen, Erasmuse@indiana.edu

Private-Value and Common-Value Auctions

In a **private-value auction**, a bidder can learn nothing about his value from knowing the values of the other bidders.

Knowing all the other values in advance would not change his estimate. It might change his bidding strategy.

SPECIAL CASE 1: **Independent private-value auction**, in which knowing his own value tells him nothing about other bidders' values.

SPECIAL CASE 2: **Affiliated private-value auction** he can use knowledge of his own value to deduce something about other players' values. (they are correlated in a certain sense)

Pure common-value auction, the bidders have identical values, but each bidder forms his own estimate on the basis of his own private information.

Auction Rules

1 Ascending (English, open-cry, open-exit);

2 First-Price (first-price sealed-bid);

3 Second-Price (second-price sealed-bid, Vickrey);

4 Descending (Dutch)

5 All-Pay

Ascending (English, open-cry, open-exit)

Rules

Each bidder is free to revise his bid upwards. When no bidder wishes to revise his bid further, the highest bidder wins the object and pays his bid.

Strategies

A bidder's strategy is his series of bids as a function of (1) his value, (2) his prior estimate of other bidders' values, and (3) the past bids of all the bidders. His bid can therefore be updated as his information set changes.

Payoffs

The winner's payoff is his value minus his highest bid ($t = p$ for him and $t = 0$ for everyone else). The losers' payoffs are zero.

Some variations:

(1) The bidders offer new prices using pre-specified **increments** such as dollars or thousands of dollars.

(2) The **open-exit** auction, in which the price rises continuously and bidders show their willingness to pay the price by not dropping out, where a bidder's dropping out is publicly announced to the other bidders.

(3) The **silent-exit** auction (my neologism), in which the price rises continuously and bidders show their willingness to pay the price by not dropping out, but a bidder's dropping out is not known to the other bidders.

(4) The **Ebay auction**, in which a bidder submits his "bid ceiling." There is a prespecified ending time.

(5) The **Amazon auction**, in which a bidder submits his bid ceiling. Prespecified ending time OR ten minutes after the last increase in the current winning bid, whichever is later.

A mechanism $[G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i})]$ takes payments t and gives an object with probability G to player i if he announces that his value is $\tilde{v}_i$ and the other players announce $\tilde{v}_{-i}$.

An ascending auction can be seen as a mechanism in which each bidder announces his value, the object is awarded to whoever announces the highest value, and he pays the second-highest announced value (the second-highest bid).

The Continuous-Value Auction

Players: One seller and two bidders.

0. Nature chooses Bidder i's value for the object, v_i, using the strictly positive, atomless density $f(v)$ on the interval $[\underline{v}, \overline{v}]$.
1. The seller chooses a mechanism $[G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i})]$ that takes payments t and gives the object with probability G to player i (including the seller) if he announces that his value is $\tilde{v}_i$ and the other players announce $\tilde{v}_{-i}$. He also chooses the procedure in which bidders select $\tilde{v}_i$ (sequentially, simultaneously, etc.).
2. Each bidder simultaneously chooses to participate in the auction or to stay out.
3. The bidders and the seller choose $\tilde{v}$ according to the mechanism procedure.
4. The object is allocated and transfers are paid according to the mechanism, if it was accepted by all bidders.

Payoffs:

The seller's payoff is

$$\pi_s = \sum_{i=1}^{n} t(\tilde{v}_i, \tilde{v}_{-i}) \tag{1}$$

Bidder i's payoff is zero if he does not participate, and otherwise is

$$\pi_i(v_i) = G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i}) \tag{2}$$

In the Continuous-Value Auction, denote the highest announced value by $\tilde{v}_{(1)}$, the second-highest by $\tilde{v}_{(2)}$, and so forth.

The highest bidder gets the object with probability

$$G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 1$$

at price

$$t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = \tilde{v}_{(2)},$$

and for $i \neq 1$, $G(\tilde{v}_{(i)}, \tilde{v}_{-i}) = 0$

$$t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 0.$$

The PROFIT-MAXIMIZING (optimal?) mechanism has a

reserve price p^* below which the object would remain unsold. Thus

$G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 1$

$t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = Max\{\tilde{v}_{(2)}, p^*\}$ if $\tilde{v}_{(1)} \geq p^*$ but $G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 0$ if $\tilde{v}_{(1)} < p^*$.

Optimal mechanisms are not always efficient. The object will go unsold if $\tilde{v}_{(1)} < p^*$.

First-Price (first-price sealed-bid)

Rules

Each bidder submits one bid, in ignorance of the other bids. The highest bidder pays his bid and wins the object.

Strategies

A bidder's strategy is his bid as a function of his value.

Payoffs

The winner's payoff is his value minus his bid. The losers' payoffs are zero.

The First-Price Auction with a Continuous Distribution of Values

Suppose Nature independently assigns values to n risk-neutral bidders using the continuous density $f(v) > 0$ (with cumulative probability $F(v)$) on the support $[0, \bar{v}]$.

A bidder's payoff as a function of his value v and his bid function $p(v)$ is, letting $G(p(v))$ denote the probability of winning with a particular $p(v)$:

$$\pi(v, p(v)) = G(p(v))[v - p(v)]. \tag{3}$$

Now go to the board.

Now let us try to find an equilibrium bid function. From equation (3), it is

$$p(v) = v - \frac{\pi(v, p(v))}{G(p(v))}. \tag{4}$$

That is not very useful in itself, since it has $p(v)$ on both sides. We need to find ways to rewrite π and G in terms of just v.

First, tackle $G(p(v))$. Monotonicity of the bid function (from Lemma 1) implies that the bidder with the greatest v will bid highest and win. Thus, the probability $G(p(v))$ that a bidder with price p_i will win is the probability that v_i is the highest value of all n bidders. The probability that a bidder's value v is the highest is $F(v)^{n-1}$, the probability that each of the other $(n-1)$ bidders has a value less than v. Thus,

$$G(p(v)) = F(v)^{n-1}. \tag{5}$$

Next think about $\pi(v, p(v))$. The Envelope Theorem says that if $\pi(v, p(v))$ is the value of a function maximized by choice of $p(v)$ then its total derivative with respect to v equals its partial derivative, because $\frac{\partial \pi}{\partial p} = 0$:

$$\frac{d\pi(v,p(v))}{dv} = \frac{\partial \pi(v,p(v))}{\partial p}\frac{\partial p}{\partial v} + \frac{\partial \pi(v,p(v))}{\partial v} = \frac{\partial \pi(v,p(v))}{\partial v}. \tag{6}$$

We can apply the Envelope Theorem to equation (3) to see how π changes with v assuming $p(v)$ is chosen optimally, which is appropriate because we are characterizing not just any bid function, but the optimal bid function. Thus,

$$\frac{d\pi(v, p(v))}{dv} = G(p(v)). \tag{7}$$

Substituting from equation (5) gives us π's derivative, if not π, as a function of v:

$$\frac{d\pi(v, p(v))}{dv} = F(v)^{n-1}. \tag{8}$$

To get $\pi(v, p(v))$ from its derivative, (8), integrate over all possible values from zero to v and include the a base value of $\pi(0)$ as the constant of integration:

$$\pi(v, p(v)) = \pi(0) + \int_0^v F(x)^{n-1}dx = \int_0^v F(x)^{n-1}dx. \tag{9}$$

The last step is true because a bidder with $v = 0$ will never bid a positive amount and so will have a payoff of $\pi(0, p(0)) = 0$.

We can now return to the bid function in equation (4) and substitute for $G(p(v))$ and $\pi(v, p(v))$ from equations (5) (9):

$$p(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}}. \tag{10}$$

Suppose $F(v) = v/\bar{v}$, the uniform distribution. Then (10) becomes

$$
\begin{aligned}
p(v) &= v - \frac{\int_0^v \left(\frac{x}{\bar{v}}\right)^{n-1} dx}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\
&= v - \frac{\Big|_{x=0}^{v} \left(\frac{1}{\bar{v}}\right)^{n-1} \left(\frac{1}{n}\right) x^n}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\
&= v - \frac{\left(\frac{1}{\bar{v}}\right)^{n-1} \left(\frac{1}{n}\right) v^n - 0}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\
&= v - \frac{v}{n} = \left(\frac{n-1}{n}\right) v.
\end{aligned}
\tag{11}
$$

What a happy ending to a complicated derivation! If there are two bidders and values are uniform on $[0, 1]$, a bidder should bid $p = v/2$, which since he has probability v of winning yields an expected payoff of $v^2/2$. If $n = 10$ he should bid $\frac{9}{10}v$, which since he has probability v^9 of winning yields him an expected payoff of $v^{10}/10$, quite close to zero if $v < 1$.

Second-Price Auctions (Second-price sealed-bid, Vickrey)

Rules

Each bidder submits one bid, in ignorance of the other bids. The bids are opened, and the highest bidder pays the amount of the second-highest bid and wins the object.

Strategies

A bidder's strategy is his bid as a function of his value.

Payoffs

The winning bidder's payoff is his value minus the second-highest bid. The losing bidders' payoffs are zero. The seller's payoff is the second-highest-bid.

Consider the following equilibrium.

$$\begin{array}{ll} p_1(v=10)=10 & p_1(v=16)=16 \\ p_2(v=10)=1 & p_2(v=16)=10 \end{array} \tag{12}$$

Since Bidder 1 never bids less than 10, Bidder 2 knows that if $v_2 = 10$ he can never get a positive payoff, so he is willing to choose $p_2(v = 10) = 1$. Doing so results in a sale price of 1, for any $p_1 > 1$, which is better for Bidder 1 and worse for the seller than a price of 10, but Bidder 2 doesn't care about their payoffs. In the same way, if $v_2 = 16$, Bidder 2 knows that if he bids 10 he will win if $v_1 = 10$, but if $v_2 = 16$ he would have to pay 16 to win and would earn a payoff of zero. He might as well bid 10 and earn his zero by losing.

Descending Auctions (Dutch)

Rules

The seller announces a bid, which he continuously lowers until some bidder stops him and takes the object at that price.

Strategies

A bidder's strategy is when to stop the bidding as a function of his value.

Payoffs

The winner's payoff is his value minus his bid. The losers' payoffs are zero.

Descending auctions are **strategically equivalent** to the first-price auction, which means there is a one-to-one mapping between the strategy sets and the equilibria of the two games.

All-Pay Auctions

Rules

Each bidder places a bid simultaneously. The bidder with the highest bid wins, and each bidder pays the amount he bid.

Strategies

A bidder's strategy is his bid as a function of his value.

Payoffs

The winner's payoff is his value minus his bid. The losers' payoffs are the negative of their bids.

The Equal-Value All-Pay Auction

Suppose each of the n bidders has the same value, v. Why is the equilibrium in mixed strategies?

Suppose we have a symmetric equilibrium, so all bidders use the same mixing cumulative distribution $M(p)$. Let us conjecture that $\pi(p) = 0$, which we will later verify.

$$M(p)^{n-1}v = p, \tag{13}$$

so

$$M(p) = \sqrt[n-1]{\frac{p}{v}} \tag{14}$$

At the extreme bids that a bidder with value v might offer, $M(0) = \sqrt[n-1]{\frac{0}{v}} = 0$ and $M(v) = \sqrt[n-1]{\frac{v}{v}} = 1$, so we have found a valid distribution function $M(p)$. Moreover, since the payoff from one of the strategies between which it mixes, $p = 0$, equals zero, we have verified our conjecture that $\pi(p) = 0$ in the equilibrium.

$$M(p) = \sqrt[n-1]{\frac{p}{v}}, \tag{15}$$

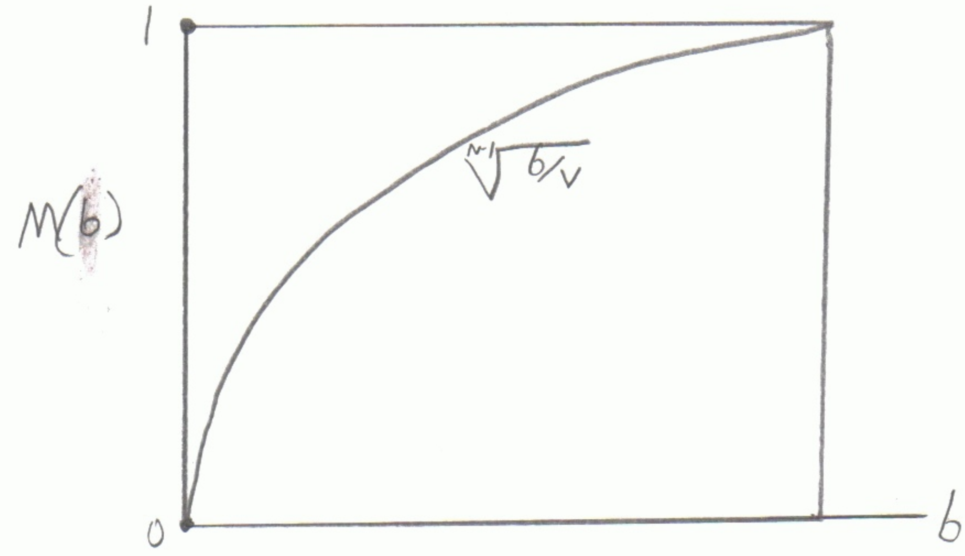

Figure 2: The Bid Function in an All-Pay Auction with Identical Bidders

The Continuous-Value All-Pay Auction

Suppose each of the n bidders picks his value v from the same density $f(v)$. Conjecture that the equilibrium is symmetric, in pure strategies, and that the bid function, $p(v)$, is strictly increasing. The equilibrium payoff function for a bidder with value v who pretends he has value z is

$$\pi(v, z) = F(z)^{n-1}v - p(z), \tag{16}$$

since if our bidder bids $p(z)$, that is the highest bid only if all $(n-1)$ other bidders have $v < z$, a probability of $F(z)$ for each of them.

The function $\pi(v, z)$ is not necessarily concave in z, so satisfaction of the first-order condition will not be a sufficient condition for payoff maximization, but it is a necessary condition since the optimal z is not 0 (unless $v = 0$) or infinity and from (16) $\pi(v, z)$ is differentiable in z in our conjectured equilibrium. Thus, we need to find z such that

$$\frac{\partial \pi(v, z)}{\partial z} = (n-1)F(z)^{n-2}f(z)v - p'(z) = 0 \tag{17}$$

In the equilibrium, our bidder does follow the strategy $p(v)$, so $z = v$ and we can write

$$p'(v) = (n-1)F(v)^{n-2}f(v)v \tag{18}$$

Integrating up, we get

$$p(v) = p(0) + \int_0^v (n-1)F(x)^{n-2}f(x)x dx \tag{19}$$

This is deterministic, symmetric, and strictly increasing in v, so we have verified our conjectures.

We can verify that truthelling is a symmetric equilibrium strategy by substituting for $p(z)$ from (19) into payoff equation (16).

$$\begin{aligned} \pi(v, z) &= F(z)^{n-1}v - p(z) \\ &= F(z)^{n-1}v - p(0) - \int_0^z (n-1)F(x)^{n-2}f(x)x dx \\ &= F(z)^{n-1}v - p(0) - F(z)^{n-1}z + \int_0^z F(x)^{n-1}dx, \end{aligned} \tag{20}$$

where the last step uses integration by parts ($\int gh' = gh - \int hg'$, where $g = x$ and $h' = (n-1)F(x)^{n-2}f(x)$). Maximizing (20) with respect to z yields

$$\frac{\partial \pi(v, z)}{\partial z} = (n-1)F(z)^{n-2}f(z)(v - z), \tag{21}$$

which is maximized by setting $z = v$. Thus, if $(n-1)$ of the bidders are using this $p(v)$ function, so will the remaining bidder, and we have a Nash equilibrium.

Let's see what happens with a particular value distribution. Suppose values are uniformly distributed over [0,1], so $F(v) = v$. Then equation (19) becomes

$$\begin{aligned} p(v) &= p(0) + \int_0^v (n-1)x^{n-2}(1)x dx \\ &= p(0) + \Big|_{x=0}^{v} (n-1)\frac{x^n}{n} \\ &= 0 + \left(\frac{n-1}{n}\right) v^n, \end{aligned} \tag{22}$$

where we can tell that $p(0) = 0$ because if $p(0) > 0$ a bidder with $v = 0$ would have a negative expected payoff. If there were $n = 2$ bidders, a bidder with value v would bid $v^2/2$, win with probability v, and have expected payoff $\pi = v(v) - v^2/2 = v^2/2$. If there were $n = 10$ bidders, a bidder with value v would bid $(9/10)v^{10}$, win with probability v^9, and have expected payoff $\pi = v(v^9) - (9/10)v^{10} = v^{10}/(10)$. As we will see when we discuss the Revenue Equivalence Theorem, it is no accident that this is the same payoff as for the first-price auction when values were uniformly distributed on [0,1], in equation (**??**).

THE REVENUE EQUIVALENCE THEOREM. Let all players be risk-neutral with private values drawn independently from the same atomless, strictly increasing distribution $F(v)$ on $[\underline{v}, \bar{v}]$. If under either Auction Rule A_1 or Auction Rule A_2 it is true that:

(a) The winner of the object is the player with the highest value; and

(b) The lowest bidder type, $v = \underline{v}$, has an expected payment of zero;

then the symmetric equilibria of the two auction rules have the same expected payoffs for each type of bidder and for the seller.

Proof of RET. Let us represent the auction as the truthful equilibrium of a direct mechanism in which each bidder sends a message z of his type v and then pays an expected amount $p(z)$. (The Revelation Principle says that we can do this.) By assumption (a), the probability that a player wins the object given that he chooses message z equals $F(z)^{n-1}$, the probability that all $(n-1)$ other players have values $v < z$. Let us denote this winning probability by $G(z)$, with density $g(z)$. Note that $g(z)$ is well defined because we assumed that $F(v)$ is atomless and everywhere increasing.

The expected payoff of any player of type v is the same, since we are restricting ourselves to symmetric equilibria. It equals

$$\pi(z, v) = G(z)v - p(z). \tag{23}$$

The first-order condition with respect to the player's choice of type message z (which we can use because neither $z = 0$ nor $z = \bar{v}$ is the optimum if condition (a) is to be true) is

$$\frac{d\pi(z; v)}{dz} = g(z)v - \frac{dp(z)}{dz} = 0, \tag{24}$$

so

$$\frac{dp(z)}{dz} = g(z)v. \tag{25}$$

We are looking at a truthful equilibrium, so we can replace z with v:

$$\frac{dp(v)}{dv} = g(v)v. \tag{26}$$

Next, we integrate (26) over all values from zero to v, adding $p(\underline{v})$ as the constant of integration:

$$p(v) = p(\underline{v}) + \int_{\underline{v}}^{v} g(x)x dx. \tag{27}$$

We can use (27 to substitute for $p(v)$ in the payoff equation (23), which becomes, after replacing z with v and setting $p(\underline{v}) = 0$ because of assumption (b),

$$\pi(v, v) = G(v)v - \int_{\underline{v}}^{v} g(x)x dx. \tag{28}$$

Equation (28) says the expected payoff of a bidder of type v depends only on the $G(v)$ distribution, which in turn depends only on the $F(v)$ distribution, and not on the $p(z)$ function or other details of the particular auction rule. But if the bidders' payoffs do not depend on the auction rule, neither does the seller's. Q.E.D.

The symmetric equilibria of the ascending, first-price, second-price, descending, and all-pay auctions with continuous values all satisfy the two conditions of the Theorem: (a) the winner is the bidder with the highest value, and (b) the lowest type makes an expected payment of zero. Thus, the following corollary is true.

A REVENUE EQUIVALENCE COROLLARY. Let all players be risk-neutral with private values drawn from the same strictly increasing, atomless distribution $F(v)$. The symmetric equilibria of the ascending, first-price, second-price, descending, and all-pay auctions all have the same expected payoffs for each type of bidder and for the seller.

TWO CAVEATS

(1) Although the different auctions have the same expected payoff for the seller, they do not have the same realized payoff.

(2) We need INDEPENDENT private values for the Revenue Equivalence Theorem.

Consider what happens if there are two bidders, both with values drawn uniformly from [0,10], but interdependently, with $v_2 = 10 - v_1$.

If we put aside equilibria with weakly dominated strategies (e.g., for a player to bid 0 if his value is less than 5), the second-price auction yields revenue equal to $p = v_{(2)}$, the second-highest value.

The seller can extract more revenue by using the auction rule that the winner is the highest bidder, and he pays 10 minus the second-highest bid: both players bid their values then, and the winning payment equals the highest value.

Common-Value Auctions and the Winner's Curse

The Winner's Curse: If bidders in a pure common-value auction all bid up to their valuations, the winner will be the one who overestimated the value the most, and he will have a negative payoff.

One way to think about a bidder's conditional estimate is to think about it as a conditional bid.

The bidder would like to submit a bid of

$[X$ *if I lose, but* $(X-Y)$ *if I win*$]$,

where X is his value estimate conditional on losing and $(X-Y)$ is his estimate conditional on winning.

If he still won with a bid of $(X-Y)$ he would be happy. If he lost, he would be relieved.

But Smith can achieve the same effect by simply submitting the bid $(X-Y)$ in the first place, since when he loses, the size of his bid is irrelevant.

Another way to look at the Winner's Curse is based on the Milgrom (1981) definition of "bad news".

Suppose the government is auctioning off the mineral rights to a plot of land with common value v and that bidder i has value estimate $\hat{v}_i$. Consider a symmetric equilibrium.

If Bidder 1 wins with a bid $p(\hat{v}_1)$ that is based on his prior value estimate $\hat{v}_1$, his posterior value estimate $\tilde{v}_1$ is

$$\tilde{v}_1 = E(V|\hat{v}_1, p(\hat{v}_2) < p(\hat{v}_1), \ldots, p(\hat{v}_n) < p(\hat{v}_1)). \tag{29}$$

The news that $p(\hat{v}_2) < \infty$ would be neither good nor bad, since it conveys no information.

The information that $p(\hat{v}_2) < p(\hat{v}_1)$ is bad news, since it rules out values of p more likely to be produced by large values of $\hat{v}_2$. Hence,

$$\tilde{v}_1 < E(V|\hat{v}_1) = \hat{v}_1, \tag{30}$$

Table 1 Bids by Serious Competitors in Oil Auctions

Offshore Louisiana 1967 Tract SS 207	**Santa Barbara Channel 1968** Tract 375	**Offshore Texas 1968** Tract 506	**Alaska North 1969** Tract 2
32.5	43.5	43.5	10.5
17.7	32.1	15.5	5.2
11.1	18.1	11.6	2.1
7.1	10.2	8.5	1.4
5.6	6.3	8.1	0.5
4.1		5.6	0.4
3.3		4.7	
		2.8	
		2.6	
		0.7	
		0.7	
		0.4	

Strategies in Common-Value Auctions

Milgrom & Weber (1982) found that when there is a common-value element in an auction and signals are "affiliated" then revenue equivalence fails.

The first-price and descending auctions are still identical, but they raise less revenue than the ascending or second-price auctions. If there are more than two bidders, the ascending auction raises more revenue than the second-price auction. (In fact, if signals are affiliated then even in a private value auction, in which each bidder knows his own value with certainty, the first-price and descending auctions will do worse.)

Suppose n signals are independently drawn from the uniform distribution on $[\underline{s}, \overline{s}]$. Denote the j^{th} highest signal by $s_{(j)}$. The expectation of the kth highest value happens to be

$$Es_{(k)} = \underline{s} + \left(\frac{n+1-k}{n+1} \right) (\overline{s} - \underline{s}) \qquad (31)$$

This means the expectation of the very highest value is

$$Es_{(1)} = \underline{s} + \left(\tfrac{n}{n+1}\right) (\overline{s} - \underline{s}) \qquad (32)$$

The expectation of the second-highest value is

$$Es_{(2)} = \underline{s} + \left(\tfrac{n-1}{n+1}\right) (\overline{s} - \underline{s}) \qquad (33)$$

The expectation of the lowest value, the n'th highest, is

$$Es_{(n)} = \underline{s} + \left(\tfrac{1}{n+1}\right) (\overline{s} - \underline{s}) . \qquad (34)$$

Let n risk-neutral bidders, $i = 1, 2, ...n$ each receive a signal s_i independently drawn from the uniform distribution on $[v - m, v + m]$, where v is the true value of the object to each of them. Assume that they have "diffuse priors" on v, which means they think any value from $v = -\infty$ to $v = \infty$ is equally likely and we do not need to make use of bayes's rule. The best estimate of the value given the set of n signals is

$$Ev|(s_1, s_2, ..., s_n) = \frac{s_{(n)} + s_{(1)}}{2}. \qquad (35)$$

The estimate depends only on two out of the n signals— a remarkable property of the uniform distribution. If there were five signals $\{6, 7, 7, 16, 24\}$, the expected value of the object would be 15 (=[6+24]/2), well above the mean of 12 and the median of 7, because only the extremes of 6 and 24 are useful information.

Someone who saw just signals $s_{(n)}$ and $s_{(1)}$ could deduce that v could not be less than $(s_{(1)} - m)$ or greater than $(s_{(n)} + m)$. Learning the signals in between would be unhelpful, because the only information that, for example, $s_{(2)}$ conveys is that $v \leq (s_{(2)} + m)$ and $v \geq (s_{(2)} - m)$, facts which our observer had already figured out from $s_{(n)}$ and $s_{(1)}$.

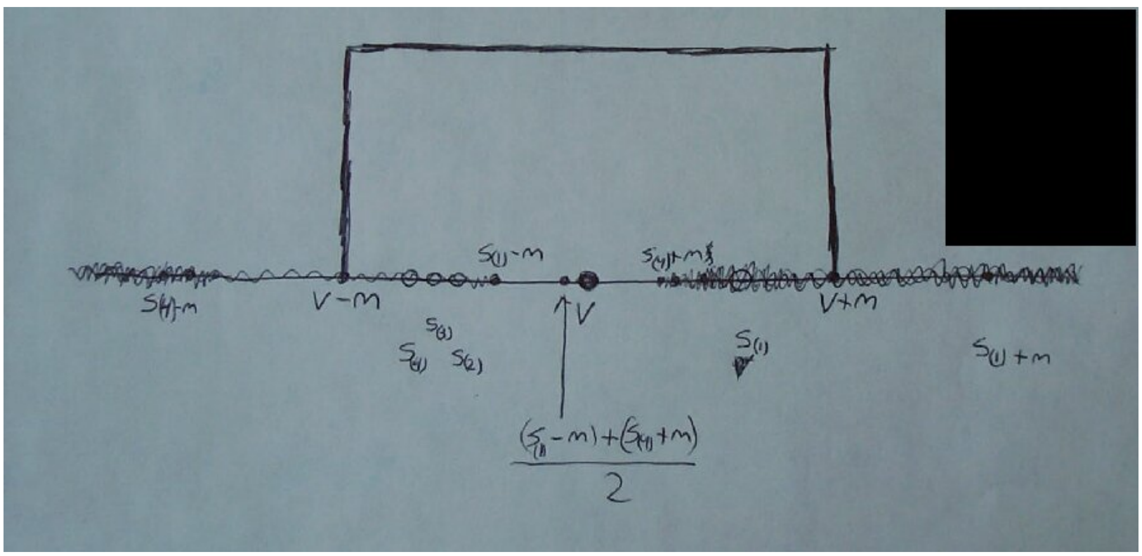

Figure 4: Extracting Information From Uniformly Distributed Signals

The Ascending Auction (open-exit)

Equilibrium: If no bidder has quit yet, Bidder i should drop out when the price rises to s_i. Otherwise, he should drop out when the price rises to $p_i = \frac{p_{(n)}+s_i}{2}$, where $p_{(n)}$ is the price at which the first dropout occurred.

Explanation: If no other bidder has quit yet, Bidder i is safe in agreeing to pay his signal, s_i. Either (a) he has the lowest signal, or (b) everybody else has the same signal value s_i too, and they will all drop out at the same time. In case (a), having the lowest signal, he will lose anyway. In case (b), the best estimate of the value is s_i, and that is where he should drop out.

Once one bidder has dropped out at $p_{(n)}$, the other bidders can deduce that he had the lowest signal, so they know that signal $s_{(n)}$ must equal $p_{(n)}$. Suppose Bidder i has signal $s_i > s_{(n)}$. Either (a) someone else has a higher signal and Bidder i will lose the auction anyway and dropping out too early does not matter, or (b) everybody else who has not yet dropped out has signal s_i too, and they will all drop out at the same time, or (c) he would be the last to drop out, so he will win. In cases (b) and (c), his estimate of the value is $p_{(i)} = \frac{p_{(n)}+s_i}{2}$, since $p_{(n)}$ and s_i are the extreme signal values and the signals are uniformly distributed, and that is where he should drop out.

The price paid by the winner will be the price at which the second-highest bidder drops out, which is $\frac{s_{(n)}+s_{(2)}}{2}$. The expected values are, from equations (33) and (34),

$$\begin{aligned} Es_{(n)} &= (v-m) + \left(\frac{n+1-n}{n+1}\right)((v+m)-(v-m)) \\ &= v + \left(\frac{1-n}{n+1}\right) m \end{aligned} \tag{36}$$

and

$$\begin{aligned} Es_{(2)} &= (v-m) + \left(\frac{n+1-2}{n+1}\right)((v+m)-(v-m)) \\ &= v + \left(\frac{n-3}{n+1}\right) m. \end{aligned} \tag{37}$$

Averaging them yields the expected winning price,

$$\begin{aligned} Ep_{(2)} &= \frac{\left[v+\left(\frac{1-n}{n+1}\right)m\right]+\left[v+\left(\frac{n-3}{n+1}\right)m\right]}{2} \\ &= v - \left(\frac{1}{2}\right)\left(\frac{1}{n+1}\right)2m. \end{aligned} \tag{38}$$

If $m = 50$ and $n = 4$, then

$$Ep_{(2)} = v - \left(\frac{1}{10}\right)(100) = v - 10. \qquad (39)$$

Expected seller revenue increases in n, the number of bidders (and thus of independent signals) and falls in the uncertainty m (the inaccuracy of the signals). This will be true for all three auction rules we examine here.

It is not always true that the bidders can deduce the lowest signal in an ascending auction and use that to form their bid. Their ability to discover $s_{(n)}$ depended crucially on the open-exit feature of the auction— that the player with the lowest signal had to openly drop out, rather than lurk quietly in the background. A secret-exit ascending auction would behave like a second-price auction instead.

The Second-Price Auction

Equilibrium: Bid $p_i = s_i - \left(\frac{n-2}{n}\right) m$.

Explanation: In forming his strategy, Bidder i should think of himself as being tied for winner with one other bidder, and so having to pay exactly his bid. Thus, he imagines himself as the highest of $(n-1)$ bidders drawn from $[v-m, v+m]$ and tied with one other. On average, if this happens,

$$s_i = (v-m) + \left(\frac{([n-1]+1-1)}{[n-1]+1}\right)([v+m]-[v-m])$$

$$= v + \left(\frac{n-2}{n}\right)(m). \tag{40}$$

He will bid the value v which solves equation (40), yielding the optimal strategy, $p_i = s_i - \left(\frac{n-2}{n}\right)(m)$.

On average, the second-highest bidder actually has the signal $Es_{(2)} = v + \left(\frac{n-3}{n+1}\right) m$, from equation (37). So the expected price, and hence the expected revenue from the auction, is

$$\begin{aligned} Ep_{(2)} &= [v + \left(\tfrac{n-3}{n+1}\right) m] - \left(\tfrac{n-2}{n}\right)(m) \\ &= v - \left(\tfrac{n-1}{n}\right)\left(\tfrac{1}{n+1}\right) 2m. \end{aligned} \tag{41}$$

If $m = 50$ and $n = 4$, then

$$Ep_{(2)} = v - \left(\frac{3}{4}\right)\left(\frac{1}{5}\right)(100) = v - 15. \tag{42}$$

If there are at least three bidders, expected revenue is lower in the second- price auction.