APPENDICES: SOMETHING HILBERT GOT WRONG AND EUCLID GOT RIGHT: THE METHOD OF SUPERPOSITION AND THE SIDE-ANGLE-SIDE AXIOM IN PROPOSITIONS 1 TO 4 OF BOOK I OF THE ELEMENTS

February 10, 2012

Eric B. Rasmusen

Abstract

This file just contains the appendices.

Rasmusen: Dan R. and Catherine M. Dalton Professor, Department of Business Economics and Public Policy, Kelley School of Business, Indiana University. BU 438, 1309 E. 10th Street, Bloomington, Indiana, 47405-1701. (812) 855-9219. Fax: 812-855-3354. mailto:erasmuse@indiana.edu erasmuse@indiana.edu, http://www.rasmusen.org.

This paper: http://www.rasmusen.org/papers/euclid-rasmusen-appendices.pdf.

Appendices: Systems of Axioms for Euclidean Geometry

This appendix will list various sets of axioms for Euclidean geometry— Euclid's, Rasmusen's, Hilbert's, Birkhoff's, Tarski's, Veblen's and the SMSG's.

Appendix I: Euclid's Axioms

The definitions used in the postulates are boldfaced.

http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#posts

Definition 1.
A point is that which has no part.

Definition 2.
A line is breadthless length.

Definition 3.
The ends of a line are points.

Definition 4.
A straight line is a line which lies evenly with the points on itself.

Definition 5.
A surface is that which has length and breadth only.

Definition 7.
A plane surface is a surface which lies evenly with the straight lines on itself.

Definition 8.
A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9.
And when the lines containing the angle are straight, the angle is called rectilinear.

Definition 10.
When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Definition 11.
An obtuse angle is an angle greater than a right angle.

Definition 12.
An acute angle is an angle less than a right angle.

Definition 13.
A boundary is that which is an extremity of anything.

Definition 14.
A figure is that which is contained by any boundary or boundaries.

Definition 15.
A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16.
And the point is called the center of the circle.

Definition 19.
Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Definition 20.
Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definition 21.
Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22.
Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Definition 23
Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulate 1.
To draw a straight line from any point to any point.

Postulate 2.
To produce a finite straight line continuously in a straight line.

Postulate 3.
To describe a circle with any center and radius.

Postulate 4.
That all right angles equal one another.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common notion 1.

Things which equal the same thing also equal one another.

Common notion 2.
If equals are added to equals, then the wholes are equal.

Common notion 3.
If equals are subtracted from equals, then the remainders are equal.

Common notion 4.
Things which coincide with one another equal one another.

Common notion 5.
The whole is greater than the part.

Tarski and Given note that to decide the simplicity of an axiomatic system, you need to incorporate the definitions into it. That's right.

RASMUSEN'S AXIOMS ,

Definition 1.
A point is that which has no part.

Definition 2.
A curve is breadthless length.

Definition 3.
The ends of a curve are points.

Definition 4.
A segment is a curve which lies evenly with the points on itself.

Definition 5.
A surface is that which has length and breadth only.

Definition 7.
A plane surface is a surface which lies evenly with the segments on itself.

Definition 8.
A plane angle is the inclination to one another of two curves in a plane which meet one another and do not lie in a segment.

Definition 9.
And when the curve containing the angle are straight, the angle is called rectilinear.

Definition 10.
When a segment standing on a segment makes the adjacent angles equal to one another, each of the equal angles is right, and the segment standing on the other is called a perpendicular to that on which it stands.

Definition 11.
An obtuse angle is an angle greater than a right angle.

Definition 12.

An acute angle is an angle less than a right angle.

Definition 13.
A boundary is that which is an extremity of anything.

Definition 14.
A figure is that which is contained by any boundary or boundaries. [connected?]

Definition 15.
A circle is a plane figure contained by one line such that all the segments falling upon it from one point among those lying within the figure equal one another.

Definition 16.
And the point is called the center of the circle.

Definition 19.
Rectilinear figures are those which are contained by segments, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four segments.

Definition 20.
Of trilateral figures, an equilateral triangle is that which has its three sides congruent, an isosceles triangle that which has two of its sides alone congruent, and a scalene triangle that which has its three sides noncongruent.

Definition 21.
Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22.
Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And

let quadrilaterals other than these be called trapezia.

Definition 23
Parallel segments are segments which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulate 1.
To draw a segment from any point to any point.

Postulate 2.
To produce a finite segment continuously in a straight line.

Postulate 3.
To describe a circle with any center and radius.

Common notion 1.
Things which equal the same thing also equal one another.

Common notion 2.
If equals are added to equals, then the wholes are equal.

Common notion 3.
If equals are subtracted from equals, then the remainders are equal.

Common notion 4.
Things which coincide with one another equal one another.

Common notion 5.
The whole is greater than the part.

Tarski and Given note that to decide the simplicity of an axiomatic system, you need to incorporate the definitions into it. That's right.

Order of Propositions goes here too.

HILBERT'S AXIOMS , as taken from the 1950 English edition at http://en.wikipedia.org/wiki/Hilbert%27s_axioms.

I. Combination

1. Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.

Instead of "determine," we may also employ other forms of expression; for example, we may say "A lies upon a", "A is a point of a", "a goes through A and through B", "a joins A to B", etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines a and b have the point A in common," etc.

2. Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.

3. Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = \alpha$.

We employ also the expressions: "A, B, C, lie in α"; "A, B, C are points of α", etc.

4. Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.

5. If two points A, B of a straight line a lie in a plane α, then every point of a lies in α. In this case we say: "The straight line a lies in the plane α," etc.

6. If two planes α, β have a point A in common, then they have at least a second point B in common.

7. Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.

II. Order

1. If a point B is between points A and C, B is also between C and A, and there exists a line containing the points A, B, C.

DIAGRAM 1

ASDFSFSFD (AXIOM XXX)

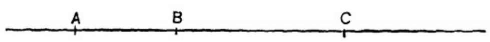
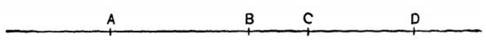


DIAGRAM 2

ASDFSFSFD (AXIOM XXX)



2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.

3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

4. Pasch's Axiom:

Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

DEFINITION. xxx

III. Parallels

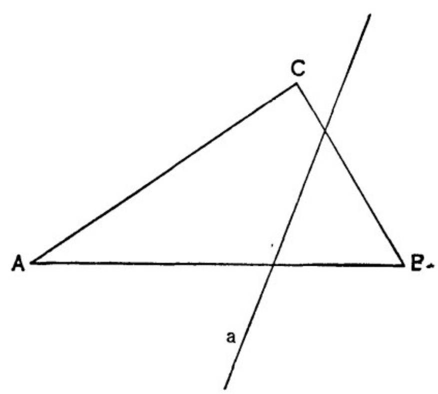
1. In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.

IV. Congruence

1. If A, B are two points on a straight line a, and if A' is a point upon the same

or another straight line a' , then, upon a given side of A' on the straight line a' , we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment $A'B'$. We indicate this relation by writing $AB \cong A'B'$. Every segment is congruent to itself; that is, we always have $AB \cong AB$. We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in one and only one way.

DIAGRAM 3
ASDFSFSFD (AXIOM XXX)

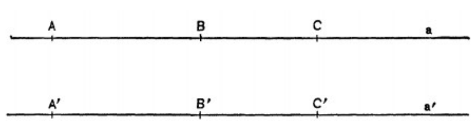


2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \cong A'B'$ and $AB \cong A''B''$, then $A'B' \cong A''B''$.

3. Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, we have $AC \cong A'C'$.

DEFINITIONS: xxx

DIAGRAM 4
ASDFSFSFD (AXIOM XXX)



4. Let an angle (h, k) be given in the plane α and let a straight line a' be given in a plane α'. Suppose also that, in the plane α', a definite side of the straight line a' be assigned. Denote by h' a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a'.

We express this relation by means of the notation $\measuredangle(h,k) \cong (h',k')$

Every angle is congruent to itself; that is, $\measuredangle(h,k) \cong (h,k)$ or $\measuredangle(h,k) \cong (k,h)$

5. If the angle (h,k) is congruent to the angle (h',k') and to the angle (h'',k''), then the angle (h',k') is congruent to the angle (h'',k''); that is to say, if $\measuredangle(h,k) \cong (h',k')$ and $\measuredangle(h,k) \cong (h'',k'')$, then $\measuredangle(h',k') \cong (h'',k'')$.

6. If, in the two triangles ABC and $A'B'C'$ the congruences $AB \cong A'B', AC \cong A'C', \measuredangle BAC \cong \measuredangle B'A'C'$ hold, then the congruences $\measuredangle ABC \cong \measuredangle A'B'C'$ and $\measuredangle ACB \cong \measuredangle A'C'B'$ also hold.

V. Continuity

1. Axiom of Archimedes.

Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points $A_2, A_3, A_4, ...$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments $AA_1, A_1A_2, A_2A_3, A_3A_4, ...$ be equal to one another. Then, among this series of points, there always exists a certain point An such that B lies between A and A_n.

2. Line completeness

To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

Hilbert (1899) included II-4, but R. L. Moore proved that this axiom is redundant, in 1902.

The Parallel and Continuity axioms are sometimes omitted, as not necessary for a "Hilbert plane" and neutral geometry. Nor are the Continuity axioms necessary for Euclidean geometry, if certain propositions such as Proposition 1 are sacrificed and Euclid's proofs for certain other propositions are replaced. (The present paper shows that III, 6 is also unnecessary, for Euclidean geometry).

BIRKHOFF'S AXIOMS http://en.wikipedia.org/wiki/Birkhoff%27s_axioms

Postulate I: Postulate of Line Measure.
A set of points $\{A, B, ...\}$ on any line can be put into a 1:1 correspondence with the real numbers $\{a, b, ...\}$ so that $|b - a| = d(A, B)$ for all points A and B.

Postulate II: Point-Line Postulate.
There is one and only one line, l, that contains any two given distinct points P and Q.

Postulate III: Postulate of Angle Measure.
A set of rays $\{\ell, m, n, ...\}$ through any point O can be put into 1:1 correspondence with the real numbers $a(mod 2\pi)$ so that if A and B are points (not equal to O) of ℓ and m, respectively, the difference $a_m - a_\ell (mod 2\pi)$ of the numbers associated with the lines ℓ and m is $\measuredangle AOB$. Furthermore, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV: Postulate of Similarity
Given two triangles ABC and $A'B'C'$ and some constant $k > 0, d(A', B') = kd(A, B), d(A', C') = kd(A, C)$ and $\measuredangle B'A'C' = \pm\measuredangle BAC$, then $d(B', C') = kd(B, C), \measuredangle C'B'A' = \pm\measuredangle CBA$, and $\measuredangle A'C'B' = \pm\measuredangle ACB$.

Birkhoff's Postulate I says that the Euclidean metric is to be used, I think. Thus, Birkhoff can prove the Pythagorean Theorem using his 4 postulates. And he can prove the parallel postulate.(See his 1932 paper.)

I am puzzled that he thinks he needs his Postulate IV, SAS. He doesn't prove independence of his axioms.

TARSKI'S AXIOMS (Tarski & Given [1999])

$\to, \leftrightarrow, \neg, \wedge, \vee. \bigwedge, \bigvee. \exists \forall$

Ax. 1. Reflexivity Axiom for Equidistance
$ab \equiv ba.$

Ax. 2. Transitivity Axiom for Equidistance
$ab \equiv pq \wedge ab \equiv rs \to pq \equiv rs.$

DIAGRAM 5
AX. 4. AXIOM OF SEGMENT CONSTRUCTION

DIAGRAM 6
AX. 5. FIVE-SEGMENT AXIOM

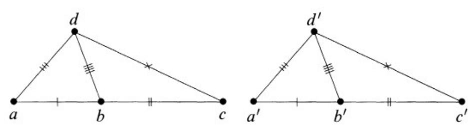
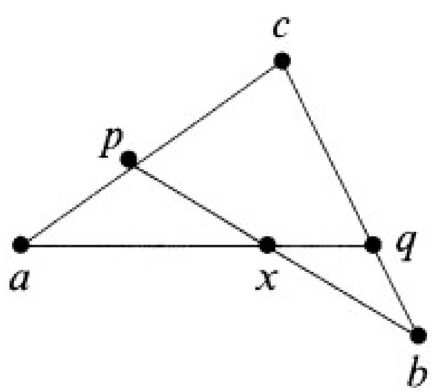


DIAGRAM 7
IDENTITY AXIOM FOR EQUIDISTANCE (AX 3)



Ax. 3. Identity Axiom for Equidistance
$ab \equiv cc \to a = b.$

Ax. 4. Axiom of Segment Construction
$\exists x(B(qax) \wedge ax \equiv be).$

Ax. 5. Five-Segment Axiom
$[a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd'] \to cd \equiv c'd'.$

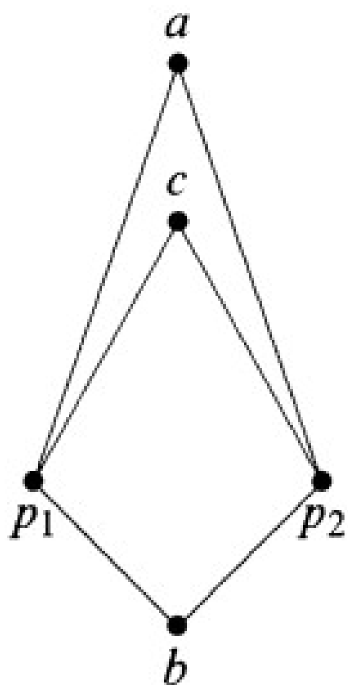
Ax. 6. Identity Axiom for Betweenness
$B(aba) \to a = b.$

Ax. 7. First (or Inner) form of the Pasch Axiom
$B(apc) \wedge B(bqc) \to \exists x[B(pxb) \wedge B(qxa)].$

Ax. 8(1. Lower 1-Dimensional Axiom

$\exists B \exists C(a \neq b)$.

DIAGRAM 8
IDENTITY AXIOM FOR
BETWEENNESS (AX 6)

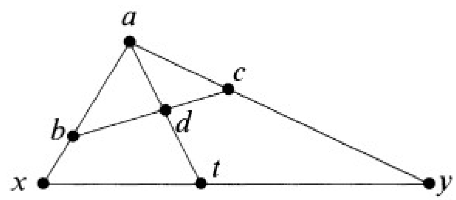


Ax. $9^{(0)}$. Upper 0-Dimensional Axiom
$a = b$.

Ax. 10. First Form of Euclid's Axiom
$B(adt) \wedge B(bdc) \wedge a \neq d \to \exists x \exists 3y[B(abx) \wedge B(acy) \wedge B(xty)]$.

Ax. 11. Axiom of Continuity
$\exists B \forall x \forall y[x \in X \wedge y \in Y \to B(axy)]$
$\to \exists b \forall x \forall y[x \in X \wedge y \in Y \to B(sby)]$.

DIAGRAM 9
UPPER O-DIMENSIONAL AXIOM
(AX9G)



Ax. 12. Reflexivity Axiom for Betweenness $B(abb)$.

DIAGRAM 10
AXIOM OF CONTINUITY

Ax. 13. $a = b \rightarrow B(aba)$.

DIAGRAM 11
TRANSITIVITY AXIOM FOR BETWEENNESS

Ax. 14. Symmetry Axiom for Betweenness $B(abc) \rightarrow B(cba)$.

DIAGRAM 12
SYMMETRY AXIOM FOR BETWEENNESS (AX 14)

Ax. 15. Inner Transitivity Axiom for Betweenness
$B(abd) \wedge B(bcd) \rightarrow B(abc)$.

Ax. 16. Outer Transitivity Axiom for Betweenness $B(abc) \wedge B(bcd) \wedge b \neq c \rightarrow B(abd)$.

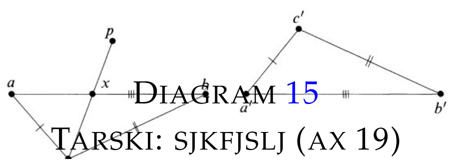
Ax. 17. Inner Connectivity Axiom for Betweenness $B(abd) \wedge B(acd) \rightarrow [B(abc) \vee$

$B(acb)]$.

DIAGRAM 13
XXX

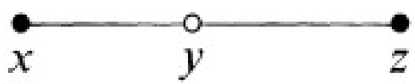
Ax. 18. Outer Connectivity Axiom for Betweenness
$B(abc) \wedge B(abd) \wedge a \neq b \rightarrow [B(acd) \vee B(adc)]$.

DIAGRAM 14
EXISTENCE AXIOM FOR TRIANGLE CONSTRUCTION



Ax. 19. $a = b \rightarrow ac \equiv bc$.

DIAGRAM 15
TARSKI: SJKFJSLJ (AX 19)



Ax. 20. Uniqueness Axiom for Triangle Construction
$[ac \equiv ac' \wedge bc \equiv bc' \wedge B(adb) \wedge B(ad'b) \wedge B(cdx) \wedge B(c'd'x)$
$\wedge\, d \neq x \wedge d' \neq x] \rightarrow c = c'.c$

DIAGRAM 16
UNIQUENESS AXIOM FOR TRIANGLE CONSTRUCTION (AX 20)

Ax. 21. Existence Axiom for Triangle Construction
$ab \equiv a'b' \rightarrow \exists c \exists x (ac \equiv a'c' \wedge bc \equiv b'c' \wedge B(cxp)$
$\wedge\, [B(abx) \vee B(bxa) \vee B(xab)]).c'$

Ax. 22. Density Axiom for Betweenness

$x \neq z \to \exists y[x \neq y \wedge z \neq y \wedge B(xyz)]$.

Ax. 23. $[B(xyz) \wedge B(x'y'z') \wedge xy \equiv x'y' \wedge yz \equiv y'z'] \to xz \equiv x'z'$.

Ax. 24.$B(xyz) \wedge B(x'y'z') \wedge xz \equiv xz' \wedge yz \equiv y'z' - \to xy \equiv x'y'$

Maybe I need to add other n-dimesnatinoal versions of some of these aximos. ALso, some come in variants.

VEBLEN'S AXIOMS

The axioms are twelve in number; they presuppose only the validity of the operations of logic and of counting (ordinal number).

Axiom I.
There exist at least two distinct points.

Axiom II.
ğğ If points A, B, C are in the order ABC, they are in the order CBA.

Axiom HI.
If points A, B, C are in the order ABC, they are not in the order BCA.

Axiom IV.
If points A, B, C are in the order ABC, then A is distinct from C.

Axiom V.
If A and B are any two distinct points, there exists a point C such that A, B, C are in the order ABC.

Def. 1.
The line AB (A + B) consists of A and B and all points X in one of the possible orders ABX, AXB, XAB. The points X in the order AXB constitute the segment AB. A and B are the end-points of the segment.

Axiom VI.
If points C and B (C + B) lie on the line AB, then A lies on the line CB.

Axiom VII.
If there exist three distinct points, there exist three points A, B, C not in any of the orders ABC, BCA, or CAB.

Def. 2.
Three distinct points not lying on the same line are the vertices of a triangle ABC, whose sides aré the segments AB, BC, CA, and whose boundary consists of its

vertices and the points of its sides.

Axiom VIII.
If three distinct points A , B, and C do not lie on the same line, and B and E are two points in the orders BCB and CFA, then a point F exists in the order AFB and such that B, E, F lie on the same line.

Def. 5.
A point O is in the interior of a triangle if it lies on a segment, the end-points of which are points of different sides of the triangle. The set of such points O is the interior of the triangle.

Def. 6.
If A, B, C form a triangle, the plane ABC consists of all points collinear with any two points of the sides of the triangle.

Axiom IX.
If there exist three points not lying in the same line, there exists a plane ABC such that there is a point B not lying in the plane ABC.

Def. 7.
If A, B, C, and B are four points not lying in the same plane, they form a tetrahedron ABCB whose faces are the interiors of the triangles ABC, BCB, CBA , BAB (if the triangles exist)* whose vertices are the four points, A, B, C, and D, and whose edges are the segments AB, BC, CB, BA, AC, BB. The points of faces, edges, and vertices con- stitute the surface of the tetrahedron.

Def. 8.
If A, B, C, B are the vertices of a tetrahedron, the space AB CD consists of all points collinear with any two points of the faces of the tetra- hedron.

Axiom X.
If there exist four points neither lying in the same line nor lying in the same plane, there exists a space AB CD such that there is no point E not collinear with two

points of the space, AB CD.

Axiom XI.
If there exists an infinitude of points, there exists a certain pair of points A C such that if [cr] * is any infinite set of segments of the line A C, having the property that each point which is A, C or a point of the segment A C is a point of a segment cr, then there is a finite subset cr1, a2, ęŢ Ţ, an with the same property.

Axiom XII.
If a is any line of any plane a there is some point C oí a through which there is not more than one line of the plane a which does not inter- sect a.

We can do Crude Geometry without either the Parallel Postulate or the Triangle Postulate. This will not allow propositions about angles. But we can still do interesting things. I think the effect of getting rid of the Triangle Postulate is to allow local changes in curvature and metric, as in my example earlier.

School Mathematics Study Group, Geometry. New Haven: Yale University Press, 1961.

Postulate 1. (Line Uniqueness) Given any two distinct points there is exactly one line that contains them.

Postulate 2. (Distance Postulate) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

Postulate 3. (Ruler Postulate) The points of a line can be placed in a correspondence with the real numbers such that:

To every point of the line there corresponds exactly one real number. To every real number there corresponds exactly one point of the line. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Postulate 4. (Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Postulate 5. (Existence of Points)

Every plane contains at least three non-collinear points. Space contains at least four non-coplanar points.

Postulate 6. (Points on a Line Lie in a Plane) If two points lie in a plane, then the line containing these points lies in the same plane.

Postulate 7. (Plane Uniqueness) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

Postulate 8. (Plane Intersection) If two planes intersect, then that intersection is a line.

Postulate 9. (Plane Separation Postulate) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:

each of the sets is convex; if P is in one set and Q is in the other, then segment PQ intersects the line.

Postulate 10. (Space Separation Postulate) The points of space that do not lie in a given plane form two sets such that:

Each of the sets is convex. If P is in one set and Q is in the other, then segment PQ intersects the plane.

Postulate 11. (Angle Measurement Postulate) To every angle there corresponds a real number between 0ř and 180ř.

Postulate 12. (Angle Construction Postulate) Let be a ray on the edge of the half-plane H. For every r between 0 and 180, there is exactly one with P in H such that $mPAB = r$.

Postulate 13. (Angle Addition Postulate)
If D is a point in the interior of ÐBAC, then $mBAC = mBAD + mDAC$.

Postulate 14. (Supplement Postulate)
If two angles form a linear pair, then they are supplementary.

Postulate 15. (SAS Postulate)
Given a one-to-one correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 16. (Parallel Postulate)
Through a given external point there is at most one line parallel to a given line.

Postulate 17. (Area of Polygonal Region)
To every polygonal region there corresponds a unique positive real number called

the area.

Postulate 18. (Area of Congruent Triangles)
If two triangles are congruent, then the triangular regions have the same area.

Postulate 19. (Summation of Areas of Regions)
Suppose that the region R is the union of two regions R1 and R2. If R1 and R2 intersect at most in a finite number of segments and points, then the area of R is the sum of the areas of R1 and R2.

Postulate 20. (Area of a Rectangle)
The area of a rectangle is the product of the length of its base and the length of its altitude.

Postulate 21. (Volume of Rectangular Parallelpiped)
The volume of a rectangular parallelpiped is equal to the product of the length of its altitude and the area of its base.

Postulate 22. (Cavalieri's Principle) Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane the two intersections determine regions that have the same area, then the two solids have the same volume.