Something Hilbert Got Wrong and Euclid Got Right: The Method of Superposition and the Side-Angle-Side Axiom in Propositions 1 to 4 of Book I of the Elements

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Abstract

Euclid's method of superposition can be made valid using his other methods. It really depends on the parallel postulate, because that is equivalent to imposing the Euclidean metric and defining the size of an angle clearly. As a result, Hilbert's and Birkhoff's Side-Angle-Side axioms are not independent of their other axioms– SAS can be proved using the Parallel Postulate. For Euclidean geometry, the Side-Angle-Side axiom is redundant, though for neutral geometry it is not.

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This paper: http://www.rasmusen.org/papers/euclid-rasmusen.pdf. The Appendices are in a separate file.

Keywords: xxx

I would like to thank xxx

Definition 9B
The *size of a rectilinear angle* **is the base of an isosceles triangle formed from that angle. Define a reference segment DE for measuring the size of angle $\angle BAC$. Extend or contract the segments AB and AC to length DE to create segment $AF{=}DE$ and $AG{=}DE$. The size of $\angle BAC$ is the segment FG.**

DROP: Postulate 4.
That all right angles equal one another.

DROP: Postulate 5. [(*Euclid's Parallel Postulate* 5 **)]**
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Postulate 5_2
Two angles are equal if and only if they have equal size.

Postulate 5B (The Intersection Postulate)
Let a circle's center be A and its radius equal AB. If and only if a figure contains a point whose distance to A is less than AB and a point whose distance is greater than AB then it intersects with the circle, containing a point whose distance is exactly AB.

PROPOSITIONS

Proposition I-0B.
If and only if the distance between the centers A and B of two circles is no less than the radius of either but no greater than the sum of their radii then the two circles have at least one point in common.

Proposition 3^2: To cut off from the greater of two given unequal straight lines a straight line equal to the less; or to extend the lesser to be equal to the greater.

Prop I-20 (the triangle inequality): (same proposition, new place in the ordering)

Proposition I-23 (reproducing an angle): (same proposition, new place in the ordering)

Proposition $3B$: If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base.

1. Introduction

Hilbert set out in his Groundwork for xxx (Grundlagen) to fix up Euclid. But if we confine ourselves to Euclidean geometry, there is one issue on which Euclid wins. Euclid uses the "method of superposition," which really amounts to handwaving, to prove his proposition I-4, that if two triangles have two sides and the angle between them of the same size, then the remaining angles and side will be the same too. Hilbert, re-axiomatizing geometry in his *Grundlagen*, makes a slightly weakened form of proposition I-4 (that side-angle-side =ruence implies the other two angles also match) one of his 23 axioms. Birkhoff does the same in his re-axiomatization, which assumes the geometry is in a metric space and so only requires four further axioms— one of which is Side-Angle-Side. But Hilbert and Birkhoff assume more than we need if the task is to re-axiomatize Euclid rather than axiomatize all possible geometries. It is true, as Hilbert shows, that Side-Angle-Side cannot be proven using his other congruence axioms. It is false, however, that Side-Angle-Side cannot be proven using his other axioms in their entirety. What we need for the proof is the celebrated Parallel postulate, as I will show below. Hilbert includes the Parallel postulate in his axioms, even though it is an axiom that he knows he will want to drop for many geometries. His disproof of Side-Angle-Side's provability violates the Parallel postulate by using a non-Euclidean metric, as I will explain below. Hilbert aimed to make the other 22 postulates stand independent of the parallel postulate, and to do so requires making Side-Angle-Side an axiom. In Euclidean geometry, we are allowed the Parallel postulate, and so we don't need Side-Angle Side.

Here are four points I will make in this paper.

1. Hilbert does not need all 20 of his axioms for Euclidean geometry. One is superfluous (besides the superfluous one found in 1902 that reduced his 21 to 20). For a similar reason, Bischoff can dispense with one of the 4 axioms for the Euclidean geometry he proposes on top of assuming a metric space.

2. It is generally thought that Euclid does not need the Parallel postulate until

Proposition I-29, when he first invokes it. To the contrary, his proofs of earlier theorem implicitly rely on the Parallel postulate, even if the propositions happen to be true even without it. (Joyce says the first 15 propositions are true in elliptic geometry and the first 28 in hyperbolic.)

3. Of the many equivalent statements of the parallel postulate, the one that gets to its essence is that the metric does not change with location, not the Playfair Postulate (which is the one that states it as that two parallels never meet). The Euclidean metric is sufficient both not necessary. What is necessary and sufficient, I think, is that the metric not change depending on location; it can be purely ordinal, for example. The truest to the spirit of Euclid would be an axiom which does not rely on the idea of a metric or testing whether something happens at infinite distances. An example is that any point can be part of a Diagram with four right angles.

4. Euclid's much-criticized Method of Superposition, with which he proves Proposition I-4 and which is therefore the foundation of most of what comes later, was vaguely described but proposition I-4 can be proved constructively anyway.

Before going further it may be useful to try to explain how I, an obscurity, could find a mistake in well-known and much-studied works of famous mathematicians Hilbert and Birkhoff. First, note that this would not be the first mistake to be found in Hilbert's list of axioms. In 1902, three years after the *Grundlagen*, Prof. Moore found that one of its axioms was redundant. So Hilbert was perhaps not at his most careful in the Grundlagen.

In the case of SAS, here is what I think happened. Before the discovery of non-Euclidean geometry, nobody tried to increase the number of axioms in Euclidean geometry. Instead, much attention was given to trying to reduce the number of axioms, by proving the parallel postulate using the other axioms. After non-Euclidean geometry was discovered, as a consequence of dropping the parallel postulate or replacing it with a different postulate, mathematicians did turn their minds to axiomatizing geometry. Hilbert did not take as his task the problem of axiomatizing Euclidean geometry, however. Rather, his task was to assemble a minimal number of axioms with which to prove the propositions of neutral geometry— geometry without the parallel postulate— and then to add the parallel postulate or a substitute for it to make Euclidean geometry and hyperbolic geometry into special cases. With his attention on the axiomatization of neutral geometry, Hilbert a did not notice that the parallel postulate could replace other axioms if axiomatization was restricted to the special case of Euclidean geometry.

I find Birkhoff's mistake harder to understand. His Postulate 1 assumes the Euclidean metric, and thus substitutes for the Parallel Postulate. Why, then, does he think he needs his Postulate 4, Side-Angle-Side? I do not know. He makes no attempt to prove the independence of his axioms, however.

2. The Various Postulates

Euclid's *Elements* was the most important textbook in history in any field, and also important as a treatise on geometry. Euclid starts with definitions, postulates, and "common notions," and demonstrates propositions from them by logical argument and diagrammatic construction. He shocks the novice by telling him we must prove self-evident propositions such as the triangle inequality, that two sides of a triangle add up to more than the remaining side. There are rough spots in the Elements immediately, however. Proposition I-1, Book I implicitly assumes that two overlapping circles intersect. That is perhaps forgiveable— I certainly didn't spot it on my own— but proposition I-4 is more troubling. If the reader skims the proof for logical steps, he will see two gaps.

Proposition I-4.
If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proof:
Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF respectively, namely AB equal to DE and AC equal to DF, and the angle BAC equal to the angle EDF.

I say that the base BC also equals the base EF, [ADD THE QUESTIONALBE STUFF HERE] the triangle ABC equals the triangle DEF, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle ABC equals the angle DEF, and the angle ACB equals the angle DFE.

If the triangle ABC is superposed on the triangle DEF, and if the point A is placed on the point D and the straight line AB on DE, then the point B also coincides with E, because AB equals DE.

Again, AB coinciding with DE, the straight line AC also coincides with DF, **because the angle BAC equals the angle EDF.**

Hence the point C also coincides with the point F, because AC again equals DF.

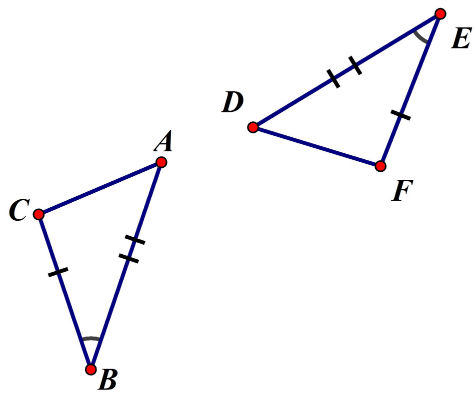
But B also coincides with E, hence the base BC coincides with the base EF and equals it.

Thus the whole triangle ABC coincides with the whole triangle DEF and equals it.

And the remaining angles also coincide with the remaining angles and equal them, the angle ABC equals the angle DEF, and the angle ACB equals the angle DFE.

Therefore if two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. (Euclid, Elements, from Joyce)

DIAGRAM 1
SIDE-ANGLE-SIDE (PROP I-4)



I will start with the second gap, about angles. Euclid has previously defined "angle," but not the size of an angle, or how to tell if one angle equals another.

Definition 8.
A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

This definition can be tightened up without much trouble, as I will do later and many others did long ago. It is the kind of fuzziness one would expect to find in an ancient mathematician.

The other gap, however, has been criticized for centuries, and seems to have made Euclid himself uneasy. What's this about "If the triangle ABC is superposed on the triangle DEF, and if the point A is placed on the point D and the straight line AB on DE"? After carefully using compass and straightedge to physically construct the most trivially obvious diagrams in propositions I-1, I-2, and I-3, now Euclid just says to slap down one triangle on top of another so it fits right. This is known as Euclid's "method of Ssuperposition". It is based on common notion 3.

Common Notion 3.
Things which coincide with one another equal one another.

That's fine, though I think that insofar as it is meaningful it is equivalent (as will be explained later) to postulate 5, the parallel postulate. But Euclid hasn't shown that the two triangles coincide. What is needed is a careful transfer of one triangle to atop another using straightedge and compass— which I will provide later.

Instead, Euclid relies on the Method of Superposition. He uses it for only two propositions, I-4 and I-8, which has led scholars to think that he was not proud of the method. Unfortunately, proposition I-4 itself is foundational for the rest of the *Elements*. As diagram 2, taken from Dodgson (1885) shows, it is necessary for every proposition in Book I except I-1, 2, 3, and 22.[1]

[2]

> "The fourth proposition is a tissue of nonsense. Superposition is a logically worthless device; for if our triangles are spatial, not material, there is a logical contradiction in the notion of moving them, while if they are material, they cannot be perfectly rigid, and when superposed they are certain to be slightly deformed from the shape they had before." Russell, Bertrand (1902)"The Teaching of Euclid, The Mathematical Gazette 2 (33) (1902), 165-167, http://www-groups.dcs.st-and.ac.uk/~history/Extras/Russell_Euclid.html.

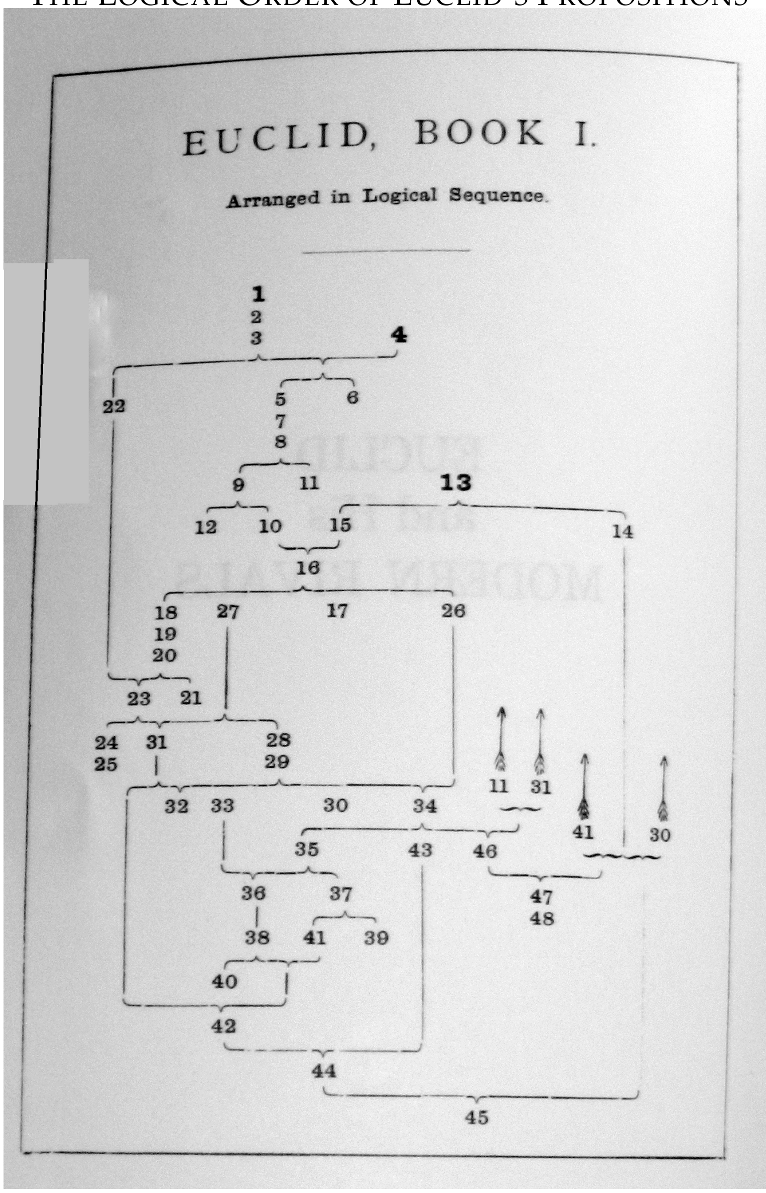
The Literature on Fixing Up Euclid

Hilbert, Birkhoff, and Tarski have produced re-axiomatizations of Euclid's Elements that are each useful in their own way. Hilbert used 21 axioms, one of which was soon discovered to be redundant (Halsted xxx)). Birkhoff assumed the properties of a metric space and added four other axioms. (I think he uses compass and ruler, so he really is not Euclidean— he can square the circle and trisect the angle). SMSG is a simpler version of Birkhoff. Tarski came up with a formulation in first-order logic.

[1]Dodgson (1885) notes that a number of 19th-century writers of elementary geometry texts were not so fastidious, and made heavy use of the moethod of superposition.

[2]xxx Heath, p. 225 ,of gth Metho fo Super: "Nor as a matter of fact do we find in the ancient geometers any expression of doubt as to the legitimacy of the method."

DIAGRAM 2
THE LOGICAL ORDER OF EUCLID'S PROPOSITIONS



The parallel postulate has received a lot of attention. After many fruitless attempts to prove it, mathematicians started assuming its falsity and invented the fields of hyperbolic and elliptic geometry. Neutral (or absolute) geometry refers to those propositions which can be generated without any version of the parallel postulate.

For more than a thousand years, students have learned mathematics from Euclid's elements. Its importance is not so much in the mathematical results as in the idea of the proof from first principles. It is therefore dismaying that even in the limited domain of circles and rectilinear Diagrams in planes Euclid fails to be rigorous. Or, rather, he is rigorous in proving some obvious propositions but loosely assumes the obvious in proving other propositions. It is worth patching up Euclid for two reasons. One is to show how to construct an axiomatic system to prove his propositions. The other is to show how his method can be used rigorously by students learning proofs. The first of these objectives has long ago been tackled, most famously in Hilbert's set of axioms. Hilbert, however, tried to escape from the diagrammatic style which produces the essential flavor of Euclid's geometry.

Recently a literature has sprung up trying to axiomatize that diagrammatic approach, keeping the diagrams but adding rigorous rules for drawing them (Barwise, Manders, Mumma, Miller). The systems of Mumma and Miller are valuable, but although they provide foundations for Euclid's approach, they complexify it greatly and they depart from the stark style of Euclid.

We have a more modest objective: to patch up some of the more obvious and grating problems in Euclid. This allows at least a rigorous introduction to Euclid, and it will not require changes to Euclid's essential style. Specifically, we will address three problems. (1) Euclid does not define the size of an angle (except for right angles), which leaves it unclear when two angles can be of the same size. (2) Euclid lacks a postulate saying when figures intersect— his only postulate dealing with intersections is the famous postulate 5 about the intersection of non-parallel lines. (3) Very early on, in proposition I-4 (the foundation for many later propositions), Euclid uses the "method of superposition," in a key proof about the =ruence of triangles (that Side-Angle-Side defines a triangle).

We argue that the method of superposition is unnecessary, provided that we clean up the definition of an angle and add the circle intersection postulate necessary to prove proposition I-1. The method reaches valid results; the problem is that it skips steps. Rather than superimposing figure DEF on figure ABC directly, we will construct a duplicate GHI of DEF on top of ABC with straightedge and compass, so that the proof is to show that GHI equals DEF and GHI also equals ABC so DEF equals ABC. To do the construction, we will need to build an angle of size equal to an existing angle but in a different location, To build such an angle, we will need to use the intersection of circles.

4. A Minimal Fix-Up of Euclid: Definitions and Postulates

4. The Changes

We will accept Euclid's xxx Book I definitions, his 5 postulates, and his 5 "common notions," with certain changes and additions. I will follow the suggestion of Dodgson (1885, p. 12) and number new definitions, postulates, and propositions by adding capital letters starting with B to the numbers of the statements of Euclid that precede them. I will denote new versions of existing statements in the same style as Tarski does for his variant axioms, by adding superscripts starting with 2, e.g. there will be existing postulate 5 and new postulates 5_2, and 5_3.

Missing Definitions

Euclid does not actually use the term "radius". He uses xxxx (from Joyce or Heath). He does omit to define several terms used in the postulates.

Definition X.
The *radius* of a circle is xxx

Definition X.
interior angle

Definition X.
side

We will not not use these terms in our postulates.

I should define INSIDE of a circle and OUTSIDE.

Note that Euclid really is a lot more complicate than it looks, because it has a lot of the axioms in the implicit definitions. Cite Tarski and GIven on simplicity in axiomatic systems.

The Definition of Lines

Definition 1.
A point is that which has no part.
Definition 2.
A line is breadthless length.
Definition 2B. A line is a set of points such that

II-1 **If a point B is between points A and C, B is also between C and A, and there exists a line containing the points A, B, C.**

II-2 **For any two points on a line, there exists at least one point lying between them.**

II-3 **Of any three points A, B, C situated on a line, one of the three and only one lies between the other two.**

I take these three characteristics from Hilbert's Grundlagen, where variants of them are in his second set of three axioms, the Axioms of Betweenness. Why have all three? II-1 says that order doesn't depend on whether you read from right to left or left to right. II-2 says that the points in the collection are dense:there are no holes in the line. II-3 says that the ordering is a full ordering, not a partial ordering: there are no ties.

Note that we have not included any characteristics that require a line to be straight. A "line" for Euclid could be a curve.

These two characteristics of a straight line are inspired by Hilbert's Axioms of Order. I-1 says that only one of the many lines connecting A and B is a straight line. I-2 says that each closed subset of a straight line's points is also a straight line.

Definition 3.
The ends of a line are points.

Definition 3B. An end point of a line is a point which is not between two other points on it.

Note that definition 3B, like the others, is just a definition and does not say that the thing defined actually exists. End points may well not exist— we allow for lines to be open sets, lacking boundaries. That does not require us to use the real number system, which we will not do, because it is true even for rational numbers that between any two points there exists an infinite number of other points.

DIAGRAM 3
DEFINING A LINE (DEF 2B))

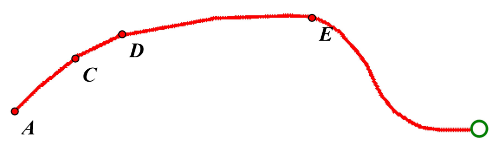
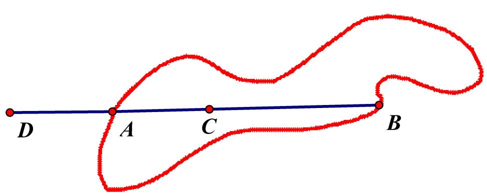


DIAGRAM 4
DEFINING A STRAIGHT LINE (DEF 4B))



Definition 4.
A straight line is a line which lies evenly with the points on itself.

Definition 4B. A *straight line* is a line such that

I-1 Taking any two points, one and only one line has them as boundary points.

I-2 Any two points in a straight line are the endpoints of some straight line.

Definition 4C. A *continuation of a straight line* is a straight line that contains all of that line and more points besides.

We will need straight lines to exist, and for them to allows be continuable. Postulates 1 and 2B will take care of that.

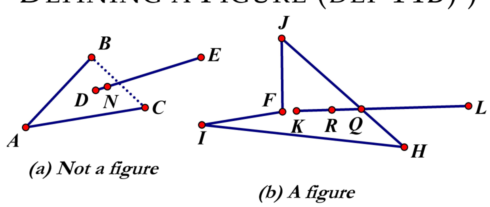
Definitions of Boundaries

Definition 13.
A boundary is that which is an extremity of anything.

Definition 13B.
A boundary of a collection of points consists of certain end points of lines entirely in the collection, such that if the lines are continued beyond those end points they will no longer be entirely in the collection.

Definition 14.
A figure is that which is contained by any boundary or boundaries.

DIAGRAM 5
DEFINING A FIGURE (DEF 14B))



Definition 14B.
A figure is a collection of points such that any line that includes any of those points will, if continued long enough in either direction include a boundary point.

Euclid omits to define "side," and this is the natural place to put the definition.

Definition 14C.
A *side* of a figure is a straight line which is part of its boundary.

Equality

We will need to define ”equal” for definition 15 below to make sense. Here is a definition

DEFINITION 14D: Two segments are equal if they can both be radii of the same circle. Two circles are equal if their radii are equal. Two figures are equal if the lines and circles that compose them are equal and intersect in the same way.

Definition 14D essentially says that the compass defines which segments are equal. That is proper, since the compass is really just a line-transferring device, as Hilbert says. [cite].

But I don't have a circle defined yet! I need to fix this up somehow.

DEFINITION 14D2: two figures are equal if there is a one-to-one mapping between them (does that preserve distance? No, I think. Hmmm.) PASCH

The usual definition is based on numbers, and so we can't use it:

Definition $14D_3$
The binary relation *equals* between two of any three magnitudes x, y and z says that if $x = y$, the following three things are true:

Reflexivity: For each x, x = x.

Symmetry: If x = y, then y = x.

Transitivity: If x = y and y = z, then x = z.

Can I use this definition where x and y are not magnitudes, but lines? Actually, this does not seem strong enough. Lines have have more dimensions, more features, than numbers do. So this is not strong enough.

Definition 15.
A circle is a plane figure contained by one line such that all the straight lines falling

upon it from one point among those lying within the figure equal one another.

The Definition of Angle Size

The existing definitions 8 and 9 are:

Definition 8.
A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

and

Definition 9.
And when the lines containing the angle are straight, the angle is called *rectilinear.*

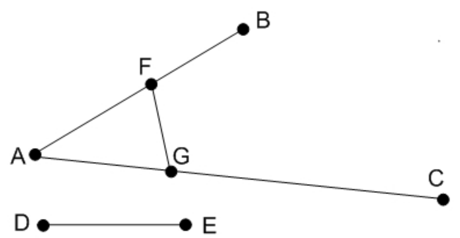
From later definitions and postulate 4 we see that Euclid takes the right angle as his reference angle, so we may say that $1\ rightangle = 90\ degrees = \frac{\pi}{2}\ radians$. We could go further and say that angles can be added and subtracted like numbers, so adding two right angles we get 2 rightangles = 180 degrees but subtracting one right angle from another we get 0 rightangles= 0 degrees. This has two problems. (1) This mapping of angles to numbers only allows angles to be measured that can be constructed by a finite number of operations on a right angle, thus losing a constructive mapping for angles of irrational size. (2) This mapping of angles to numbers does not link angle size to segment size, which is important if we are to analyze what happens to equal figures on the non-flat surfaces of non-Euclidean geometries.

A new definition that avoids these problems is 9B:

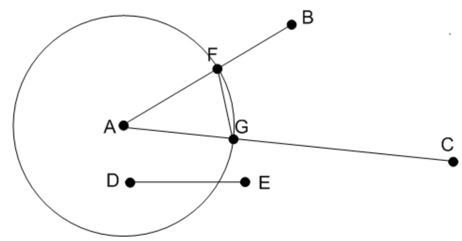
Definition 9B
The *size of a rectilinear angle* is the base of an isosceles triangle formed from that angle. Define a reference segment DE for measuring the size of angle $\angle BAC$. Extend or contract the segments AB and AC to length DE to create segment $AF{=}DE$ and $AG{=}DE$. The size of $\angle BAC$ is the segment FG.

DIAGRAM 6

THE SIZE OF AN ANGLE (DEFINITIONS 9B AND 9C)



(a) Definition 9B



(b) Definition 9C

The size of an angle can be found from the law of cosines:

$$s_1^2 = s_2^2 + s_3^2 - 2s_2s_3cos(angle\ facing\ s_1) \tag{1}$$

If we take the reference segment to equal one then $s_2 \equiv s_3 = 1$. Define the function $angle(x) \equiv x$ and we can rewrite the angle's size as

$$s_1 = \sqrt{2 - 2cos(angle(s_1))} = \sqrt{2 - 2\frac{1}{1 + s_4}}, \tag{2}$$

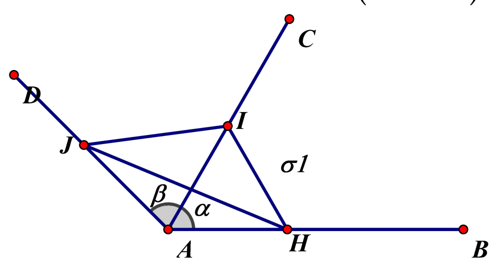
where s_4 is the distance that needs to be added to 1 to get the length of the hypotenuse of the right triangle formed by dropping a perpendicular from the extension of s_2 down to s_3. That doesn't seem very useful.

Angle size thus defined is ordinal, not cardinal. It is a concave function of the angle, so although $\angle\alpha + \angle\beta = \angle(\alpha + \beta)$, $size(\angle\alpha) + size(\angle\beta) > size(\angle(\alpha + \beta))$. This is illustrated in diagram 7, where $\angle BAC + \angle CAD = \angle BAD$ but $IH + JI > JH$ by the triangle inequality.

Heath tells us (p. 228) that Veronese (1891) came up with a postulate that is essentially the same definition as 9B:

Definition $9B_2$
Let AB, AC, and $A'B', A'C'$ **be two pairs of straight lines intersecting at** A, A'**, and let there be determined upon them the congruent segments** $AB, A'B'$ **and the congruent segments** $AC, A'C'$**;**
then, if $BC, B'C'$ **are congruent, the two** *pairs of straight lines* **are congruent.**

DIAGRAM 7
ADDING ANGLE SIZES (DEF 9B)



Questioni riguardanti le Matematiche Elementari, 1900, Bologna Vol. 1, parte 8 F Enriques - VI, Zanichelli,(ristampa 1983)

If you want a definition such that when two angles are combined, the size of the combined angle is the sum of the sizes of each angle, definition 9C, based on arc length, will do:

Definition 9C
The *additive size of a rectilinear angle* **BAC is defined relative to some reference line DE. Extend or contract the lines AB and AC to length DE to create lines AF and AG. Draw a circle centered at A with radius DE. The additive size of the angle BAC is the length of that part of the circle lying between F and G.**

There is a one-to-one mapping between the size of an angle and its additive size. We do not need the additive size of an angle for Book I of the *Elements*. It would useful for talking about the length of arcs. For there to be a difference in usefulness between the size and additive size we must be able to measure the length of an arc. We can do that fairly easily by bisecting some reference angle— say, the right angle— and continuing to bisect to form smaller units of measurement, but that only gets us a countable number of angle sizes, a number that does not even include all the rational angle sizes since we cannot trisect an angle in Euclidean geometry. The conventional definition of arc length takes the limit of linear approximations, which is quite contrary to the spirit of Euclidean geometry.

A similar way to get a definition for cardinal angle size is simply to start with that property and use outside knowledge. Associate with each angle a real number. Define size(1 rightangle)=1 and size (2 rightangles) =2. Then all the angles of Euclid (that is, from 0 to 180 degrees) will have additive sizes between 0 and 2. This is much like Birkhoff's method for axiomatizing Euclidean geometry by associating segment lengths with the real numbers. It suffers from the same drawback: it is contrary to the Euclidean spirit of constructive geometry and is concealing much complexity behind a seemingly brief postulate.

> **Definition** $9C_2$
> **The** *additive size of a rectilinear angle* **BAC is defined relative to some reference segment DE. Extend or contract segment AB to length DE to create segment AF. Extend AC to some AG sufficiently long (picking G so that** $AG = AC + 2DE$ **will do the trick). Drop a perpendicular from F to AG and call the intersection H. The additive size of angle BAC is FH.**

Definition $9C_2$ is constructive and does not need any new definitions, unlike definition 9C. It is more elaborate than definition 9B, however. Worse, $9C_2$ requires postulate 5B to guarantee existence of the intersection H, and the construction of a perpendicular, not shown until proposition xxx. Still worse, proposition xxx requires proposition 4 for its proof, and proposition 4 requires definition $9C_2$. Nonetheless, if some other definition could be used to get to proposition xxx, $9C_2$ would be attractive thereafter, since it is additive and since by the definition of the sine function, if we normalize by setting the length of the reference segment equal to 1, this definition yields

$$s(angle) = sin(angle) \tag{3}$$

for angles smaller than right angles, and

$$s(angle) = 1 + sin(angle) \tag{4}$$

for angles bigger than right angles.

One can achieve the same result in a different way, closer to the spirit of Euclid though still somewhat modern. Start with definition 9B, the constructive ordinal definition, $s(angle)$. As explained above, $s(angle) = \sqrt{2 - 2cos(angle)}$. Since this is an ordinal definition, any ordinal transformation $f(s)$ of $s(angle)$ will retain its essential property of ordering angles. Thus, let us choose an ordinal transformation with the cardinal properties we desire. Specifically, we want

$$s^*(angle(s)) = f(s(angle)) = angle \tag{5}$$

where we want $s^*(1\ rightangle) = 1$, $s^*(2\ rightangles) = 2$ for two right angles added together, and $s^*(0rightangles) = 0$ for one right angle subtracted from another.

If we let $f(s) \equiv s/\sqrt{2 - 2cos(angle(s))}$, then $s^*(angle) = angle$.

The mapping $s^*(angle)$ takes us back to Euclid's mapping, but now our mapping is constructive. We start by constructing $s(angle)$. We then use mapping $f()$ to map s(angle) to $s^*(angle)$. This could, I think, even be done constructively, though that would be tedious. We just need the cosine and square root functions, both of which can be made constructive because they can be constructed via the Pythagorean theorem. Both functions can map not just rational lines but irrational lines.

I don't know if I'm right here. Transcendental numbers and such may be involved.

The First Three Postulates

The first three postulates use terms defined outside of the system. I will not try to fix that, only to fix features needed later in the *Elements*.

I will not change the first and third postulates. They correspond to allowing the use of an unmarkable straightedge and a collapsing compass to construct figures. Even though our diagrams are mere physical representations and not exact, they are postulated to be exact for the analysis.

Postulate 1.
To draw a straight line from any point to any point.

It has been suggested that postulate 1 be amended to say that there is only one straight lines between any two points. I have snuck that in with Definition 2B, which makes uniqueness one of the defining properties of a line.

Postulate 2.
To produce a finite straight line continuously in a straight line.

Postulate 2 says that we can always make a continuation of a straight line. It does not say that infinite continuations exist, only that we can always construct a longer line.

Using 2 and the uniqueness of a line I can use Heath's proof that two segmetns cannot have a common segement (because if AC and BC overlap in segment DC, then extending DC would yield two straight lines, not one).

Can I prove that a segment does not intersect with itself?

Postulate 3.
To describe a circle with any center and radius.

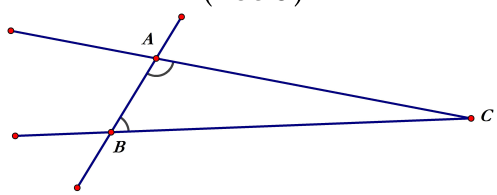
The Angle Parallel Postulate

Euclid defines parallels thus:

Definition 23
Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

I drop

DIAGRAM 30
EUCLID'S PARALLEL POSTULATE
(POS 5)



Postulate 4.
That all right angles equal one another.

and

Postulate 5. [(*Euclid's Parallel Postulate* 5 **)]**
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

You might think that part of postulate 5 should be its converse; that is, the first "if" should be "if and only if",or, put another way, Euclid should have added

... but if a straight line falling on two straight lines makes the interior angles on the same side equal to two right angles, the two straight lines, if produced indefinitely, never meet.

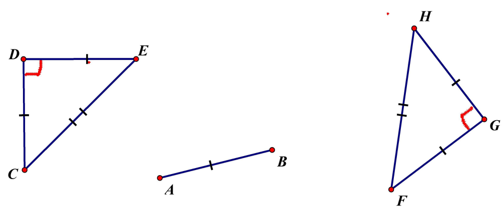
In fact, Euclid does not need to include the converse in postulate 5, because he can prove it as a proposition. He does that via proposition I-27, which concerns the properties of the angles made by a line that intersects two parallel lines, and proposition I-31, which shows how to construct a line parallel to an existing line.

probably pos 5B can be weakened to just match right angles, not all angles.

Add postulate 5_2, a replacement for postulates 4 and 5:

Postulate 5_2
Two angles are equal if and only if they have equal size.

DIAGRAM 9
THE ANGLE PARALLEL POSTULATE (POS 5_2)

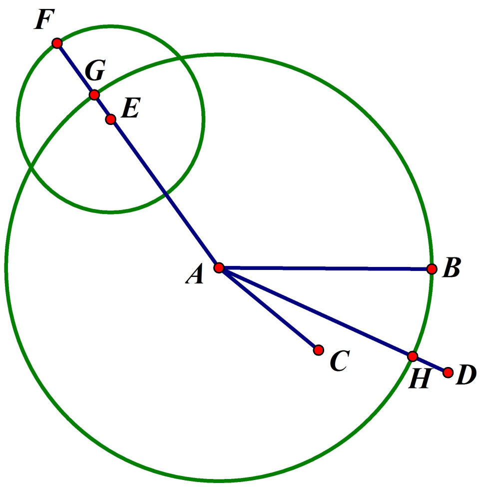


What is needed is not an axiom that a segment's length is the same everywhere on the plane (as one might think, and as one might do starting with a metric approach). Propositions 1, 2, and 3 and their proofs do not depend on that, though how one draws the figures on a Euclidean plane, with zero curvature, does depend on that. What is crucial is the relation between segments and lines, between equality of segments and equality of angles. The basic measurement unit for angles is the right angle, defined as the angle such that two right angles equal the "flat" angle produced by a single segment. The basic measurement unit for a segment is the radius of a circle. [explain better]. That is because the way segments are compared is by using AB as the radius of a circle and seeing if the successive replication of the circle centered at different points leads to CD being the radius of the same (except for where it is centered)) circle. What definition 9B and the angle parallel postulate 5_2 do is to constructively relate angle equality to segment equality. Euclid's parallel postulate 5 and its equivalents discussed in a later section do the same thing non-constructively and opaquely.

The Intersection Postulate

Postulate 5B (the intersection postulate)
Let a circle's center be A and its radius equal AB. If and only if a figure contains a point whose distance to A is less than AB and a point whose distance is greater than AB then it intersects with the circle, containing a point whose distance is exactly AB.

DIAGRAM 10
THE INTERSECTION POSTULATE
(POS 5B)



For example, in diagram 10, segment CD contains point E, and circle FG contains point H.

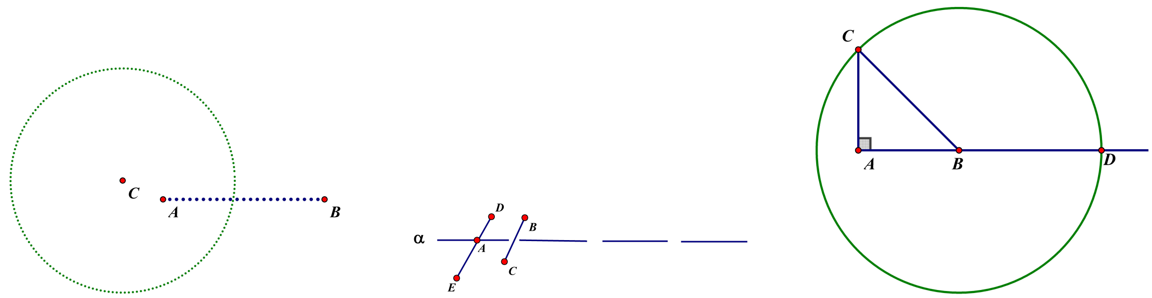
The analog of postulate 5B is usually called a continuity postulate, but since the only point for us is to establish when intersections occur, I think "intersection postulate" is more precise. The axiom does not rule out constructions with discontinuities; it really is saying something about the plane we are working in, not the figures we draw on it. The plane must not have any holes, so a convex figure drawn on it will intersect with any other convex figure it crosses.

Postulate 5B is needed to avoid the problems in diagram 11, where the two figures cross but do not intersect. In diagram 11(a), points are only defined on the dots, and the crossing happens to be where no dots occur, so the circle and the segment do not intersect. In diagram 11(b), points are defined almost everywhere (in the formal sense), but not at integer values of the reference segment. Thus, segment BC does not intersect the long horizontal segment α, but segment DE does intersect α. In Diagram 11(b), points are defined over the field of rational numbers, setting the reference segment AB as unity. Segment AC is drawn so $AB{=}AC$ and AC is perpendicular to AC, in which case the size of BC, the hypotenuse, is $\sqrt{2}AB$. The circle centered at B with radius BC therefore crosses the segment AE at D, which is not a point since $AB + BD$ is irrational.

Maybe define "inside a circle" and "outside a circle" using distance to the center.

DIAGRAM 11

The Continuity Postulate (Postulate 5B)



(a) (b) (c)

Note that postulate 5B implies the intersection not just of a circle and another figure, but also of any two figures that "overlap", including two overlapping segments, since we can draw a circle based on one of the segments so as to be able to apply postulate 5B. (xxx illustrate in teh diagram).

Euclid's only mention of intersection is in postulate 5, which we have dropped. Since we have postulate 5B, we don't need postulate 5 for intersections.

It is well-known that some postulate such as postulate 5B is needed to prove proposition I-1. This is similar to Dedekind's axiom, which is part of Hilbert's group V of postulates for continuity. Here is the form Heath (p. 236) gives us, citing Dedekind (1872/1905) p.11):

> **Dedekind's Axiom**
> **If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class there exists one and only one point which produces this division of all the points into two classes, this division of the straight line into two parts.**

Postulate 5B addresses the need for an intersection axiom more simply and directly. Note too that postulate 5B does not need the terms "'left" and "right" or "outside" and "inside".

Postulate 5B will be used to prove the new proposition I-0B (two overlapping circles intersect), which is necessary to prove the triangle inequality (prop I-20) and proposition I-1. It will guarantee that there is an intersection of the two circles in that proof.

Euclidean numbers or constructible numbers are the rationals plus those you can get from quadratic expressions (square root based).

5. A Minimal Fix-Up of Euclid: Propositions

Now we are equipped to tackle the propositions. Our major goal is to prove proposition I-4 properly. On the way to that goal we will prove the new proposition I-0B, fix the proof of proposition I-1, fix the statement of proposition I-3, prove the new propositions I-3B and I-20B, and bring forward the ordering of propositions I-20 and I-23. For completeness, we will include proposition I-2 even though it will not need changing.

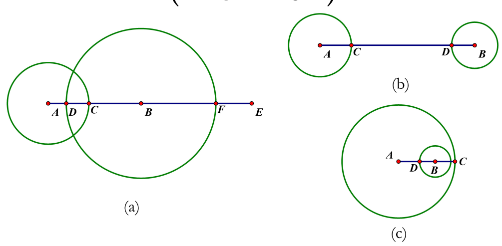
Now I will explain the changes in the Propositions.

> **Proposition I-0B.**
> If and only if the distance between the centers A and B of two circles is no less than the radius of either but no greater than the sum of their radii then the two circles have at least one point in common.

Proof.
Draw the segment AB (pos 1). Cut AB down or extend it up to create segment AC, where AC is equal to the radius of the circle centered at A. Cut AB down or extend it up to create segment BD, where BD is equal to the radius of the circle centered at B. Extend AB beyond B to some point E which is further than D from B.

Oops– I do not have Prop 3 yet, for extending a line. Do soemthing about that in the proof.

DIAGRAM 12
WHEN TWO CIRCLES INTERSECT
(PROP I-0B)



By hypothesis, we have $AB \succ AC$, $AB \succnsim BD$, and $AB \precnsim AC\widetilde{+}BD$.

Since $AB \succ AC$ and$AB \precnsim AC\widetilde{+}BD$. it follows that $AD \succnsim AC$. Since $AB \succnsim BD$ and $AB \precnsim AC\widetilde{+}BD$. it follows that $AD \succnsim AC$. Thus, the circle centered on B contains a point D which is closer to A than the radius AC of the A-circle.

Consider the segment BE. It contains the point B which is closer to B than the radius BD of the B-circle. It also contains the point E which by construction is further than BD from B. Thus, BE also contains a point F which is also part of the B-circle (pos 5B).

The B-circle contains the point C which is less than AD from the center of the A-circle. The B-circle also contains the point F which is more than AD from the center of the A-circle since $AF > AB > AD$. Thus, the B-circle must contain some point in common with the A-circle (pos 5B).

Now do the converse, for when there is no point of intersection.sdfsfdfdsfds

Q.E.D.

Heath (p. 238) proves the following using Dedekind's Axiom:

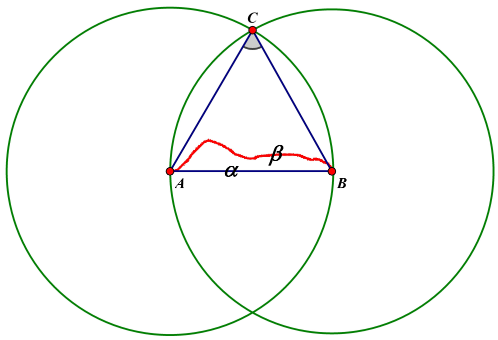
Heath's Circle Proposition:
If in a given plane a circle C has one point X inside and one point Y outside another circle C', the two circles intersect at two points.

Heath's proposition is stronger than Proposition I-0a in that it states that there are exactly two points of intersection, but weaker in that it does not say when the circles will overlap or what happens if they do not overlap. It requires "inside" and "outside" to be defined. It requires some 25 paragraphs of proof plus 5 diagrams.

We will need proposition I-0B to prove propositions I-1 and I-20.

Prop I-1C:
Two given points form the end points of only one segment.

DIAGRAM 13
ONLY ONE SEGMENT CONNECTS ANY TWO POINTS (PROP I-0B)



Proof:

Suppose not, and two segments α and β have points A and B as their boundaries, as in Figure 13. Draw two circles of radius α centered at A and B (pos 3). These will intersect, I hope (pos 5B). Call the intersection point C. The size of the angle $\measuredangle ACB$ is, using α as the reference segment, both α and β. But a single angle cannot have two sizes (pos 5_2). Thus it must be that α is both equal to β and identical in position.

Q.E.D.

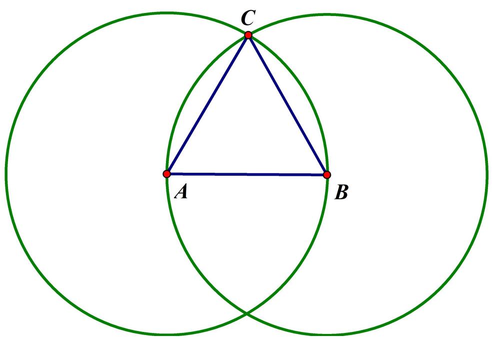
I can expand 0-1C to include why no two lines share a segment, and prove both very easily with the new Pos 2B, using the fact that the extension of a line is unique.

Note that the proof is still valid even if C lies on the segment β.

We will need this proposition to prove proposition 4.

Proposition 1
To construct an equilateral triangle on a given finite straight line.

DIAGRAM 14
CONSTRUCTING AN EQUILATERAL TRIANGLE (PROP I-1)



Proof.
Postulates 5_2? and 5B fix holes in this proof.

> "Join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B."

Euclid had no postulate to show that such a point C existed. Proposition I-0a says that it does.

Another well-known flaw is that Euclid does not say why the lines CA and CB might not have a common segment.(?) Proposition 0b rule that out, because .... why? Maybe I need the triangle inequality first. Or use Pos 2, as is commonly done?

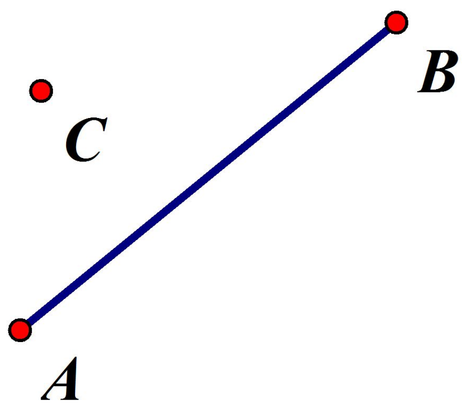
Q.E.D.

Proposition I-2
To place a straight line equal to a given straight line with one end at a given point.

Proof.
sdfsdfsdfsdfsd

Q.E.D.

DIAGRAM 15
COPYING A LINE SEGMENT (PROP 2)



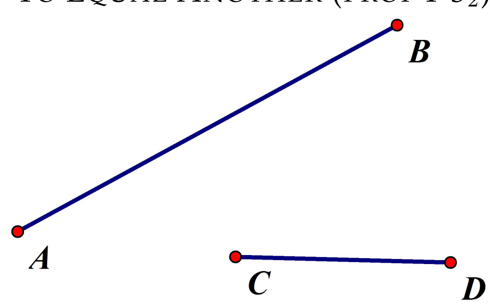
The proof of proposition I-2 does not require postulate 5B, since it does not assume the existence of points of intersection. Nor does it need postulate 5. Although proposition I-2 is transferring a length using circles, that can be done even if the metric depends on the line's location; it just means that on paper the circles won't look circular and lines of the same length will not look equal.

Proposition I-3
To cut off from the greater of two given unequal straight lines a straight line equal to the less.

It SHOULD say

Proposition I-3_2
To cut off from the greater of two given unequal straight lines a straight line equal to the less; or to extend the lesser to be equal to the greater.

DIAGRAM 16
MODIFYING ONE LINE SEGMENT TO EQUAL ANOTHER (PROP I-3_2)



Proof.
sdfsdfsdfsdfsd

Q.E.D.

I bet Euclid uses it that way all the time.

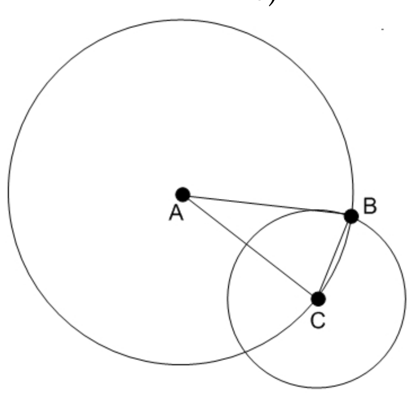
Next, move up Proposition 20:

Prop I-20 (the triangle inequality)
In any triangle the sum of any two sides is greater than the remaining one.

The proof is by contradiction. Suppose side AC of triangle is longer than BC plus AB. Draw a circle centered at A with radius AB and a circle centered at C with radius CB. If $AC > AB + BC$ then the distance between the two centers is greater than the sum of the radii. Thus, they will not intersect (prop I-0B). But we know that point B is in both circles, so we have a contradiction.

Q.E.D.

DIAGRAM 17
THE TRIANGLE INEQUALITY (prop I-20)



We will also move up proposition I-23:

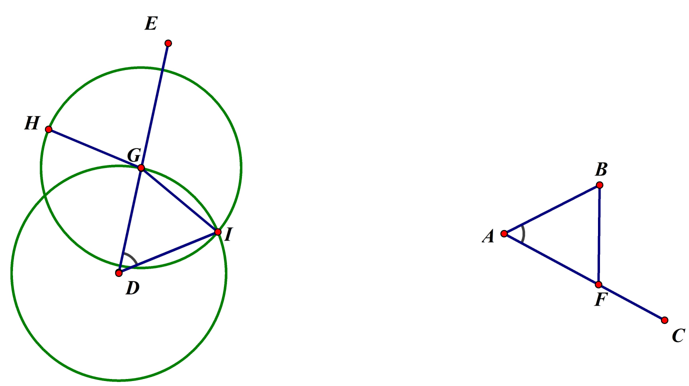
Proposition I-23
To construct a rectilinear angle equal to a given rectilinear angle on a given straight line [segment] and at a point on it.

Proof.
Start with angle BAC and segment DE. We will construct an angle equal to BAC with D as the vertex and lying on DE.

Make segment AB our reference segment size. Draw point F, shortening or lengthening AC so $AF{=}AB$ (prop 3). Draw segment BF, which is the size of angle BAC (def 9B).

Draw point G, cutting or lengthening DE so $DG{=}AB$ (prop 3).

DIAGRAM 18
COPYING AN ANGLE



Draw a circle centered at D with radius DG (pos 3).

Draw a segment $GH{=}BF$ starting at G (prop 2).

Draw a circle with radius GH centered at G (pos 3). It will intersect the D-circle (pos 5B). Label the intersection as I.

Draw the segment DI. Since it connects the center D to the circle of radius $DG{=}AB$, it follows that $DI{=}AB$.

Since $DG{=}AB$ and $DI{=}AB$, the angle GDI has size $GI{=}BF$. But that means the two angles are equal (pos 5_2).

Q.E.D.

[3]

---

[3]xxx What is this?

Draw the lines DI and FI (pos 1). Note that $FI{=}BC$, by construction of the F-circle. Also $DF{=}DI$, because both are on the circle centered at D, and $DF{=}AB$ by construction. Thus, angle FDI has the size FI, which equals BC, the size of the angle BAC (Def of angle size). So we've constructed the desired angle size with vertex at D.

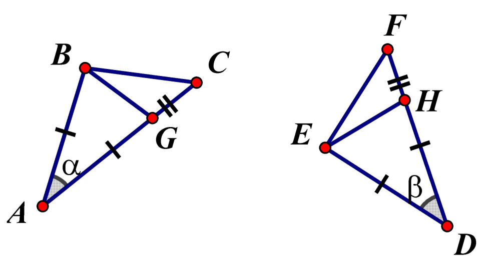
Next we will move the angle to align with DE. Draw a line GJ the length of IF but starting at G (prop 2). Draw a circle with radius GJ centered at G. The G-circle and the D-circle will intersect. By postulate 5B that is true, because D and G are distance $DG{=}AB{=}AC$ apart and their radii are DG and $GL{=}BC$, where by the triangle inequality (prop 20) $AB + BC > AC$. Label the intersection (either intersection) K. Since K is on both circles, it is distance $DG{=}AB$ from D and distance $GL{=}BC$ from G. Since GD =AB, the reference length for the angle BAC, the size of the angle GDK is GL=BC, the same as that of the angle BAC.

We are now almost equipped to prove proposition I-4 by an improved method of superposition, but first we will use it in a weaker form of that proposition, I-3B.

> **Proposition I-3B (Weak Side-Angle-Side)**
> **If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base.**

Proof.

DIAGRAM 19
WEAK SIDE-ANGLE-SIDE I (PROP I-3B)



Use the same notation as Euclid did in his proof of proposition I-4, as in diagram 19. Euclid says, "Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF respectively, namely AB equal to DE and AC equal to DF, and the angle BAC equal to the angle EDF."

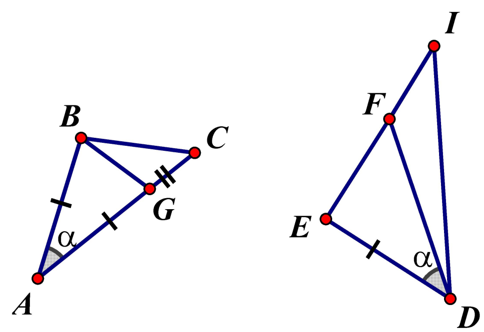
Make AB our unit length for angle size, and measure angle $\alpha \equiv \angle BAC$ by cutting the extension of line AC to length $AG{=}AB$ (prop I-3_2). Then BG is the size of α (def 9B).

Use proposition I-23 to construct an angle of the same size as α and with side lengths equal to AB and AG, but with the vertex A moved to D and the AB side moved to lie on top of DE. There will be two ways to construct the angle. Choose the one which puts the AG side on the same side of DE as DF (it will then, as we will show, lie on top of DF). Label the outer boundary of the new angle as I (see diagram 20).

Look again at diagram 19. By hypothesis we know that $\beta \equiv \angle HDE{=}\alpha$, $FD{=}AC$, and $DE{=}AB$. Since the angles are equal, their sizes are equal, so $BG{=}EH$

(pos 5_2). By subtracting $AG{=}DH$ from $AC{=}AF$,, we can deduce that $GC{=}HF$ (c.n. xxx).

DIAGRAM 20
WEAK SIDE-ANGLE-SIDE II (PROP I-3B)



We now have BAC superposed on DEF and must show that the bases are equal, i. e. that $FE{=}BC$.

Draw the segment EI of length BC on the extension of EF starting at E (prop I-3_2). Either $EI > EF$, as in diagram 20, or $EI < EF$ or $EI{=}EF$. Our aim is to show that $EI{=}EF$, i.e. that $I = F$ (equals in location, not just is equal to F).

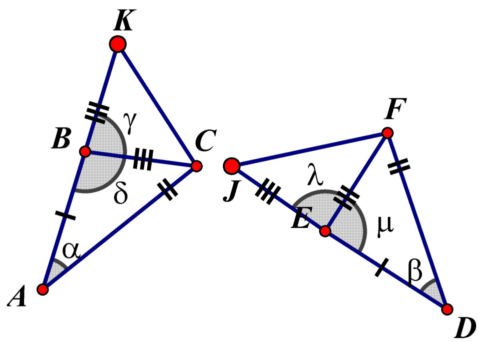
We know $AC{=}DF$ by hypothesis. Consider the triangle IFD in diagram 20. If I does not equal F, then the distance from I to D is ID, which is different from IF+DF by the triangle inequality (prop I-20). But then we have $AC{=}DF{=}IF + DF$, which is absurd. So it must be that I is the same as F. We have thus proven a subproposition: that if two triangles have two sides and the intervening angle equal, then their third sides are equal too.

Q.E.D.

Proposition I-4 is a strengthened form of I-3B.

Proposition I-4: [Side-Angle-Side]
If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

DIAGRAM 21
SIDE-ANGLE-SIDE (PROP I-4)



Proof.

Use the same notation as Euclid did in his proof of proposition I-4, as in diagram 19. Euclid says, "Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF respectively, namely AB equal to DE and AC equal to DF, and the angle BAC equal to the angle EDF."

If two sides and the angle between are equal between two triangles, then so is the third side (prop I-3B). Thus, $BC{=}EF$.

Without loss of generality, extend the segment AB out by amount BC to create segment BK, and extend the corresponding segment DE out by amount EF to create segment EJ.

Take the reference segment to be BK. Then the size of angle $\gamma \equiv \angle CBK$) is KC (def 9B), and since $BK = EJ = DF$, the size of angle $\lambda \equiv \angle EJF$ is JF (def 9B).

By hypothesis, $\alpha = \beta$ and $AC = DF$. By adding equal segments, $AB + BK = AK = DE + EJ = DJ$. Thus, the triangles AKC and DJF have two sides and the angle between that are equal, and their third sides, CK and JF, must be equal (prop I-3B). CK is the size of γ and KF is the size of angle λ, so those two angles must be equal: $\lambda = \gamma$ (pos 5B).

Adding γ to δ yields two right angles since AK is a segment and therefore straight (def ???). Adding λ to $\mu \equiv \angle DEF$) yields two right angles since DJ is a segment and therefore straight (def ???). Thus, δ and μ must be equal (c.n. xxx).

We have thus proved that if two triangles have three sides and one angle the same, one of the other angles must be the same in both triangles. But if that is true, it is also true that if two triangles have three sides and two angles the same, the third angle must be the same in both. Thus, all three angles of the two triangles ABC and DEF correspond.

Better: Use prop I-0C, the prop that one straight line and only one goes between two points. And look to Euclid's proof adn the[] interpolation about not enclosing a space.

Q.E.D.

Fix prop16 too. See Venema objection pn page 10. Venema is quite good.

I need to jsutify my axioms. I do NOT use the real line. Make everything constructive. I talk about addnig distances, but I really mean I am putting together two segments. I cannot get any real number that cannot be gotten that way. Perhaps pi is impossible for me, for example– can I construct a segemet the same as a full arc? No– I do not have an arc length definition.

Euclid uses the method of superposition for just one other proposition, proposition I-8. We can replace it with Proclus's proof, however, as given in Heath's book.

Proposition I-8 (Side-Side-Side) [replace "equal" with "equal"]
If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proof.

DIAGRAM **??**

SIDE-SIDE-SIDE (PROP I-8)

This is a pla

Philo's proof of I. 8.

This alternative proof avoids the use of I. 7, and it is elegant; but it is inconvenient in one respect, since three cases have to be distinguished. Proclus gives the proof in the following order (pp. 266, 15—268, 14).

Let ABC, DEF be two triangles having the sides AB, AC equal to the sides DE, DF respectively, and the base BC equal to the base EF.

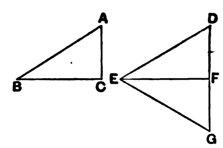
Let the triangle ABC be applied to the triangle DEF, so that B is placed on E and BC on EF, but so that A falls on the opposite side of EF from D, taking the position G. Then C will coincide with F, since BC is equal to EF.

Now FG will either be in a straight line with DF, or make an angle with it, and in the latter case the angle will either be *interior* (κατὰ τὸ ἐντός) to the figure or *exterior* (κατὰ τὸ ἐκτός).

I. Let FG be in a straight line with DF.

Then, since DE is equal to EG, and DFG is a straight line,

DEG is an isosceles triangle, and the angle at D is equal to the angle at G. [I. 5].



DEG is an isosceles triangle, and the angle at D is equal to the angle at G. [I. 5].

II. Let DF, FG form an angle *interior* to the figure.

Let DG be joined.

Then, since DE, EG are equal,

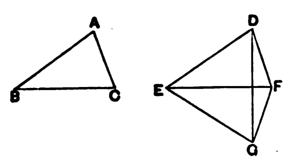
the angle EDG is equal to the angle EGD.

Again, since DF is equal to FG,

the angle FDG is equal to the angle FGD.

Therefore, by addition,

the whole angle EDF is equal to the whole angle EGF.



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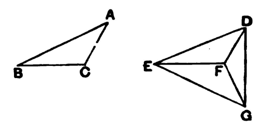
III. Let DF, FG form an angle *exterior* to the figure.

Let DG be joined.

The proof proceeds as in the last case, except that subtraction takes the place of addition, and

the remaining angle EDF is equal to the remaining angle EGF.

Therefore in all three cases the angle EDF is equal to the angle EGF, that is, to the angle BAC.



It will be observed that, in accordance with the practice of the Greek geometers in not recognising as an "angle" any angle not less than two right angles, the re-entrant angle of the quadrilateral $DEGF$ is ignored and the angle DFG is said to be *outside* the figure.

Q.E.D.

Note that by direct application of proposition I-4, it follows that the other two angles of the two triangles are equal, not just one.

[4]

Also, Pasch's Axiom was invented in Pasch (1882) to correct Euclid's proof of proposition 21 (reference). We can prove Pasch's Axiom as a proposition using postulate 5B.

Proposition I-16 is sometimes said to require a new axiom. I think postulate 2, broadly interpreted, does the trick.

> **Proposition I-19B (Pasch's Axiom):**
> **Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.**

Proof:
See diagram 23. Draw a segment DE on the line a so that DE is equal to the longest side of the triangle (prop 3_2). Label as F some point further than DE from D in the direction of E (pos 2).

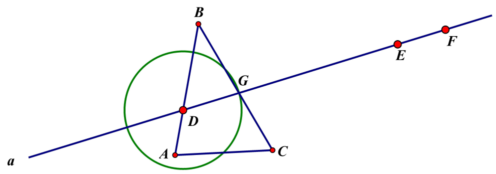
The line a will cross either side BC or side AC of the triangle (why? Triangle Inequality?). Assume without loss of generality that it crosses BC. We must show that there is a point lying on both a and BC. Label the crossing as G.

Draw a circle of radius DG centered at D. Point D in segment DF is closer than DG to D. $DF > DE$, (2) no point on a triangle can be further than the longest side of the triangle from any other point on the triangle (needs proof), Thus, point F is further than DG from D. Since DF contains a point closer to the center of the D-circle than its radius and a point further from the center than the radius, there exists a point G which is in both the A-circle and AF (pos 5B).

[4]Probably the only axiom I need for pos 5B is that right angles all ahve the same size— I could get that all angles have a unique size as a proposition.

Q.E.D.

DIAGRAM 23
PASCH'S AXIOM: A LINE GOING THROUGH A TRIANGLE I (PROP I-20B)



The "longest side" is BC in the diagram, but it could be AB or AC and the proof still works.

This is Hilbert's Axiom II, 5 (grundlagen) shown by Moore to be redundant (Halsted [1902]).

Maybe My axiom 5b is wrong, WHy??? Something about short short lines?

Now, change the order a bit. Put proposition I-31 after proposition I-28. Neither of these needs the parallel postulate. Hartshorne is good at showing the lgoic of this, on page 38. He has a proof of Euclid's 5 using Playfair, too.

Proposition 27.
If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28.
If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Prop 28 is not needed to prove prop 31; I put it here for completeness.

Proposition 31.
To draw a straight line through a given point parallel to a given straight line.

Now add:

Proposition 31B.
Playfair's axiom.

Proof: Use prop 31 and pos 5_2.

Proposition 31C.
Euclid's parallel postulate.

Proof: Use prop 31B.

Then don't prove Prop. 40. Rather, use it to prove prop 5_2.

Proposition 29.
A straight line falling on parallel straight lines makes the alternate angles equal to

one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proof: requires Euclid's parallel postulate.

6. Some Intuition, and Hilbert's Mistake

It is useful to think about the properties of a line AB. I can think of only four properties.

(1) Size, the length or distance of a line, which is measured relative to some referent line. This is the property used in =ruence.
(2) Location, the precise locations of A and B, which is independent of any units.
(3) Orientation, the angle it makes relative to a referent line.
(4) History, when it was drawn relative to other objects.

People say (footnote here) that thinking about distances is contrary to the spirit of Euclid, because he never used numbers. They misunderstand Euclid. He of course deals with numbers throughout the Elements, and he even does some number theory, in Book XXX. The whole purpose of the compass is to measure distance. The essence of Euclid is using only compass and straightedge and doing everything constructively. All distances are relative to a given line. But whenever he talks about longer and shorter, bigger and smaller, he is in effect using numbers.

The standard description of Euclid's approach to geometry is as limiting constructions to the use of the rule and compass, to a straight edge that can draw lines and a compass which can draw circles of a given radius but collapses if you pick it off the paper and try to use it to match a distance directly. This is a little troubling because a rule is a very simple device and a compass with its two arms and movable parts seems in a different league. A better way to think of it is that Euclid allows you to use an unmarked straightedge but to mark it to use some given distance as a unit of measurement with which to construct a new line of the same length. As with the compass, we must say that the straightedge can't be moved away from its original starting point, just rotated. After Proposition 3 we don't even need that limitation since that proposition shows us how to get around that limitation.

Let us now think about Hilbert's method for dealing with proposition I-4, which is to introduce it as a new axiom, axiom IV-6.[5]

IV, 6 [(Hilbert's Triangle Axiom)]
If, in the two triangles ABC and ABC the equalities

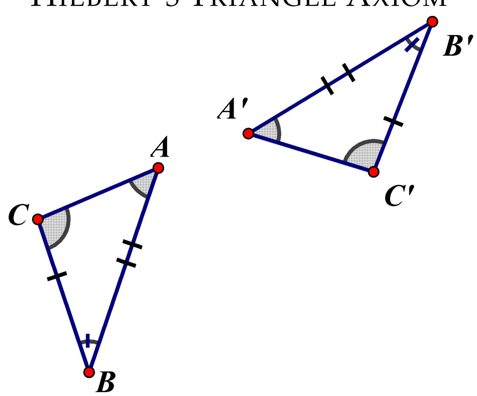
$$AB{=}A'B', AC{=}A'C', \measuredangle BAC{=}\measuredangle B'A'C'$$

hold, then the equalities

$$\measuredangle ABC{=}\measuredangle A'B'C' and \measuredangle ACB{=}\measuredangle A'C'B'$$

also hold.

DIAGRAM 24
HILBERT'S TRIANGLE AXIOM



In words, if two triangles have two lengths and the angle between of the same size, then they must have the remaining two angles of the same size. The last side also matches (as Euclid prop I-4 says), but that can be proved as a proposition.

To show that axiom IV-6 cannot be proved as always resulting necessarily from his other axioms, Hilbert gave an example of a geometry that satisfies other axioms but not IV-6. Using the notation above (and reducing Hilbert's definition from 3 dimensions to 2 in the obvious way), he defines distance as

$$d = \sqrt{(x_1 - x_2 + y_1 - y_2)_2 + (y_1 - y_2)_2}$$

Hilbert says, "It is at once evident that, in the geometry of space thus defined, the axioms I, II, III, IV 1–2, 4–5, V are all fulfilled. (He considers the fulfillment of

[5]There were ten German editions of the *Grundlagen* and at least two English editions. The numbering of the axioms changed across editions. Here, I use the numbering of the 1950 English edition.

axiom IV-3 less evident and treats it separately.) The last part of his proof consists in showing that axiom IV-6 fails in an example using this metric.

Hilbert's metric is not the Euclidean metric, though, so he must be wrong in saying that axiom III is fulfilled. It is a fine point. What Hilbert's metric does is make horizontal spans less important than vertical spans in making distances bigger. If you construct two parallel lines through $x = 0, x = 1$ that are always distance 1 apart as they rise from $y = 0$, then $d = \sqrt{(1+y)_2 + y_2} = \sqrt{(1+2y+2y_2}$.

Birkhoff uses the same axiom as Hilbert. It is

> Postulate IV: Postulate of Similarity. [(it Birkhoff's Triangle Axiom)]
> Given two triangles ABC and $A'B'C'$ and some constant $k > 0, d(A', B') = kd(A, B), d(A', C') = kd(A, C)$ and $\measuredangle B'A'C' = \pm\measuredangle BAC$, then $d(B', C') = kd(B, C), \measuredangle C'B'A' = \pm\measuredangle CBA$, and $\measuredangle A'C'B' = \pm\measuredangle ACB$.

See p. 9 of Moise (2nd edition), where he shows the independent of hte SAS postulate– but not, I think, from the parallel postulate, just from the ruler postulate.

SECTION 1: THE PARALLEL POSTULATE

The term "line" is confusing in Euclid. He uses it for line segments, for line segments indefinitely extended in one direction, for line segments indefinitely extended in both directions, and for curves. I will quote him as translated and continue in his usage, since his setting really does not allow for infinite rays and lines, just ones that are "long enough" for the particular use— which is perhaps why he chose his angle version of Postulate 5 rather than Playfair's parallel lines version.

> **Postulate 5. [(*it Euclid's Parallel Postulate* 5)]**
> **That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.**

Think about testing Postulate 5 in some setting, to see if you are within the scope of Euclidean geometry. If it is the Playfair Axiom, then you must show that two parallel lines never meet. How do you do that constructively? You need to go out to infinity on both sides. If Postulate 5 is Euclid's version, you need to show that any two non-parallel lines cross. That requires checking an infinite number of lines, some of which will only cross near infinity, but at least we don't quite reach infinity. Better yet would be this version, which appears as Proposition I-46 in Euclid:

> **Postulate** 5_3
> **Any point lies on some square.**

See http://en.wikipedia.org/wiki/Lambert_quadrilateral. http://en.wikipedia.org/wiki/Saccheri_quadrilateral The "any point" requirement is there because a space might mostly use a Euclidean metric but not everywhere— see the example below in the parallel postulate appendix.

Proposition I-46.
To describe a square on a given straight line.

Postulate 5_3 is nice because unlike many other Parallel Postulates, it can be tested locally, in the following sense: pick a point and you can test the postulate using only points within a ball of any given finite radius, however small.

Postulate 5_3 is related to Lambert and Sacceri quadrilaterals. See http://en.wikipedia.org/wiki/Lambert_quadrilateral. http://en.wikipedia.org/wiki/Saccheri_quadrilateral The "any point" requirement is there because a space might mostly use a Euclidean metric but not everywhere— see the example below in the parallel postulate appendix. Once we have it, we also have Euclid's Parallel Postulate (5), Playfair's Parallel Postulate, and others.

Hilbert states it differently, as the only axiom in the third of the five groups into which he organizes his 21 axioms (reduced to 20 in 1902 when E.H. Moore found that one was redundant (Halsted (1902)).

Group III. [(*Hilbert's Parallel Postulate* 5_4)]
In a plane α there can be drawn through any point$A$, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.

Hilbert is using a form of Playfair's Axiom. Hilbert's Axiom III is not quite equivalent to the Parallel postulate—it is stronger — but Euclid's axioms as a group imply Axiom III (source?).

The following are equivalent axioms:

1. The Pythagorean Theorem
2. The Euclidean metric measures distance

3. Euclid's Parallel Postulate
4. The shortest distance between two points is a straight line
5. The shortest distance between two points follows a unique path

Proof. This is easy, if we have Theorems 1 and 2 already. First, it is well-known that the Euclidean metric is equivalent to the Pythagorean Theorem. Second, it is well-known that the Parallel Postulate is also equivalent to the Pythagorean Theorem. Third, Theorems 1 and 2 say that axioms 4 and 5 are true only if the Pythagorean Theorem is true. Fourth, it is well-known that if the Pythagorean Theorem is true, so are axioms 4 and 5. Thus, all five axioms are equivalent to the Pythagorean Theorem and to each other. QED.

It is well known that Postulate 5^5 below is equivalent to the parallel postulate.[6]

Postulate 5^5
If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.

Note that this can *sometimes* be true outside of Euclidean space, but for the space to be Euclidean we require it of EVERY quadrilateral.

[6] Wikipedia, "Parallel Postulate" or page 470 of Non-Euclidean Geometry: A Historical Interlude Jürgen Richter-Gebert Springer, 2011. http://www.springerlink.com/content/r23u748233175760/

The Choice of Metric and the Parallel Postulate

The Parallel Postulate has many equivalent or close statements. It is equivalent to the Pythagorean Theorem, for example, which fails in non-Euclidean spaces. In Cartesian geometry, the Pythagorean Theorem can be viewed as a statement of the metric of a space, since it takes the horizontal and vertical projections of two points and yields the length of the line segment between them. If the Pythagorean Theorem holds, the distance between points A_1 and A_2 is $(x_1-x_2)_2+(y_1-y_2)_2 = d_2$; that is

$$d = \sqrt{(x_1 - x_2)_2 + (y_1 - y_2)_2}$$

If the metric is different, we are not in a Euclidean geometry because the Pythagorean Theorem does not hold, so the the Parallel Postulate does not hold.

It will be useful to add new definitions to distinguish between distance and length.

Can we even use the compass to draw circles if we do not have the Euclidean metric? Distances will vary from the compass-width over the plane. A circle defined as a set of points equidistant from the center will look distorted. See http://williewong.wordpress.com/2009/07/01/straightedge-and-com . One can still do equivalent constructions in non-Euclidean space, but the diagrams are more purely heuristic than ever; they are not only flattenings onto a Euclidean plane, but distorted flattenings too.

FIX THIS UP TO NOTE CHANGES

Definition 1.
A point is that which has no part.

Definition 2.
A line [curve] is breadthless length.

Definition 3.
The *ends of a line* [curve] are points.

Definition 4.
A *straight line* [segment] is a line [curve] which lies evenly with the points on itself.

I will use the terms "segment" where Euclid uses "straight line" and reserve "line" for an segment extended indefinitely in both directions.

Definition 4B.
The *length* of the segment AB is I NEED A DEFINITION FOR THE LOOK ON HTE PAGE

Definition 4C.
The *distance* between points A and B is a mapping with the following properties:

HERE PUT THE PROPERTIES OF A METRIC.

Prop I-3D. The distance beteen two points is the length of the segment between them.

For 1, it would be good to say in the PP that one and only one parallel line exists.

Or, I need something else to say that the metric has to be Euclidean given Pos 5.
Aha ! Postulate 1 SHOULD say that any two points describie one and only one line segment. Joyce notes that Euclid left out the uniqueness, but uses it.Really, it coudl ahve been a definition that a segment is the unique thing pinned down by two points. WAIT– it won't be unique except with the parallel postulate. I'm too tired to Diagram this out now though.

Consider the following metric:

Let the *slowness* of a segment be for most segments on the plane the Euclidean metric, so $s(a,b) = \sqrt{(x_a - x_b)_2 + (y_a - y_b)_2}$ if we use Cartesian coordinates. For any segments lying on the $y = 1$ line, however, the slowness is $s(a,b) = .1\sqrt{(x_a - x_b)_2 + (y_a - y_b)_2} = .1(x_a - x_b)$.

This creates a turnpike, because distances are shorter if you travel partly on the $y = 1$ line.

The metric, or distance, is the minimum slowness of the path between a and b using no more than three segments, i.e.

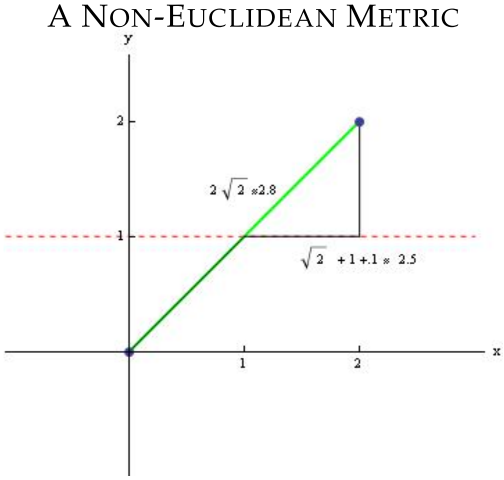
$$d(a,b) = \overset{Min}{c, d, e}\ [s(a,c) + s(c,d) + s(e,b)]$$

This is a legitimate metric, because it has the properties:

1. d(a,b) is a function, i.e. (a, b) maps to a unique d.
2. $d(a, b) \geq 0$.
3. $d(a, b) = 0$ if and only if $a = b$.
4. $d(a, b) = d(b, a)$.
5. $d(a, b) \leq d(a, c) + d(c, b)$ (The Triangle Inequality).

The shortest distance, as the diagram shows, is not a straight line between $a = (0, 0)$ and $b = (2, 2)$. We must go back to the definition of straight line to see this, since if one defines a straight line as the shortest distance between two points, the black path in the Diagram is indeed straight.

DIAGRAM 25
A NON-EUCLIDEAN METRIC



A simple example is to use the Manhattan Distance: $d \equiv |x_a - x_b| + |y_a - y_b|$. Then there is an infinite number of shortest paths that are not straight lines. But maybe the diagonal is still faster– this is connected with the $1 = 2$ fallacious theorem.

As another example of equivalence, use a metric where speeds, in both dimensions are faster in the negative x zone. Then the shortest distance between (1,1) and (-2, 2) will not be a Euclidean straight line— there will be a turnpike effect of wanting to get into the fast zone quickly.

Theorem 1. The shortest distance between two points is a straight line only if the Pythagorean Theorem is true for all right triangles, Corollary 1. The shortest distance between two points is a straight line only if the Parallel Postulate holds true.

Theorem 2. The shortest distance between two points follows a unique path only if the Pythagorean Theorem is true for all right triangles,

Corollary 2. The shortest distance between two points follows a unique path only if the Parallel Postulate holds true.

Proposition 28B: The Angle Parallel Postulate, 5_2, implies the Square Parallel Postulate, 5_3.

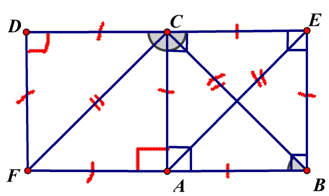
Proof:
We need to show that

> **Postulate 5_2**
> **If and only if two angles are equal they are of equal size.**

implies

> **Postulate 5_3**
> **Any point lies on some square.**

DIAGRAM 26
POSTULATE 5_3 IMPLIES
POSTULATE 5_2— I



We will do that constructively, as in Euclid's Proposition I-46. Note as a preliminary that propositions 4,5, 11, 31, and 33 do not require Euclid's Parallel Postulate for their proof, so they are available for our use here.

Start with any point, which we will label A as in Diagram 26. Choose an arbitrary point B and draw the segment AB, which will be our reference segment (pos 1).

Now draw a segment perpendicular to segment AB touching it at point A, and shorten or lengthen it to end at point C where it equals AB (prop 11, prop 3). The angle $\measuredangle CAB$ is a right angle by the definition of a perpendicular (def xx). It has size AC, since AB is the reference segment (Def 9b).

DIAGRAM **??**
POSTULATE 5_2 IMPLIES
POSTULATE 5_3— II

This is a place-holder

DIAGRAM **??**
POSTULATE 5_2 IMPLIES
POSTULATE 5_3– III

This is a place-holder

Draw a line passing through C that is parallel to the line passing through segment AB (prop 31) and cut off a segment CD =AB in the direction furthest from B (prop 3).

Draw a segment CE equal to AB that is lying on the line CD and has E closer to B than to A (prop I-3).

Draw a segment AF equal to AB that is lying on the extended line AB (which exists by pos 2) where F is closer to A than to B (prop 3).

By the definition of a right angle (def xxx), if $\measuredangle ACD$ is a right angle then so is $\measuredangle ACE$, since they add up to a straight line.

Draw the segments EB and DF. These segments are equal to AB (prop 33). Thus, the figures ACEB and ACDF have seven equal sides together. Figure ACEB has four equal sides, one of the requirements of a square.

Since all right angles have equal sizes, CF (the size of $\measuredangle CAB$) equals AE (the size of $\measuredangle CAB$) (pos 5_2).

Since angles $\measuredangle CEB$ and $\measuredangle CDF$ have size AE, they both are right angles (pos 5_2).

The triangles EAB, BEC, FAC, and FDC all have one right angle and two sides adjacent to it that equal AB. Thus, the other two angles of each triangle also match (prop 4). Each triangle is an isosceles triangle since it has two equal sides (def xx), so it those two angles are equal for each individual triangle (prop 5).

Angles $\measuredangle FCD$, $\measuredangle FCA$,$\measuredangle ACB$ and $\measuredangle BCE$ are therefore equal to each other. Those four angles add up to a straight line. Therefore the two equal angles that add to the straight line (and thus are right angles— def xxx) each are equal to the sum of two of the isosceles angles. In particular, angle $\measuredangle ACE$ is a right angle.

The size of $\measuredangle ACE$ is CB. But that is also the size of $\measuredangle ABE$, so $\measuredangle ABE$ is a right angle (pos 5_2). Thus, all four angles of the figure ABCE are the same. We earlier showed that all four sides of ABCE are the same, so ABCE must be a square (def xxx).

Q.E.D.

Proposition 0C: The Square Parallel Postulate, 5_3, implies the Angle Parallel Postulate, 5_2.

Proof:
We need to show that

> **Postulate** 5_3
> **Any point lies on some square.**

implies

> **Postulate** 5_2
> **If and only if two angles are equal they are of equal size.**

jkjkjlkjkljklj

Q.E.D.

To continue to use Euclid's proofs. we also need to show that the Angle Parallel Postulate implies Euclid's Parallel Postulate.

> **Postulate** 5. **[(***Euclid's Parallel Postulate* **)]**
> **That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.**

Proposition 28C: The Angle Parallel Postulate 5_2 implies Euclid's Parallel Postulate 5

DIAGRAM **??**
POSTULATE 5_3 IMPLIES
POSTULATE 5_2

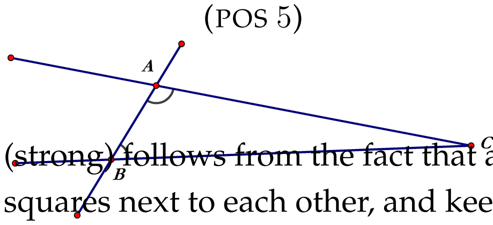
Proof:
We need to show that

> **Postulate 5_2**
> **If and only if two angles are equal they are of equal size.**

implies

> **Postulate 5. [(*Euclid's Parallel Postulate*)]**
> **That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.**

DIAGRAM 30
EUCLID'S PARALLEL POSTULATE
(POS 5)



First, note that if it is true that "if and only if two angles are equal they are of equal size" (pos 5_2), then it is true that "any point lies on some square" (pos 5_3).

Let us start by demonstrating that the second part of Euclid's parallel postulate (strong) follows from the fact that any point lies on some square. Set up a series of squares next to each other, and keep going forever, as in diagram dfkjsklsjd.

sdfsdfsdfsdf

Q.E.D.

Mayb Veblen does what I am doing, with his 12 axioms.

Concluding Remarks

These still need to be written up.

Diagram 31
The Square Parallel Postulate Implies that Parallel Lines Never Meet

Appendices: Systems of Axioms for Euclidean Geometry (These are in a separate file)

This appendix will list various sets of axioms for Euclidean geometry— Euclid's, Rasmusen's, Hilbert's, Birkhoff's, Tarski's, Veblen's and the SMSG's.

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By all accounts, that work is difficult in the extreme.REVIEW OF EHRLICK BOOK.

“It is almost impossible, for instance, to find a clearly defined pattern in the axioms he proposes. They are divided into the geometrical and the practical, and they alternate with philosophical observations that certainly do not help to clarify the presentation.... Veronese, who had been one of KleinŠs students, was still anchored to the empiricist conception of geometry that had been at the forefront of research in the two decades that preceded his work. Therefore, just like Pasch, Veronese attributed an empirical nature to geometrical axioms, without contradicting the abstract nature of the discipline:" Foundations of Geometry in Italy before Hilbert Aldo Brigaglia http://semioweb.msh-paris.fr/f2ds/docs/geo_2004/Aldo_Brigaglia_1.pdf

MAYBE ADD:

Geometric Structures in Dimension two Bjørn Jahren November 14, 2011 175 pages, available electronically on the web at http://folk.uio.no/bjoernj/kurs/4510/gs.pdf, 2011.

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NOTES

Crude Geometry: Incidence Geometry.

I have to worry about elliptical geometry fro props after 15. Euclid's proof of Prop 16 is wrong.

Postulate 1.
To draw a straight line from any point to any point.

To rule out elliptical geometry, people suggest e.g. Heath at p. 280,

Postulate 1.
One and only one segment can be drawn between two points.

OR, Postulate 2.
To produce a finite straight line continuously in a straight line.

Interpret this to say that the line does not come back upon itself. THen, I think Euclid's proof is correct. My Pos 5-2 rules out elliptical geometry too. So just put it in Prop. 16's proof to guarantee that there is just one point. Again: WIth the parallel postulate, there is no need to fix up Pos 1 or pos 2. Maybe add a proposition to that effect:

I am doing to Hilbert what others tried to do to Euclid: to prove his ugly axiom, the SAS one in this case.

Hartshorne, p. 92: SAS "is essentially what tells us that the plane is homogeneous. Geometry is the the same at different places in the plane."

The RULER is a way to see if two segments are equal. That cannot be done just with compass and straightedge. The straightedge allows us to reproduce a given segment on top of another, but not to check if they are equal.

EUclid Prop 22 shows how to build a triangle from 3 given sides. THus, for Prop 4 we can get SSS angle.

We do not need iff here. We only need one point (and that's all we rpvoe here).

$AB \tilde{\succeq} BD$.

$AB \precsim AC$

$\not\prec$.

$\not\succ$,

$cong \ncong \measuredangle$,

http://mirror.math.ku.edu/tex-archive/info/symbols/comprehensive/symbols-a4.pdf

Integer geometry: use not a field, but the integers. ANy angles without integer sizes don't count as angles (we will let the reference segment be multiplied). The Continuity postulate will fail dramatically. Parallel, OK. Pos 1,2,OK. Postulate 3 OK if modified maybe.

The assumptions make it clear our plane is like a piece of flat paper, with no holes in it, adn certainly not just a fishing net, and no bumps or sticky spots.

The Square Postulate is nice because it is locally testable. You DO have to do the local test everywhere tho, unless you want to add teh SAS Axiom to guarantee homogeneity– and then you ahve to test THAT everywhere. SAS really says: a figure I make in one area of the plane can be moved fixedly to anywhere else in the plane.

For neutral geometry, drop 5.2 and add Postulate 5C, the SAS postulate of Hilbert. Or, for 5.2 use 5.3 in various forms for different geometries and add SAS for non-Euclidean.

Postulate 5C: The Side-Angle-Side Postulate. If in the two triangles ABC and $A'B'C'$ the equalities $AB{=}A'B'$, $AC{=}A'C'$, $\measuredangle BAC{=}\measuredangle B'A'C'$ hold, then $\measuredangle ABC{=}\measuredangle A'B'C'$ and $\measuredangle ACB{=}\measuredangle A'C'B'$.

Note that this is weaker than Proposition I-4 because it does not say that the remaining sides BC and $B'C'$ are equal.

How do I turn a rectangle into a different rectangle with one side equal to the reference segment, thus transferring an area to a segment?

Proposition I-45 (quadrature) To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

REWRITE AS:

Proposition I-45B (quadrature) To construct a rectangle equal to a given rectilinear figure in a given rectilinear angle and with one side equal to the reference segment.

Proposition I-45C (area) To construct a segment equal to a given rectilinear figure. (maybe make this a corollary)

Proof:

Use I-45B. The side not equal to the reference segment will equal the given rectilinear figure.

Always make AB the reference segment.