Notes on Exponents

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http:\\rasmusen.org\papers\exponents.pdf
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The usual way people learn about exponents is to learn a basic idea— multiplying something by itself a bunch of times— and some rules for multiplying exponents and taking the exponent of an exponent. The notation 10^3 means $10 \cdot 10 \cdot 10$, which is 1,000, with $\cdot$ being a multiplication symbol like $\times$ or * or using (10)(10)(10). More generally, $x^3 = x \cdot x \cdot x$. Then, you learn some rules. I like examples better than rules, since they're concrete and teach the same idea, so here are a few of them.

$$10^2 \cdot 10^3 = 10^{2+3} = 10^5$$

$$(10^2)^3 = 10^{(2)\cdot(3)} = 10^6$$

$$10^0 = 1$$

$$10^{.5} = \sqrt{10} \approx 3.16 (\approx \text{ means``about''})$$

$$10^{1/3} = \sqrt[3]{10} \approx 2.15$$

$$10^{-1} = \frac{1}{10^1} = \frac{1}{10} = .1$$

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = .001$$

The pattern to remember for multiplying exponents is

$$10^2 \cdot 10^3 \quad = \quad 10^5$$

$$100 \cdot 1,000 \quad = \quad 100,000$$

so

$$x^2 \cdot x^3 \quad = \quad x^5$$

The best way to remember this is not to memorize it outright, but to try out examples. The common problem you will run into is that you will ask yourself, "Do I add the exponents, or do I multiply them?" If you try particular numbers like $x = 10$, you will realize that you should add them, because

$$10^2 \cdot 10^3 = 100 \cdot 1,000 = 100,000 = 10^5 \neq 10^{(3 \cdot 2)} = 10^6 = 1,000,000.$$

Or, you can try a simpler example to pin down the rule in your mind:

$$\begin{aligned} 10^1 \cdot 10^1 &= 10^{1+1} = 10^2 \\ 10 \cdot 10 &= 100, \\ 10^1 \cdot 10^1 &\neq 10^{1 \cdot 1} = 10^1 = 10 \end{aligned}$$

What's even better than using numerical examples is to understand the rule, because while you can memorize the multiplication rule, there are others too, so your memory is likely to fail eventually. Also the process of trying may acts as a mnemonic (pronounced, "nemonic"), a help to memory.[1] Oddly enough, attaching a few other words to the thing to be remembered, even if only vaguely related, helps us to remember the thing. I think this is probably because the attachments act as extra "handles" to the thing in the memory, so it is easier to grab onto it and then to haul in the thing itself. Anyway, a logical reason for something acts as a mnemonic for it.

Moreover, the logical reason for all those rules is interesting. Think about how you'd do it, if you were going to pick some rules for doing things with exponents. Here we go.

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[1] *How do you spell mnemonic*
It's practically demonic.
You put an M before the N;
And then it's just phenomic!
The poem is itself a mnemonic.

What if we multiply exponents, e.g. $10^2 \cdot 10^3$*?*

The rule for multiplying exponents has a pretty obvious justification. We need to have $(10 \cdot 10) \cdot (10 \cdot 10 \cdot 10) = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, because that's how multiplication works. But that's exactly the same as saying $10^2 \cdot 10^3 = 10^5$.

What if we take an exponent of an exponent, e.g. $(10^2)^3$*?*

The rule for exponents of exponents can be justified in a similar way. We need to have $(10 \cdot 10) \cdot (10 \cdot 10) \cdot (10 \cdot 10) = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, with six 10's, but that's the same as saying $(10^3)^2 = 10^6$.

What if the exponent is zero, e.g. 10^0*?*

What about the rule that says $10^0 = 1$? It seems arbitrary, but it's not. We *must* have that rule, or we'd violate basic arithmetic. Here's why. Our multiplication rule says that

$$10^a \cdot 10^b = 10^{a+b}$$

That implies

$$10^2 \cdot 10^0 = 10^{2+0} = 10^2 = 100.$$

So any definition we might have of 10^0 requires that $100 \cdot 10^0 = 100$. But there's only one number x that leaves 100 unchanged when you multiply by it to get $100x$, which is $x = 1$! So the only definition we can use is that $10^0 = 1$.

What if the exponent is a negative number, e.g. 10^{-2}*?*

How about the rule that says $10^{-2} = \frac{1}{100}$? The laws of arithmetic require that too. Our multiplication rule says that

$$10^a \cdot 10^b = 10^{a+b}$$

That implies

$$10^{-2} \cdot 10^2 = 10^{-2+2} = 10^0 = 1.$$

So any definition we might have of 10^{-2} requires that $10^{-2} \cdot 100 = 1$. But there's only one number x that takes you to 1 when you multiply by it to get $100x$, and that is $x = \frac{1}{100}$, the inverse of 100! So the only definition we can use is that $10^{-2} = \frac{1}{100}$. Note that we need two steps for this, though— the exponent rule, and the fact that if we use the exponent rule, we also need to have $10^0 = 1$.

What if the exponent is a fraction, e.g. $10^{1/2}$*?*

How about the standard exponentiation rule that says $10^{1/2} = \sqrt{10}$? Can we justify that rule? Let's start with an easier example: $9^{1/2} = \sqrt{9} = 3$. The laws of arithmetic require 3 to be the result. Our multiplication rule says that

$$9^a \cdot 9^b = 9^{a+b}$$

This implies

$$9^{1/2} \cdot 9^{1/2} = 9^{1/2+1/2} = 9^1 = 9.$$

So any definition we might have of $9^{1/2}$ requires that when multiplied by itself it equals 9— which is exactly the definition of the square root of nine.

To be sure, this needs a warning attached. The number 9 has two square roots, +3 and -3. So we've got a little leeway. The standard definition is to take the positive number, +3, as the one denoted by the symbol $\sqrt{x}$. Thus, $3 = \sqrt{9}$ and $-3 = -\sqrt{9}$. In the same way, $3 = 9^{1/2}$ and $-3 = -9^{1/2}$.

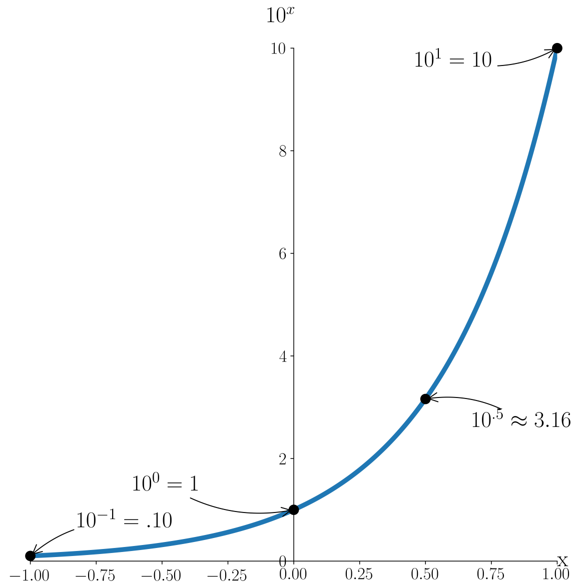
How about more complicated fractions such as $10^{2/3}$? Well, using our earlier rules, that's the same as $(10^2)^{1/3}$. Thus, it's the same as $\sqrt[3]{100}$,which is about 4.6. It will also be the same as $(10^{1/3})^2$, which is about 2.15^2 and reaches the same result. The fractional exponent $10^{2/3}$ means to square ten and then take the cube root, or to take the cube root of ten and then square it–both reach the same answer.

Thus, we can calculate any number of the form $10^{a/b}$ using our existing rules. Put differently, if the exponent is a rational number q

(where q being rational means it can be represented as $q = a/b$ for integers a and b), we can calculate the exponent to the qth power.

If we apply all those rules, we have a function $f(x) = 10^x$ not just for x being in the natural numbers— 1, 2, 3, and so forth, but all rational numbers, including fractions, zero, and negative numbers. Our derivations above all depended on the exponent being a fraction a/b, since we were using the rules of arithmetic. The next section, after the graph will look at what happens when the exponent is an irrational numbers such as π or $\sqrt{2}$ and we are trying to find some number such as 10^π or $10^{\sqrt{2}}$. Putting that aside, the graph will be Figure 1. The graph curves so sharply that I've only graphed it between -1 and 1.

Figure 1:
The Function $f(x) = 10^x$



What if the exponent is an irrational number, e.g. 10^π*?*

Irrational numbers, such as π, e and $\sqrt{2}$, cannot be represented as fractions. They have an infinite number of digits as a decimal, which is a fraction of powers of ten, and they can't be represented by any other kind of fraction either. The logic used above fails because there are no whole numbers a and b such that $\pi = a/b$. We have to go to quite a bit more work to justify exponents that are irrational numbers, such as 10^{π}.

It's pretty easy to see what exponents to irrational numbers *ought* to be. We have already shown how to figure out $10^{3.14}$ and $10^{3.15}$. Surely $10^{3.14159...}$ should be in between $10^{3.14}$ and $10^{3.15}$. It should be bigger than $10^{3.14}$ and smaller than $10^{3.15}$, so 10^x is a strictly monotonic function. The problem is that what *ought* to be the case isn't always what *is* the case, and we also want our numbers with irrational exponents to have the nice properties our numbers with rational exponents have. What we need to do is show that some function $f(x)$ exists such that (a) $f(x) = 10^x$ when x is rational, (b) $f(x)$ is continuous and monotonically increasing, and (c) our standard rules apply even if x is irrational, e.g. $[f(x)]^y = f(xy)$.

The easiest way to do this is to construct an $f(x)$ with the properties we want. There are two approaches to that.

Approach 1. Use the following function of k and x, an infinite series of simple terms (note that $n!$, "n factorial", means $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! \equiv 1$ and that though it's not obvious, this series does converge):

$$\begin{aligned} f(kx) &= \tfrac{kx^0}{0!} + \tfrac{kx^1}{1!} + \tfrac{kx^2}{2!} + \tfrac{kx^3}{3!} + \ldots \\ &= k + kx + \tfrac{kx^2}{2} + \tfrac{kx^3}{6} + \ldots \end{aligned}$$

This turns out to be $f(kx) = e^{kx}$ for Euler's number $e \approx 2.7$. It can be shown that this has all the properties we want for exponentiation of rational numbers. It's clearly increasing monotonically and continuous, since each term in the series is increasing and continuous. We'll see that $[f(x)]^y = f(xy)$, and this leads to all the other properties we need.

As a first step, we need to use a rule from basic calculus: that the slope of $h(x) = x^a$ is $h'(x) = ax^{a-1}$. Applying that rule to our $f(kx)$

function, we get

$$\begin{aligned} f'(x) &= \tfrac{0kx^{-1}}{0!} + \tfrac{1kx^{1-1}}{1!} + \tfrac{2kx^{2-1}}{2!} + \tfrac{3k^{3-1}}{3!} + \tfrac{4kx^{4-1}}{4!} + \ldots \\ &= 0 + k + kx + \tfrac{kx^2}{2!} + \tfrac{kx^3}{3!} + \ldots \\ &= kf(kx) \end{aligned}$$

If we define $e^{kx} \equiv f(kx)$, it follows that since the slope of $f(kx)$ is $f'(kx) = kf(kx)$, the slope of e^{kx} is ke^{kx}.

To show that $f(x+y) = f(x)f(y)$, let's set $k = 1$ and define another function, $r(x)$, which we will show equals 1 for any x.

$$r(x) \equiv \frac{e^x e^y}{e^{x+y}}$$

Its derivative is

$$\begin{aligned} r'(x) &= \tfrac{e^x e^y}{e^{x+y}} - \tfrac{e^x e^y e^{x+y}}{(e^{x+y})^2} \\ &= \tfrac{e^x e^y}{e^{x+y}} \left(1 - \tfrac{e^{x+y}}{e^{x+y}}\right) \\ &= \tfrac{e^x e^y}{e^{x+y}} (1-1) \\ &= 0 \end{aligned}$$

But if $r'(x) = 0$, that means $r(x)$ is constant for all x, including $x = 0$. Since $r(0) = \frac{e^0 e^y}{e^{0+y}} = 1$, we know that $r(x) = 1$, which means that $e^x e^y = e^{x+y}$. We have the multiplication rule.[2]

When we had whole numbers, we could use the multiplication rule to get the exponent of exponents rule easily, but now we can't, because x isn't a whole number. So let's derive the exponent of exponents rule separately.

[2]Don't be bothered by the fact that we used $(e^{x+y})^2 = e^{x+y} \cdot e^{x+y}$. Since 2 is a whole number exponent, we've already found that $z^2 = z \cdot z$.

To show that $f(x)^y = f(xy)$, let's set $k = 1$ and define another function, $g(x)$, which we will show equals 1 for any x.

$$g(x) \equiv \frac{(e^x)^b}{e^{xb}}$$

Now use another calculus rule: that $\frac{d}{dx}h(m(x)) = \frac{dh}{dm}\frac{dm}{dx}$. Then if $h(m) = m^b$ and $m(x) = e^x$, we have $\frac{d}{dx}h(m(x)) = \frac{dh}{dm}\frac{dm}{dx} = bm^{b-1} \cdot e^x = b(e^x)^b$, so

$$\begin{aligned} g'(x) &= \tfrac{b(e^x)^b}{e^{xb}} - \tfrac{(e^x)^b b e^{xb}}{(e^{xb})^2} \\ &= \tfrac{b(e^x)^b}{e^{xb}}\left(1 - \tfrac{e^{xb}}{e^{xb}}\right) \\ &= \tfrac{b(e^x)^b}{e^{xb}}(1-1) \\ &= 0 \end{aligned}$$

But if $g'(x) = 0$, that means $g(x)$ is constant for all x, including $x = 1$. Since $g(1) = \frac{(e^b}{e^b} = 1$, we know that $g(x) = 1$, which means that $(e^x)^b = e^{xb}$. We have our exponent of exponents rule.

So we can extend this to the real numbers too. There is some more work in getting from e^x to 8^x, though, because we have to define $ln(x)$ as the inverse of e^x. That is, $8 = e^{something}$ because e^x is a monotonically increasing function so the number *something* exists and is unique: if we start out with $e^0 = 1 < 8$ and continue to $e^{100} > 8$, we'll find *something* in between 0 and 100. We'll say that $ln\ 8 \equiv something$. If we do that, then $8 = e^{ln\ 8}$ and we can write:

$$8^x = (e^{ln\ 8})^x = e^{ln\ 8x},$$

and go on with the usual multiplying-exponentsrule to use our e^x function to find what 8^x is.

A variant on this approach is to use the Binomial Theorem to expand out the $f(x)$ series for $f(x)f(y)$ and $f(x)^y$. I don't like combinatorics and so prefer the calculus approach.

Approach 2. Build such a function by first defining the log function thus (for the natural, base e, logarithm):

$$ln(x) = \int_1^x \frac{1}{t} dt$$

This can be integrated as a Reimann integral, which needs only the rationals to make sense (you take the limits to get a Reimann integral in a countable number of steps). Then it's true, since $c = e^{ln(c)}$ for some number e that turns out to be Euler's number, that

$$c^d = e^{d \cdot ln(c)}$$

So we want to set our $f(x)$ function like this:

$$f(x) = e^x$$

and show that $f(x)f(y) = f(x + y)$ and $f(x)^y = f(xy)$, as we did in approach 1. I'd have to think more or look back at my source (Binmore, I think), but we can try showing that $lnx + lny = ln(xy)$ and $xlny = lny^x$, I think and somehow that will do it for us.

This log from integral idea I found from David Joyce at quora.com/How-do-you-interpret-irrational-exponents-for-example-3-pi-I-know-how-to-interpret-fractional-exponents-and-Ive-only-ever-interpreted-decimal-exponents-in-terms-of-fractions. It's standard, though– it's in Binmore's real analysis text.