Notes on Imaginary Exponents

Eric Rasmusen, erasmuse61@gmail.com

http:\\rasmusen.org\papers\imaginary_exponents.pdf

February 26, 2020.

What is 10^i? We know that $10^3 = 10 \cdot 10 \cdot 10$, and after some effort we can figure out that logic requires $10^{-3} = \frac{1}{10^3}$ and $10^{1/3} = \sqrt[3]{10}$. But how do we even get started with an imaginary number i in place of 3?

Let's take a first step which makes things a bit more complicated but will help in the end. Note that $10^i = e^{log(10)i}$, where e is Euler's number, special because if $f(x) = e^x$, $f'(x) = e^x$ also: the function f's slope is the same as its height. So if we can figure out what $e^{ilog(10)}$ should be, we've figured out 10^i too.

So let's see if we can calculate what e^{im} is, for any number m.

We can write the answer as a complex number $e^{im} = g(m)+ih(m)$, where maybe $h = 0$ or $g = 0$, and this makes our goal to find the functions $g(m)$ and $h(m)$.

We want this to be consistent with our existing mathematics. One part of our existing mathematics is that if $f(m) = e^{am}$, then $f'(m) = ae^{am}$. Let $a = i$, and we have $f(m) = e^{im}$ and $f'(m) = ie^{im}$.

Using our g and h functions, we also have $e^{im} = g(m) + ih(m)$ so

$$f'(m) = ie^{im} = ig(m) + i^2h(m) = ig(m) - h(m). \tag{1}$$

But we could also write the derivative as

$$f'(m) = g'(m) + ih'(m) \tag{2}$$

Putting together these last two ways to write $f'(m)$, we see that $g'(m) = -h(m)$ and $ih'(m) = ig(m)$, so $g''(m) = -h''(m) = -g(m)$. There is only one pair of functions whose derivatives flip around like that: $g(m) = cos(m)$ (with $g'(m) = -sin(m)$) and $h(m) = sin(m)$ (with $h'(m) = cos(m)$). Going back to $e^{im} = g(m) + ih(m)$, we get

Euler's Formula,

$$e^{im} = cos(m) + sin(m)i. \tag{3}$$

This works out quite nicely, because we also, for consistency, want $e^0 = 1$, and indeed, $e^{0i} = cos(0) + sin(0)i = 1 + (0)i = 1$.

Our formula for imaginary exponents thus tells us that e^{im} starts with $e^{i \cdot 0} = 1$ and then starts going counterclockwise around the unit circle in the imaginary plane with real numbers on the $x - axis$ and imaginary numbers on the vertical axis. It's on the unit circle because the coefficients are $(x, y) = (cos(m), sin(m))$ and from trigonometry we know $cos^2(m) + sin^2(m) = 1$, which means $x^2 + y^2 = 1$, the equation for the unit circle. As m gets bigger, e^{im} just keeps going around the circle, over and over.

Speaking of going around the ciricle, one thing remains. We need to define units for m, which is a measure of angle size. If there are 360 degrees to the circle, then $e^{180i} = cos(180) + sin(180)i = -1 + (0)i = -1$. If, instead, there are 720 degrees to the circle, then $e^{180i} = cos(180) + sin(180)i = 0 + (1)i = i$. The convention is to use 2π radians to the circle, so $e^{\pi i} = cos(\pi) + sin(\pi)i = -1$, which is Euler's Identity. Note that we could start off and make any real number we like to produce a simple exponent with i. We could, for example, start with $20^i = -1$. If we do that, we are in effect assuming that $e^{log(20)i} = -1$, so we are making the very ugly assumption that there are $2 * log(20)$ degrees in the circle. We would end up with $40^i = -1$ too, though, and $60^i = -1$, and so forth.

Figure 1 illustrates this. It show the imaginary plane, and various exponents, with m as an angle, the real part of the x-axis, and the imaginary part on the y-axis. Figure 2 shows the same information, but with m on the horizontal axis instead of as an angle, and m^i on the vertical axis, where the vertical axis has two curves, for the real part and the imaginary part of m^i.

FIGURE 1: $f(m) = e^{im}$, IN REAL-IMAGINARY SPACE

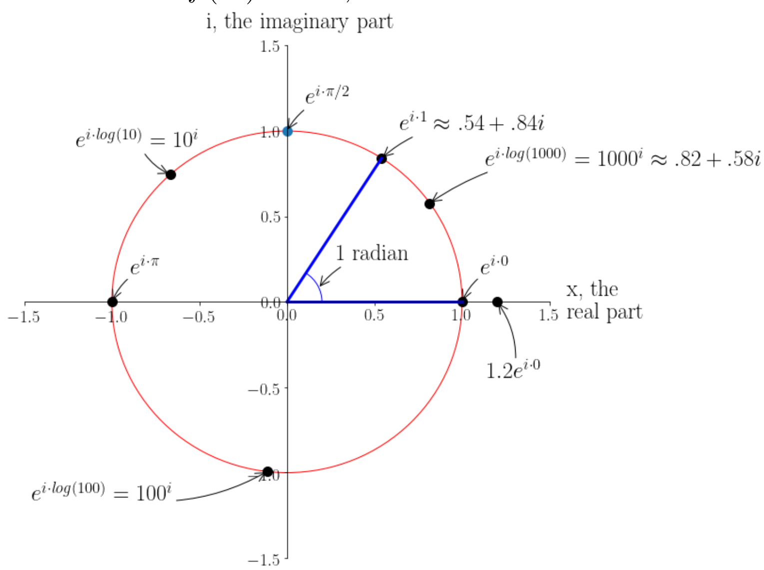
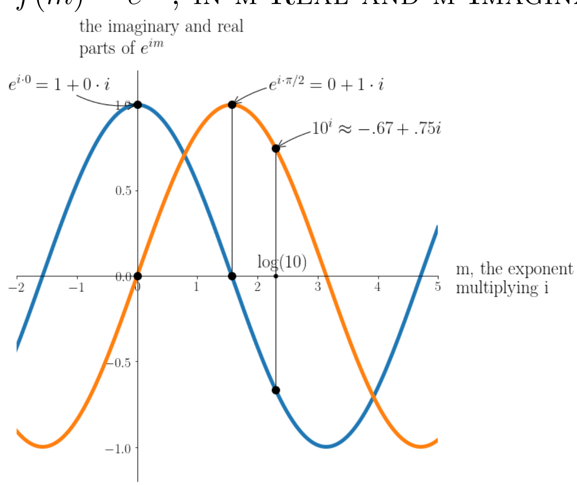


FIGURE 2: $f(m) = e^{im}$, IN M-REAL AND M-IMAGINARY SPACE



A second way to derive imaginary exponents is to start with a Taylor Series for e^x around 0, adapt it to e^{ix}, and then show that the resulting series is the sum of Taylor series's for $sin(x)$ and $cos(x)$. In general, a Taylor Series around $x = 0$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f''''(0)}{4!}(x-0)^4 + \ldots \tag{4}$$

If $f(x) = e^x$, then $f'x) = e^x$, $f''(x) = e^x$, and so forth. Since $e^0 = 1$, this means $f'(0) = 1$, $f''(0) = 1$, $f'''(0) = 1$, and so forth. Thus,

$$\begin{aligned} e^x &= e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \frac{e^0}{4!}x^4 + \frac{e^0}{5!}x^5 + \ldots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \end{aligned} \tag{5}$$

Put in ix instead of x, and we get

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^4}{5!} + \ldots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^4}{5!} + \ldots \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} + \frac{x^4}{5!} + \ldots) \end{aligned} \tag{6}$$

That last grouping of the real terms and the imaginary terms is important because it corresponds to Taylor Series around 0, for $cos(x)$ and for $sin(x)$. First, noting that $cos(0) = 1$ and $sin(0) = 0$, and $\frac{d}{dx}cos(x) = -sin(x)$ and $\frac{d}{dx}sin(x) = cos(x)$,

$$\begin{aligned} cos(x) &= cos(0) - sin(0)x - \frac{cos(0)x^2}{2!} + \frac{sin(0)x^3}{3!} + \frac{cos(0)x^4}{4!} - \frac{sin(0)x^5}{5!} + \ldots \\ &= 1 - (0)x - \frac{(1)x^2}{2!} + \frac{(0)x^3}{3!} + \frac{(1)x^4}{4!} - \frac{(0)x^5}{5!} + \ldots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \end{aligned} \tag{7}$$

Thus, $cos(x)$ is the first group in our e^{ix} Taylor Series. Doing the Taylor Series for $sin(x)$, we get

$$\begin{aligned} sin(x) &= sin(0) + cos(0)x - \tfrac{sin(0)x^2}{2!} - \tfrac{cos(0)x^3}{3!} + \tfrac{sin(0)x^4}{4!} + \tfrac{cos(0)x^5}{5!} + \dots \\ &= 0 + (1)x - \tfrac{(0)x^2}{2!} - \tfrac{(1)x^3}{3!} + \tfrac{(0)x^4}{4!} + \tfrac{(1)x^5}{5!} + \dots \\ &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + \dots \end{aligned} \tag{8}$$

Thus, $isin(x)$ is the second group in our e^{ix} Taylor Series. We can conclude that

$$e^{ix} = cos(x) + isin(x), \tag{9}$$

which is Euler's formula.

This second approach is more complicated, because we not only need to use the facts that $f''(x) = f(x)$ for $f(x) = e^x$ and that $f''(x) = -f(x)$ for $f(x) = sin(x)$ and $f(x) = cos(x)$, but also the idea of the Taylor series.