Splitting a Pie: Mixed Strategies in Bargaining under Complete Information

Abstract

We characterize the mixed-strategy equilibria for the bargaining game in which two players simultaneously bid for a share of a pie and receive shares proportional to their bids, or zero if the bids sum to more than 100%. Of particular interest is the symmetric equilibrium in which each player's support is a single interval. This consists of a convex increasing density $f_1(p)$ on $[a, 1-a]$ and an atom of probability at a, and is unique for given $a \in (0, .5)$. The two outcomes with highest probability are breakdown and a 50-50 split. We use the same approach to characterize all symmetric and asymmetric equilibria (such as "hawk-dove") that mix over a finite set of bids and for general sharing rules. We extend Malueg's 2010 proof of existence to uniqueness of equilibria with any "balanced" compact set $A \in (0, 1)$ as bid supports (but do not characterize them).

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This paper: `http://www.rasmusen.org/mixedpie.pdf`.

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1. Introduction

A fundamental problem in game theory is how to model bargaining between two players who must agree on shares of surplus or obtain zero payoff. In the classic bargaining problem "Splitting a Pie", two players bid simultaneously for shares of the pie. If their bids add up to more than one, both get a breakdown payoff of zero. The game has a continuum of pure strategy equilibria, for example, (0, 100) and (30,70). A multitude of other models of bargaining exist, of which the two best known are the Nash bargaining solution (1950) and the Rubinstein model (1982) (especially attractive with Baron & Ferejohn (1989) modifying it to allow for random choice of first mover). Nash takes the approach of cooperative game theory and finds axioms which guarantee a 50-50 split. Rubinstein takes the approach of finitely repeated games with discounting and finds close to a 50-50 split, with a small advantage to whichever player makes the first offer. In both models, the players always agree in equilibrium. Other models incorporate incomplete information, in which case failure to agree can occur in equilibrium when a player refuses to back down because he thinks, wrongly, that the other player's payoff function will cause him to agree to accept his proposal. Muthoo (1999) is the standard book-length treatment of games of complete information and the Ausubel, Cramton & Deneckere (2002) *Handbook of Game Theory* chapter treats incomplete information. Work continues on the foundations of bargaining theory, and in particular on why the split would often be 50-50; a recent example is Bastianello & LiCalzi (2019).

In this paper, we return to the classic Splitting a Pie and look at its mixed-strategy equilibria. Although examples of mixed-strategy equilibria appear as early as Roth (1985), they do not seem to have been studied systematically until Malueg (2010), who showed what a huge variety exists. In particular, he showed that for any "balanced" pair of supports, such that for each bid in a player's support there is a bid in the other player's support that sums to one with it, there exists an equilibrium, He does this for general utility functions and general sharing rules for what happens if the two players' bids add up to less than 1.

To go beyond existence, we must specify how bids translate into shares of the pie, something of no importance in pure-strategy equilibria since the bids add up to one, but which is crucial with mixed strategies, where if the bids sum to less than one it matters how much of the pie each player gets. The easiest sharing rule is the Nash (1950) "take what you bid" rule used in the "Nash demand game": if the bids are .20 and .30, Player 1 receives 20%, Player 2 receives 30%, and the other 50% is abandoned. Nash (1953) used the "split the difference" sharing rule that if the bids were .20 and .30, the shares would be .45 (= .2 + .5/2) and .55 (= .3+ .5/2). We believe the most natural specification is proportional sharing, often called "the Tullock rule", that if the bids are .20 and .30, the shares are .40 (= .2/(.2+.3)) and .60 (= .3/(.2+.3)) (Tullock [1980]; see also Flamand & Troumpounis [2015]). Proportional sharing is suitable if the model is to be a metaphor for players deciding how "tough" to be in negotiations. If both choose to be too tough, they fail to reach agreement, which is represented by complete breakdown and payoffs of zero. If they are less tough it is their relative toughness that matters, and they split the pie in proportion to toughness. Unfortunately, with either "split the difference" or the Tullock rule, the mathematics becomes difficult because a player's share depends not just on his own bid, but the other player's. For this reason, Malueg (2010) was content to only prove existence for general sharing rules, and restricted himself to rules like "take what you bid" for uniqueness and describing the equilibrium.

We will extend Malueg's results to prove uniqueness of equilibrium for general sharing rules and supports over compact sets in $(0, 1)$, as well as characterizing equilibria that include bids of 0 or 1. And we will characterize equilibria over a finite number of bids. What we will start with, however, and what we think the most interesting from an economic point of view, is characterization of the equilibria where each player mixes over a single interval and their bids translate into shares proportionally, using the Tullock rule. An example of this class of equilibrium is the symmetric equilibrium in which each player mixes over [.30, .70]. We will show that it is unique and will consist of a probability atom at .30 and a convexly increasing density over (.30, .70], and we will derive the size of the atom and the density function. The most probable outcome is either a 50% split (if both choose .30) or disagreement (if the shares they randomly choose add up to more than 100%), but with a probability density over a continuum of other shares between 30% and 70%, depending on how aggressively the players bid in a particular realization of the game. This combination of a modal 50-50 split, disagreement, and a large range of other possible shares, seems consistent with real-world behavior.

We will start by characterizing single-interval mixed-strategy equilibria, both symmetric and asymmetric, using a relatively nontechnical approach that draws out the intuition (Proposition 1). We will show what the symmetric equilibrium looks like and contrast it with the easier "take what you bid" game, and similarly construct an asymmetric equilibrium in which one player's expected payoff is higher than the other's even though the players are identical ex ante. Then we will turn to more technical methodology and derive the atom size and density function (Proposition 2). This provides a constructive proof for existence and uniqueness.

We then turn to proving existence and uniqueness for the more general problem with support in any compact subset of $(0, 1)$, including the case of equilibria over a finite number of bids and the mathematically curious case of equilibria over Cantor sets (Proposition 3). Separately, we characterize equilibria in which one player or the other bids 0 or 1 (Proposition 4). The simplest mixed-strategy equilibrium is the "hawk-dove equilibrium" in which the two player both mix between two bids; for example, 30% and 70%. What is less well-known, though illustrated by an example from Al Roth in Malueg (2010), is that there exist asymmetric hawk-dove equilibria such as Player 1 mixing between 10% and 20% and Player 2 mixing between 80% and 90%. We provide a formula for the mixing weights in any equilibrium that, like these, mixes over a finite number of bids (Proposition 5) for general sharing rules, and for what this simplifies to in our preferred case of the proportional sharing rule.

It will turn out that with a finite number of bids, the pattern of probabilities need not match the interval-of-bids pattern of high probability of the lowest bid (the atom) and then rising probability of larger bids in the support (the density). For our last result, we will show that if we increase the number of bids in the finite case to become dense in any compact set, the equilibrium probabilities do converge to the unique interval-of-bids atom and density (Proposition 6). Thus, mixing over a continuum really is a good way to describe mixing over a large number of finite bids, and not a mathematical oddity.

2. The Model

Players 1 and 2 simultaneously choose bids p and q in the interval [0,1]. If their bids add up to more than 100%, breakdown occurs and both get zero. Otherwise, when the bids are $p + q \leq 1$, Player 1 gets share $p/(p+q)$ as his payoff and Player 2 gets $q/(p+q)$. If $p = q = 0$, assume the shares are each .5. (We will later consider general sharing rules below; see assumptions ($\star$).)

This game has a continuum of pure strategy Nash equilibria: every permutation such that $p + q = 1$. There are also many mixed-strategy equilibria. Later we will address the general case of measures supported on arbitrary compact subsets of $[0,1]$. To establish intuition, we will start with the particularly interesting class of equilibrium in which the players bid by mixing over the intervals $[a,b]$ and $[c,d]$. We will see that in this case the unique equilibrium measure for player one supported on the entire interval $[a,b]$ consists of a single atom at a together with a measure in the Lebesgue class with strictly positive densities $f_1(p)$. The dual density for player two will be denoted by $f_2(q)$.

Let us begin with some qualitative features the equilibrium must have if it exists and the densities are differentiable (which we will establish later).

(1) The lower bounds are strictly between zero and one: $0 < a < b < 1$ *and* $0 < c < d < 1$.

If Player 1 plays the pure strategy of $p = 0$, his expected payoff is zero unless player 2 bids zero with an atom of probability $K_2 > 0$, in which case the payoff is $\frac{K_2}{2}$ because they will split the pie evenly. In that case, however, Player 2 could increase his payoff by $\frac{K_2}{2}$ if he switched to $K_2 = 0$. Thus, any equilibrium with $p = 0$ would require $K_2 = 0$. But then Player 1 can increase his payoff from 0 to something positive by deviating to p equalling any value above the minimum c that Player 2 might bid. Thus, Player 1's mixing distribution cannot include $p = 0$ in its support. The same argument shows that Player 2's support will not include $q = 0$, in which case neither player would wish to include a bid of 1, because that would cause the pie to explode and result in a payoff of zero.

(2) The supports are "balanced", to use Malueg's 2010 terminology: Player 2's support is $[c,d] = [1-b, 1-a]$.

Player 2 would not choose $c < 1-b$ in equilibrium because a bid of $q = c$ would earn a lower share than $q = 1-b$ and it would, like $1-b$, result in zero probability of the shares exceeding 1. He would not choose $c > 1-b$ because if $c = 1-b+x$ for some $x > 0$, Player 1's bids within $(b - x, b)$ would always make the sum of the bids exceed 1, yielding payoffs of zero for himself, so he could do better by reducing b so that $c = 1 - b$.

Player 2 would not choose $d > 1 - a$ because if $d = 1 - a + x$ for some $x > 0$, Player 2's bids within $(1 - a, 1 - a + x)$ would yields zero payoff with probability one, so he could do better by dropping those bids from his support. He would not choose $d < 1 - a$ because then if Player 1 bids $a + \epsilon$ for small enough ϵ, it will happen that $a + \epsilon + d < 1$, so the bid of $p = a + \epsilon$ will fail to cause the pie to explode for any value of q within Player 2's support and would yield Player 1 a higher share than $p = a$. Thus, Player 1 could increase his expected payoff by raising the lower bound of his support to $a + \epsilon$.

Note that feature (2) implies that in a symmetric equilibrium both players mix over $[a, 1 - a]$.

(3) The mixing distributions have probability atoms $K_1 > 0$ and $K_2 > 0$ at a and $1 - b$, and only there.

In a mixed-strategy equilibrium, every pure strategy within Player 1's support must have the same expected payoff as a response to Player 2's mixed strategy. Player 1's expected payoff is positive because Player 2 will not play more than $q = 1 - a$, and if Player 1 chooses $p = a$ breakdown will result and his expected payoff will be at least a. Consider Player 1's pure strategy of bidding $p = b$, the upper end of his support. This will cause breakdown unless Player 2 bids $q = 1 - b$, the lower end of his support. But if there is no probability atom for Player 2 at $1 - b$, the probability he bids $1 - b$ is infinitesimal, so breakdown will surely occur and Player 1's expected payoff from bidding b is zero. That is impossible in equilibrium, since zero is less than Player 1's positive payoff from bidding a. Hence, Player 2 must have a probability atom at $1 - b$, the low end of his support.

Suppose Player 2 has an atom of size K at some other point x in $(1 - b, 1 - a]$. This would induce Player 1 to deviate. Player 1's pure strategy of bidding $1 - x$ must have positive expected payoff, since that share is positive and breakdown will not occur unless $q > x$, which has probability less than 1. Player 1's pure strategy of bidding $p = 1 - x + \epsilon$ for sufficiently small ϵ, however, will have a lower expected payoff than from $1 - x$ because the expected loss from the increased probability of breakdown will be $(1 - x)K$ but the expected gain from having an ϵ higher share is less than ϵ. Note that this argument fails to apply if $x = 1 - b$, because then a pure-strategy bid of $1 - x + \epsilon$ is not in Player 1's support and does not require a payoff equal to $1 - x$.

A similar argument can be made to show that Player 1 must have an atom at a and only at a.

(4) The players' expected payoffs are $\pi_1 = bK_2$ and $\pi_2 = (1 - a)K_1$.

Every pure strategy within Player 1's support has the same expected payoff, so the expected payoff of the mixed strategy also takes that value. If Player 1 plays $p = b$, the top of his support, then his realized payoff is positive, equalling b only if $q = 1 - b$, which has probability K_2. Thus, his expected payoff is $\pi_1 = K_2 b$. Similarly, if Player 2 plays $1 - a$, his realized payoff is positive only if $p = a$, which has probability K_1, so his expected payoff is $\pi_2 = K_1(1-a)$.

(5) The mixing densities start at the levels $f_1(a) = \frac{aK_1}{1-a} > 0$ *and* $f_2(1-b) = \frac{(1-b)K_2}{b} > 0$.

If Player 2 is using density $f_2(q)$ and atom K_2 at $1-b$, then Player 1's expected payoff from the pure strategy of bidding p is

$$\pi_1(p) = K_2 \frac{p}{p+1-b} + \int_{1-b}^{1-p} \frac{p}{p+q} f_2(q)dq \tag{1}$$

for $p \in [a, b]$.

Since the payoffs from Player 1's pure-strategy best responses are the same for all p in the support, the derivative of the payoff in (1) equals zero. That derivative is

$$\begin{aligned} \frac{d\pi(p)}{dp} &= \frac{K_2}{p+1-b} - \frac{pK_2}{(p+1-b)^2} - pf_2(1-p) + \int_{1-b}^{1-p} \left(\frac{1}{p+q} - \frac{p}{(p+q)^2}\right) f_2(q)dq = 0 \\ &= \frac{(1-b)K_2}{(p+1-b)^2} - pf_2(1-p) + \int_{1-b}^{1-p} \left(\frac{q}{(p+q)^2}\right) f_2(q)dq = 0 \end{aligned} \tag{2}$$

We can rewrite this equation using the change of variables $x \equiv 1-p$ for $x \in [1-b, 1-a]$ as

$$f_2(x) = \frac{1}{1-x}\left(\frac{(1-b)K_2}{(2-x-b)^2} + \int_{1-b}^{x} \left(\frac{q}{(1-x+q)^2}\right) f_2(q)dq\right) \tag{3}$$

This implies that $f_2(q)$ is strictly positive on $[1-b, 1-a]$ and has starting level $f_2(1-b) = \frac{1-b}{b}K_2$.

For Player 2:

$$\pi_2(q) = K_1 \frac{q}{q+a} + \int_{a}^{1-q} \frac{q}{p+q} f_1(p)dp \tag{4}$$

for $p_2 \in [1-b, 1-a]$, with derivative

$$\frac{d\pi(q)}{dq} = \frac{aK_1}{(q+a)^2} - qf_1(1-q) + \int_{a}^{1-q} \left(\frac{p}{(p+q)^2}\right) f_1(p)dp = 0, \tag{5}$$

so setting $x \equiv 1-q$ for $x \in [a, b]$,

$$f_1(x) = \frac{1}{1-x}\left(\frac{aK_1}{(1-x+a)^2} + \int_{a}^{x} \left(\frac{p}{(1-x+p)^2}\right) f_1(p)dp\right) \tag{6}$$

with starting level $f_1(a) = \frac{a}{1-a}K_1$.

(6) The densities $f_1(p)$ *and* $f_2(q)$ *have strictly positive derivatives of all orders, e.g.,* $f_1' > 0, f_1'' > 0, f_1''' > 0, \ldots$

We can differentiate equation (6) to get

$$f_1'(x) = \frac{aK_1}{(1-x)^2(1-x+a)^2} + \frac{2aK_1}{(1-x)(1-x+a)^3} + \frac{xf_1(x)}{1-x} + \int_a^x \frac{2p}{(1-x)(1-x+p)^3} f_1(p)dp + \int_a^x \frac{p}{(1-x)^2(1-x+p)^2} f_1(p)dp \tag{7}$$

Differentiating again,

$$\begin{array}{rcl} f_1''(x) & = & \frac{2aK_1}{(1-x)^3(1-x+a)^2} + \frac{2aK_1}{(1-x)^2(1-x+a)^3} + \frac{2aK_1}{(1-x)^2(1-x+a)^3} + \frac{6aK_1}{(1-x)(1-x+a)^4} \\ & & + \frac{f_1(x)}{1-x} + \frac{xf_1'(x)}{1-x} + \frac{xf_1(x)}{(1-x)^2} + \frac{2xf_1(x)}{1-x} + \frac{xf_1(x)}{(1-x)^2} \\ & & + \int_a^x \frac{2p}{(1-x)^2(1-x+p)^3} f_1(p)dp + \int_a^x \frac{6p}{(1-x)(1-x+p)^4} f_1(p)dp \\ & & + \int_a^x \frac{2p}{(1-x)^3(1-x+p)^2} f_1(p)dp + \int_a^x \frac{2p}{(1-x)^2(1-x+p)^3} f_1(p)dp \end{array} \tag{8}$$

Note that $f_1' > 0$, since $x < 1$. Derivatives of every order will be positive, because they will all consist of fractions and integrals like those in equation (8). Taking the derivative will always leave the fractions positive: they start positive and will have the terms $(1-x)^m(1-x+a)^n$ in the denominator for some integers m and n, with the numerators being various multiples of a, K_1, x, and lower-order derivatives, which are all positive. Differentiating the integrals will always generate new integrals with the same bounds of a and x, which will thus remain positive. The positive sign of the derivatives of $f_2(q)$ can be shown similarly. As we will demonstrate later, this property alone implies that as the support approaches $[0,1]$ the continuous part f must approach the 0 function.

Proposition 1 collects these results. Note that the caveat about analyticity is required to allow for differentiability, because we have not yet proved that, as will be done later in Proposition 2.

Proposition 1. *Any equilibrium in which each player mixes over an interval consists for Player 1 of probability atom K_1 at a and density $f_1(p)$ on $[a,b]$ and for Player 2 of probability atom K_2 at $1-b$ and density $f_2(q)$ on $[1-b, 1-a]$. The densities are positive throughout, and if they are analytic they start at $f_1(a) = \frac{aK_1}{1-a} > 0$ and $f_2(1-b) = \frac{(1-b)K_2}{b} > 0$ and increase convexly, with positive derivatives of every order.*

3. The Shape of the Mixing Distribution and the Nash Demand Game

We have not yet demonstrated existence or uniqueness or found solutions for the atoms and the mixing densities, but it will be appropriate to discuss the intuition here and explain the difficulties that will be involved in finding the distributions. Figure 1 show what we will find when the densities are computed. It shows symmetric equilibria ($a = 1-a$) for four different supports: $(a = .1, K_1 = .24)$, $(a = .2, K_1 = .43)$, $(a = .3, K_1 = .64)$, and $(a = .4, K_1 = .83)$. The solution is more convex and with a smaller atom when a is smaller.

The derivative expression (2), which we repeat here, is helpful for intuition about the tradeoffs the players are making. It must hold for every p in Player 1's support.

$$\frac{d\pi_1(p)}{dp} = \frac{(1-b)K_2}{(p+1-b)^2} + \int_{1-b}^{1-p} \left(\frac{q}{(p+q)^2} \right) f_2(q)dq - pf_2(1-p) = 0 \tag{9}$$

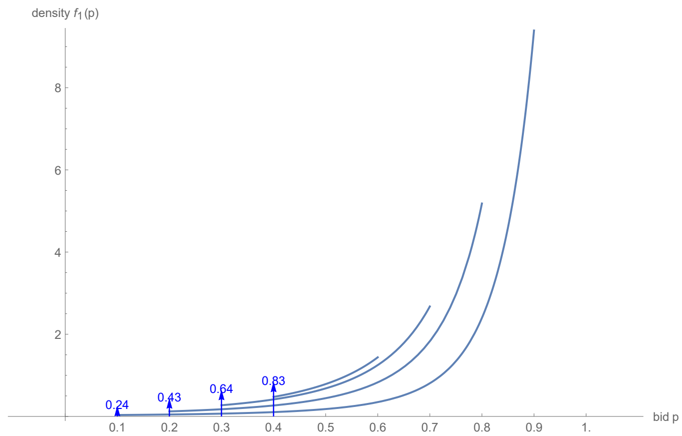

Figure 1. The Symmetric-Equilibrium Mixing Density $f_1(p)$ for $a = .10, .20, .30, .40$

The first two terms of (9) are the advantages of using a higher bid p. The first term represents the extra payoff from Player 1's share increasing when Player 2 chooses the atom bid of $1-b$, which never causes breakdown. The second term represents Player 1's extra payoff as the result of his share increasing when Player 2 bids between $1-b$ and $1-p$, an increase that is smaller (though more probable, since $f_2(q)$ is rising) for bigger q. The third term is Player 1's disadvantage from bidding higher. It represents the increase in the probability of breakdown, and the resulting loss of the share p. The crucial level of q is $q = 1-p$, since that is at the edge of breakdown. The probability of Player 2 choosing that bid is $f_2(1-p)$, and since the two player's bids add up to 1 at that bid, Player 1 lost pie share is $\frac{p}{1}$ if that happens.

To see why the equilibrium density is increasing, think of how the marginal benefits and cost change for Player 1 as p rises. Imagine the f_2 density were uniform instead of increasing. Both of the advantage terms in equation (9) would be larger for small p, because the effect on the share of increasing the bid is bigger from a lower base. Also, the second advantage term, the integral, is bigger for small p because it includes a larger probability range $[1-b, 1-p]$ over which the shares add up to less than one and breakdown is avoided. If we start with a large p,near the top of the support, then increasing it yields a greater share, but obtained with small probability; most of the expected payoff from a high p is coming from the atom of probability with which Player 2 bids $1-b$. This gives one reason why $f_2(q)$ has to be rising— so that a small p's greater range of q's that don't cause breakdown is offset by a smaller density over that range.

The second reason why f_2 must increase arises from the marginal cost of increasing p and increasing the probability of breakdown, a marginal cost of $pf_2(1-p)$. The density takes the argument $1-p$ because $q = 1-p$ is the threshold for breakdown, and hence $f_2(1-p)$ is the rate of increase of the probability of breakdown as p is increased. If f_2 were uniform, the marginal cost would increase with p, because a constant rate of increase in the probability of breakdown causes Player 1 more harm when his share is bigger. Thus, f_2 must be rising, so it is smaller for the critical $1 - p$ threshold when p is large.

There also exist asymmetric equilibria. Figure 2 shows the distributions for an equilibria in which Player 1 bids an atom of .94 at $a = .2$ and then mixes using a rising density over $[.2, .4]$. Player 2 bids an atom of .33 at $a = .6$ and then mixes using a rising density over $[.6, .8]$. These supports are balanced because $.2 + .8 = 1$ and $.4 + .6 = 1$. Player 1 must bid an atom at .2, or Player 2 would be unwilling to put positive density on his upper limit of .8, which would otherwise have zero probability of not exploding the pie. Player 2 must bid an atom at .6, or Player 1 would be unwilling to put positive density on his upper limit of .4, which would otherwise have zero probability of not exploding the pie. Player 2 will of course have a higher expected payoff, and the modal outcome will be a share of .25 for Player 1 and .75 for Player 2, as a result of bids of $p = .2$ and $q = .4$.

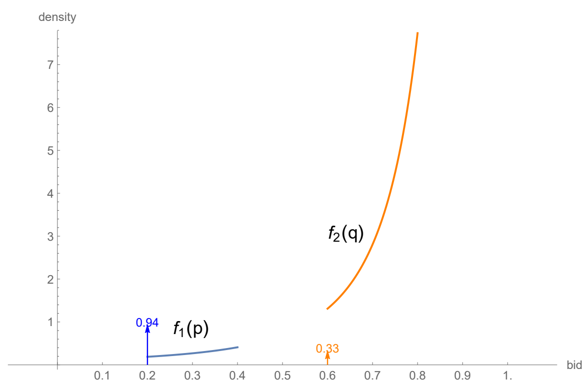

FIGURE 2. THE ASYMMETRIC-EQUILIBRIUM MIXING DENSITIES $f_1(p)$ OVER $[.2,\ .4]$ AND $f_2(q)$ OVER $[.6, .8]$

To derive the distributions we illustrate in the figures we will start with Proposition 1's conclusion that Player 2's expected payoff if the equilibrium exists is $K_1(1 - a)$. Since all pure-strategy payoffs are the same, we know that for every q in Player 2's support,

$$\pi_2(q) = K_1 \frac{q}{q+a} + \int_a^{1-q} \frac{q}{q+p} f_1(p)dp = K_1(1-a) \quad (\textit{the crucial equation}) \tag{10}$$

With the Tullock rule a player's payoff depend on the other player's bid not just through whether breakdown occurs but through its effect on his share when the bids add up to less than 1.

Consider, in contrast, the "Nash demand game" of Nash (1950), in which the players use the "take what you bid" sharing rule when the players come to agreement and almost always discard some or all of the pie in equilibrium.[1] In the Nash demand game, Player

[1]This is the game whose solution is given in Malueg (2010) as his Proposition 4.

2's share is independent of the other player's bid except via the effect on the probability of breakdown, so the equilibrium condition analogous to equation (10) is

$$\pi_2(q) = K_1 q + \int_a^{1-q} q f_1(p)dp = K_1(1-a) \tag{11}$$

A bid by Player 2 of $1-b$ yields a payoff of $1-b$ with probability 1 in the Nash demand game, so $\pi_2(1-b) = 1-b$. A bid of $1-a$ yields a payoff of $1-a$ with probability K_1 (when Player 1 bids exactly a) and 0 otherwise. Since the expected payoffs have to be equal in equilibrium, it follows that $1-b = K_1(1-a)$, and we can conclude that $K_1 = \frac{1-b}{1-a}$. Then our last equation becomes, if we let $F_1(x)$ represent the cumulative probability for $f_1(x)$,

$$\pi_2(q) = \frac{1-b}{1-a}q + F_1(1-q)q = 1-b, \tag{12}$$

which solves, using the change of variables $q = 1-p$, to

$$F_1(p) = \frac{1-b}{1-p} - \frac{b}{1-a} \tag{13}$$

so

$$f_1(p) = \frac{1-b}{(1-p)^2} \tag{14}$$

Similarly, equating Player 1's payoffs yields $\pi_1(a) = a = \pi_1(b) = K_2 b$, so $K_2 = \frac{a}{b}$. Then $\pi_1(p) = \frac{a}{b}p + F_2(1-p)p = a$ and

$$f_2(q) = \frac{a}{(1-q)^2} \tag{15}$$

In the Nash demand game, the equilibrium mixing densities are strictly greater than zero and increasing, as we found for the proportional sharing rule earlier. For Player 1, for example, the density goes from $f_1(a) = \frac{1-b}{(1-a)^2}$ to $f_1(b) = \frac{1-b}{(1-b)^2}$. Construction of the density functions demonstrates existence and uniqueness (for given a and b of equilibria that mix over intervals).

Figure (3) shows the densities for the two sharing rules for the support [.30, .70]. The atom is about .43 for the Nash demand game and .64 for the bargaining game with proportional sharing. The densities appear similar except at the upper bids near .70, but the difference is not just proportional to the difference in atoms, and using the Nash demand density for the bargaining game would result in payoffs declining in the bid. In the Nash demand game, bidding the lower limit, a, results in a payoff of only a, whereas it often yields a share of 50% in the bargaining game and it always yields more than a (except in the zero-measure case when the other player bids b). Increasing the atom increases the payoff to bidding b more than it increases the payoff to bidding a and hence is necessary for all pure-strategy payoffs to equal each other in the bargaining game.

In the next section we will use a power series approach to solve our crucial equation (10) for $f_1(p)$ and K_1 for given bid support.

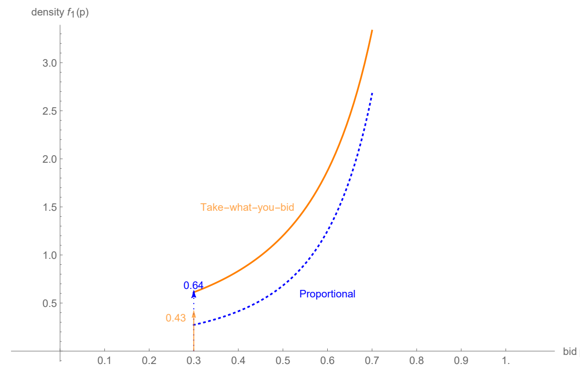

FIGURE 3. THE SYMMETRIC-EQUILIBRIUM MIXING DENSITY $f_1(p)$ FOR THE NASH DEMAND AND PROPORTIONAL SHARING GAMES WITH $a = .30$

4. FINDING THE EQUILIBRIUM MIXING DISTRIBUTION FOR CONTINUOUS SUPPORT

We will now proceed to find a solution for the mixing density. This will require somewhat more formality. Let us assume, for now, that the mixing distribution for Player 1, μ_1, is a measure on $[a, b]$ consisting of a single dirac measure at a, as is necessary, together with a positive measure in the Lebesgue class,

$$\mu_1(p) = K_1 \delta_a(p) + f_1(p)dp,$$

where $K_1 = \frac{\pi_0}{1-a}$, with expected payoff π_0, and f_1 is any function in $L^1([a, b])$ subject to the normalization that μ_1 is a probability measure. The fundamental equation (10) that characterizes the equilibrium then says that Player 1's f_1 and K_1 must be such that Player 2's expected payoff from any q in his support equals that from playing $q = 1 - a$ (with an equivalent condition for Player 2's f_2 and K_2 making Player 1's payoffs equal):

$$\pi_2(q) = \frac{\pi_0}{1-a}\frac{q}{q+a} + \int_a^{1-q} \frac{q}{q+p} f_1(p)dp = \pi_0 \quad (\textit{the crucial equation}) \tag{16}$$

Our aim will be to find the derivatives of f_1 and use them to construct a power series statement of the function. To reduce clutter, set $m(p) \equiv \frac{1-a}{a\pi_0} f_1(p)$ for some other function $m \in L^1([a, b])$. Then the equation can be rewritten as

$$\frac{\pi_0}{1-a}\frac{q}{(q+a)} + \int_a^{1-q} \frac{q}{q+p}\frac{a\pi_0}{1-a} m(p)dp = \pi_0. \tag{17}$$

Moving the first term, the contribution from the point mass, to the right-hand side and dividing by $\frac{a\pi_0}{1-a}$ yields

$$\int_a^{1-q} \frac{q}{q+v} m(p)dp = \left(\frac{1-a}{a} - \frac{q}{a(q+a)}\right) = \frac{1-a-q}{a+q} \tag{18}$$

for all $q \in [0, 1-a]$.

Formally[2] differentiating both sides of (18) with respect to the variable q yields, since the equation is true for all q in the interval $[0, 1-a]$,

$$-\,qm(1-q) + \int_a^{1-q} \left(\frac{1}{q+p} - \frac{q}{(q+p)^2} \right) m(p)\, dp = -\frac{1-a-q}{(a+q)^2} - \frac{1}{a+q} \tag{19}$$

which simplifies to

$$qm(1-q) - \int_a^{1-q} \frac{p}{(q+p)^2} m(p)\, dp = (q+a)^{-2} \tag{20}$$

We obtain, for example by setting $1-q = a$, that $m(a) = \frac{1}{1-a}$, which gives the $f_1(a) = \frac{aK_1}{1-a}$ we found in Proposition 1. Again formally taking the first derivative with respect to q of both sides of equation (20), and multiplying through by -1, we obtain

$$qm'(1-q) - (2-q)m(1-q) - 2\int_a^{1-q} \frac{p}{(q+p)^3} m(p)\, dp = 2(q+a)^{-3}$$

Recursively define $r_i(q)$ by

$$r_0(q) = qm(1-q) \quad \text{and} \quad r_i(q) = -r'_{i-1}(q) - i!\,(1-q)\,m(1-q) \quad \text{for} \quad i > 0. \tag{21}$$

Our objective is to obtain expressions for the formal derivatives of $m^{(i)}(1-q)$ that we can use to construct the f_1 density. The linear recursion relation (21) can be solved to get $r_n(q)$ in terms of all the $m^{(i)}$ from $i = 0$ to n instead of in terms of $r'_{n-1}(q)$, yielding

$$r_n(q) = qm^{(n)}(1-q) - \sum_{i=0}^{n-1} \left((i+1)(n-1-i)! + (n-i)!(1-q) \right) m^{(i)}(1-q). \tag{22}$$

Formally taking the n-th derivative of equation (20) in $1-q$ and multiplying by (-1) we obtain,

$$r_n(q) + (n+1)! \int_a^{1-q} \frac{p}{(q+p)^{n+2}} m(p)\, dp = (n+1)!(q+a)^{-(n+2)}. \tag{23}$$

Evaluating (23) at $q = 1-a$ gives $r_n(1-a) = (n+1)!$ since the integral vanishes. We can substitute $r_n(1-a) = (n+1)!$ into (22) in evaluating $r_n(q)$ at $q = 1-a$ to get

$$(n+1)! = (1-a)m^{(n)}(a) - \sum_{i=0}^{n-1} ((i+1)(n-1-i)! + (n-i)!a)m^{(i)}(a), \tag{24}$$

which after rearranging terms becomes

$$m^{(n)}(a) = \frac{(n+1)!}{(1-a)} + \sum_{i=0}^{n-1} \frac{((i+1)(n-1-i)! + (n-i)!a)}{1-a} m^{(i)}(a). \tag{25}$$

Equation (25) is a recursive formula yielding the formal derivative $m^{(n)}(a)$ in terms of $m^{(i)}(a)$ for $i < n$, and hence ultimately in terms of a. This immediately implies that all of

[2]"Formally" because we have not shown $m(p)$ is differentiable yet, which we will justify a-posteriori.

the $m^{(i)}(a)$ are positive, since $m^{(0)}(a) = m(a)$ is positive. Hence, the mixing density rises and is convex at a. We will able to use this to find a power series solution for $m(p)$ if we can find an appropriate bound for the sum of its formal derivatives, which we will do next as Lemma 1. We alert the reader that Lemma 1's proof runs a few pages. However, together with its corollaries, it yields an immediate proof of Proposition 2 which gives the explicit formula for the equilibrium strategy.

Lemma 1. *The formal derivatives $m^{(n)}(a)$ arising in the formal power series for $m(p)$ at $v = a$ satisfy the double-sided bounds*

$$\frac{n!}{(1-a)^n} \leq m^{(n)}(a) \leq \frac{(n+3)!(n+1)}{(1-a)^{n+1}}. \tag{26}$$

Proof. We will proceed inductively for both bounds, and we treat the upper bound first. First observe that the proposition is true for the base case, the original function's value $m^{(0)}(a)$:

$$m^{(0)}(a) = m(a) = \frac{1}{(1-a)} \leq \frac{(0+3)!(0+1)}{(1-a)^{0+1}}$$

We will show that if the proposition holds for all derivatives lower than n, then it holds for $m^{(n)}(a)$ too. Thus, the inductive step that we assume is that

$$m^{(i)}(a) \leq \frac{(i+3)!(i+1)}{(1-a)^{i+1}} \text{ for all } 0 \leq i \leq n-1 \tag{27}$$

Equation (25) gave us an expression for $m^{(n)}(a)$ in terms of a sum of the lower derivatives i multiplied by $\frac{((i+1)(n-1-i)!+(n-i)!a)}{1-a}$. Let us multiply both sides of (27) by that and rearrange:

$$\begin{aligned}
\frac{((i+1)(n-1-i)!+(n-i)!a)}{1-a} m^{(i)}(a) &\leq \frac{((i+1)(n-1-i)!+(n-i)!a)}{(1-a)^{i+2}}(i+3)!(i+1) \\
&\leq \frac{(n-i)!(\frac{i+1}{n-i}+a)(i+3)!(i+1)}{(1-a)^{i+2}} \\
&\leq \frac{(i+1)(n-i)!(i+3)!(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{(1-a)^{n+1}} \\
&\leq (n+3)!\frac{(i+1)(n+3-(i+3))!(i+3)!(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{(n+3)!(1-a)^{n+1}} \\
&\leq (n+3)!\frac{(i+1)(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{\binom{n+3}{i+3}(1-a)^{n+1}} \\
&\leq (n+3)!\frac{(i+1)(i+2)}{\binom{n+3}{i+3}(1-a)^{n+1}}
\end{aligned} \tag{28}$$

This last inequality holds for all positive n and $i < n$, and establishes a bound for each of the terms in the sum of derivatives making up $m^{(n)}(a)$ in (25). To check $m^{(n)}(a)$ we then sum the right hand side bounds for $i = 0, \ldots, n-1$, and add $\frac{(n+1)!}{(1-a)}$ from (25) to get,

$$
\begin{aligned}
m^{(n)}(a) &= \frac{(n+1)!}{(1-a)} + \sum_{i=0}^{n-1} \frac{((i+1)(n-1-i)! + (n-i)!a)}{1-a} m^{(i)}(a) \\
&\leq \frac{(n+1)!}{(1-a)} + \sum_{i=0}^{n-1} (n+3)! \frac{(i+1)(i+2)}{\binom{n+3}{i+3}(1-a)^{n+1}} \\
&\leq \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{(1-a)^n}{(n+3)(n+2)(n+1)} + \sum_{i=0}^{n-1} (i+1) \binom{n+3}{i+3}^{-1} \right) \\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{(1-a)^n}{(n+3)(n+2)(n+1)} + \frac{6}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} \right. \\
&\qquad \left. + \frac{2(n-1)}{(n+3)(n+2)} + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + \sum_{i=1}^{n-4} (i+1) \binom{n+3}{i+3}^{-1} \right) \\
&\leq \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{1}{(n+3)(n+2)(n+1)} + \frac{6}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} \right. \\
&\qquad \left. + \frac{2(n-1)}{(n+3)(n+2)} + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + (n-3) \sum_{i=1}^{n-4} \binom{n+3}{4}^{-1} \right) \\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{7}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} + \frac{2(n-1)}{(n+3)(n+2)} \right. \\
&\qquad \left. + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + \frac{24(n-3)(n-4)}{(n+4)(n+2)(n+1)n} \right) \\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(1 + \frac{-n^3 + 21n^2 - 181n + 288}{(n+4)(n+2)(n+1)n} \right) \\
&\leq \frac{(n+3)!(n+1)}{(1-a)^{n+1}}.
\end{aligned}
$$

The second inequality results from pulling out the $\frac{(n+3)!(n+1)}{(1-a)^{n+1}}$ factor and noting that $i+2 \leq n+1$ for $i \leq n-1$. The next equality results from expanding the $i = 0, n-1, n-2, n-3$ terms of the sum. The following inequality results from noting that $(1-a)^n \leq 1$ and also in the range $1 \leq i \leq n-4$ we have both $i+1 \leq n-3$ and $\binom{n+3}{i+3} = \binom{n+3}{n-i} \geq \binom{n+3}{4}$. The next equality simply records the fact that $(n-3)(n-4)\binom{n+3}{4}^{-1} = \frac{24(n-3)(n-4)}{(n+4)(n+2)(n+1)n}$, and the ensuing algebra gives the last equality. The final inequality follows from the fact that the numerator polynomial $-n^3 + 21n^2 - 181n + 288$ has leading negative term and a single real root in the interval $(2, 3)$. Hence it is negative for all integers $n \geq 3$. The decomposition of the sum we used, also holds only when $n \geq 4$. We have checked the $n = 0$ base case already, so it remains to check the inductive step for $n = 1, 2$, and 3.

For $n = 1$ we have,

$$
\begin{aligned}
m^{(1)}(a) &\leq \frac{(1+3)!(1+1)}{(1-a)^{1+1}}\left(\frac{(1-a)^1}{(1+3)(1+2)(1+1)} + \sum_{i=0}^{1-1}(i+1)\binom{1+3}{i+3}^{-1}\right)\\
&= \frac{(1+3)!(1+1)}{(1-a)^{1+1}}\left(\frac{(1-a)}{24} + \frac{1}{4}\right)\\
&\leq \frac{(1+3)!(1+1)}{(1-a)^{1+1}}.
\end{aligned}
$$

For $n = 2$ we have,

$$
\begin{aligned}
m^{(2)}(a) &\leq \frac{(2+3)!(2+1)}{(1-a)^{2+1}}\left(\frac{(1-a)^2}{(2+3)(2+2)(2+1)} + \sum_{i=0}^{2-1}(i+1)\binom{2+3}{i+3}^{-1}\right)\\
&= \frac{(2+3)!(2+1)}{(1-a)^{2+1}}\left(\frac{(1-a)^2}{60} + \frac{1}{2}\right)\\
&\leq \frac{(2+3)!(2+1)}{(1-a)^{2+1}}.
\end{aligned}
$$

Finally, for $n = 3$ we have,

$$
\begin{aligned}
m^{(3)}(a) &\leq \frac{(3+3)!(3+1)}{(1-a)^{3+1}}\left(\frac{(1-a)^3}{(3+3)(3+2)(3+1)} + \sum_{i=0}^{3-1}(i+1)\binom{3+3}{i+3}^{-1}\right)\\
&= \frac{(3+3)!(3+1)}{(1-a)^{3+1}}\left(\frac{(1-a)^3}{120} + \frac{41}{60}\right)\\
&\leq \frac{(3+3)!(3+1)}{(1-a)^{3+1}}.
\end{aligned}
$$

Thus we obtain the upper bound of the lemma:

$$
m^{(n)}(a) \leq \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \quad \text{for all} \quad n \geq 0. \tag{29}
$$

For the lower bound we proceed similarly, first we consider the general case assuming the inductive hypothesis.

$$
\begin{aligned}
m^{(n)}(a) &= \frac{(n+1)!}{(1-a)} + \sum_{i=0}^{n-1}\frac{((i+1)(n-1-i)! + (n-i)!a)}{1-a} m^{(i)}(a)\\
&\geq \frac{(n+1)!}{(1-a)} + \sum_{i=0}^{n-1}\frac{((i+1)(n-1-i)! + (n-i)!a)}{1-a}\frac{i!}{(1-a)^i}\\
&\geq \frac{(n+1)!}{(1-a)} + \frac{n! + a(n-1)!}{(1-a)^n} + \sum_{i=0}^{n-2}\frac{((i+1)(n-1-i)! + (n-i)!a)i!}{(1-a)^{i+1}}\\
&\geq \frac{n!}{(1-a)^n}
\end{aligned}
$$

where we have simply pulled the $n-1$ term out of the sum in the third line. Note that we will need to check just the base case, $n=0$, as the above calculation holds for all $n \geq 1$.

For the $n=0$ case we check,

$$m^{(0)}(a) = m(a) = \frac{1}{(1-a)} \geq \frac{(0)!}{(1-a)^0} = 1, \tag{30}$$

as desired. This completes the lower bound and the proof. □

Note also that Lemma 1's inequality implies that

$$\begin{aligned} m^{(n)}(a)\frac{(p-a)^n}{n!} &\leq ka\frac{(n+3)!}{n!}\frac{(1+a)^{n+2}}{(1-a)^{n+2}}(p-a)^n \\ &\leq \pi_0 \frac{1+a}{1-a}(n+1)\left(\frac{(1+a)(p-a)}{1-a}\right)^n \end{aligned} \tag{31}$$

Lemma 1 has two important corollaries:

Corollary 1.1. *The power series,*

$$\sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i \tag{32}$$

converges uniformly for all $p \in (-1+2a, 1)$. Moreover, it has a pole at $p=1$.

Proof. By Lemma 1, whenever $-(1-a) < p-a < 1-a$ or equivalently, $-1+2a < p < 1$, we have

$$\left| \frac{m^{(i)}(a)}{i!}(p-a)^i \right| \leq \frac{(i+3)(i+2)(i+1)^2}{(1-a)}\epsilon^i$$

where $\epsilon = \left|\frac{p-a}{1-a}\right| < 1$. In particular, this is the summand of an absolutely (and exponentially) convergent power series. On the other hand, the lower bound of Lemma 1 shows that

$$\frac{m^{(i)}(a)}{i!}(1-a)^i \geq 1$$

and hence the series diverges at $p=1$. □

Corollary 1.2. *The function m defined on the interval $[a,b]$ for any $0 < a < b < 1$ given by*

$$m(p) = \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i$$

where the $m^{(i)}(a)$ are computed from (25) is real-analytic on $[a,b]$ and yields the unique measurable function solving (18).

Proof. First observe that $[a, b] \subset (-1 + 2a, 1)$. Since the series converges on the domain of definition by Lemma 1, it is real-analytic there. (Indeed it converges and is analytic on a larger open interval.) Let $m_1 = m$. Any other solution m_2 that is infinitely differentiable at $v = a$ must have the same derivatives at $v = a$ by the construction above. However, an alternate solution m_2 may not have this regularity. Nevertheless, taking the difference of equation (18) for $m = m_1$ and $m = m_2$ shows that we must have for $f = m_1 - m_2$,

$$\int_a^{1-q} \frac{q}{q+p} f(p) dp = 0$$

However the functions $\frac{q}{q+p}$ in p are linearly independent in $L^1([a, 1-q])$ for all $q < 1 - a$ and hence $f = 0$ almost everywhere, yielding equivalent measures in the Lebesgue class. □

Remark 1. *In Proposition 6 we will see that we have uniqueness and existence of an equilibrium mixed strategy supported (exactly) on any desired compact subset of* $(0, 1)$ *for a more general family of payoff functions. (For example, these could be supported on uncountable sets such as Cantor sets or multiple intervals.)*

Remark 2. *One might like to have* $m(p)$ *be the solution to an ordinary linear differential equation (with preferably simple coefficient functions). However the fundamental defining equation 43 provides an integro-differential equation equating the payoffs. The complexity of our recursion equation for* m *suggests that there is no way to decouple the* $m(p)$ *into an ordinary linear differential equation.*

Applying Corollary 1.2 and our earlier results to Splitting a Pie, we obtain Proposition 2 below. We want to find our original probability measure $d\mu_1(p) = K_1\delta_a(p) + f_1(p)dp = K_1\delta_a + aK_1 m(p)dp$. Integrating over the full support $[a, b]$ gives us 1, so $K_1 = \frac{1}{1+\int_a^b am(p)dp}$. We can also put this in terms of the power series expansion for $m(p)$:

$$\begin{aligned}\int_a^b m(p)dp &= \int_a^b \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i dp = \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!} \int_a^b (p-a)^i dp \\ &= \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!} \frac{(b-a)^{i+1}}{i+1} = \sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1} \qquad (33)\end{aligned}$$

With this last manipulation, we have proven Proposition 2.

Proposition 2. *Splitting a Pie has a unique equilibrium mixing strategy among Borel measures supported on closed intervals. Players 1 and 2 use increasing, convex (with all derivatives positive), real-analytic densities $f_1(p)$ on $[a,b]$ and $f_2(q)$ on $[1-b,1-a]$ with $0<a<b<1$ and atoms of probability K_1 at a and K_2 at $1-b$. Specifically,*

$$f_1(p) = aK_1 \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i, \quad K_1 = \frac{1}{1+a\int_a^b m(p)dp} = \frac{1}{1+a\sum_{i\geq 0}\frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1}}, \tag{34}$$

and

$$f_2(q) = (1-b)K_2 \sum_{i\geq 0} \frac{m^{(i)}(1-b)}{i!}(q-(1-b))^i, \tag{35}$$

$$K_2 = \frac{1}{1+(1-b)\int_{1-b}^{1-a} m(q)dq} = \frac{1}{1+(1-b)\sum_{i\geq 0}\frac{m^{(i)}(1-b)}{(i+1)!}(b-a)^{i+1}} \tag{36}$$

with $m^{(i)}(\cdot)$ for both densities as:

$$m^{(i)}(x) = \frac{(i+1)!}{(1-x)} + \sum_{j=0}^{i-1} \frac{((j+1)(i-1-j)! + (i-j)!x)}{1-x} m^{(j)}(x). \tag{37}$$

Remark 3. *We note here that as $a \to 0$, our formula shows that f_1 tends to the constant 0 function on $[0,b]$. Hence, $K_1 = 1$, and $\mu_1 = \delta_1$. Proposition 4 will show that $\mu_2 = \delta_1$ in this case. In particular $K_2 = 0$ and f_2 vanishes on $[1-b,1)$ pointwise but limits to δ_1 in the weak-* sense. Similarly, as $b \to 1$ f_2 tends to the constant 0 function on $[0,1-a]$ and $K_2 = 1$ so $\mu_2 = \delta_0$ and Proposition 4 will show that $\mu_1 = \delta_1$. In particular $K_1 = 0$ and f_1 vanishes on $[a,1)$ pointwise but weak-* limits to δ_1. When both $a \to 0$ and $b \to 1$ then f_1 and f_2 both vanish on $[0,1)$ pointwise. Proposition 4 will again show that depending on how a and b limit as a pair we obtain that either $K_1 = 0$ and f_1 limits to $\mu_1 = \delta_1$ and $K_2\delta_a + f_2$ limits to $\mu_2 = \alpha\delta_0 + (1-\alpha)\delta_1$ for any $\alpha \in [0,1]$ or else f_2 limits to $\mu_2 = \delta_1$ and $K_1\delta_a + f_1$ limits to $\mu_1 = \alpha\delta_0 + (1-\alpha)\delta_1$ for any $\alpha \in [0,1]$.*

Also, our formula together with the bounds of Lemma 1 gives us good estimates on K_1 and K_2. Our lower bound gives,

$$\int_a^b m(p)dp = \sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1} \geq \sum_{i\geq 0} \frac{(b-a)^{i+1}}{(i+1)(1-a)^i} = (1-a)\log\frac{1-a}{1-b}.$$

Our upper bound gives,

$$\int_a^b m(p)dp = \sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1} \leq \sum_{i\geq 0} \frac{(i+3)(i+2)(i+1)}{(1-a)^{i+1}}(b-a)^{i+1} = \frac{6(1-a)^3(b-a)}{(1-b)^4}.$$

Hence we obtain,

$$\frac{1}{1+\frac{6a(1-a)^3(b-a)}{(1-b)^4}} \leq K_1 \leq \frac{1}{1+a(1-a)\log\frac{1-a}{1-b}}.$$

Similarly for K_2 we obtain,

$$\frac{1}{1+\frac{6(1-b)b^3(b-a)}{a^4}} \leq K_2 \leq \frac{1}{1+b(1-b)\log\frac{b}{a}}.$$

It follows easily that, $\lim_{b\to 1} K_1 = 0$, $\lim_{a\to 0} K_1 = 1$, $\lim_{b\to 1} K_2 = 1$ *and* $\lim_{a\to 0} K_2 = 0$. *For limits as* $a \to 0$ *and* $b \to 1$ *one of* K_1 *or* K_2 *must vanish and the other can be any value in* $[0, 1]$ *depending on the relative speed of convergence of the double limit. This removes the need to use Proposition 4 in the previous remark.*

The terms $m^{(i)}(a)$ in Proposition 2 need to be calculated recursively. We have $m^{(0)}(a) = 1/(1-a)$. Then,

$$\begin{aligned}
m^{(1)}(a) &= \frac{(1+1)!}{(1-a)} + \frac{((0+1)(1-1-0)! + (1-0)!a)}{1-a}\frac{1}{1-a} \\
&= \frac{3-a}{(1-a)^2} \\
m^{(2)}(a) &= \frac{(2+1)!}{(1-a)} + \frac{((0+1)(2-1-0)! + (2-0)!a)}{1-a}\frac{1}{1-a} + \frac{((1+1)(2-1-1)! + (2-1)!a)}{1-a}\frac{3-a}{(1-a)^2} \\
&= \frac{13-10a+3a^2}{(1-a)^3} \\
m^{(3)}(a) &= \frac{(3+1)!}{(1-a)} + \frac{((0+1)(3-1-0)! + (3-0)!a)}{1-a}\frac{1}{1-a} \\
&\quad + \frac{((1+1)(3-1-1)! + (3-1)!a)}{1-a}\frac{3-a}{(1-a)^2} + \frac{((2+1)(3-1-2)! + (3-2)!a)}{1-a}\frac{13-10a+3a^2}{(1-a)^3} \\
&= \frac{71-89a+55a^2-13a^3}{(1-a)^4}
\end{aligned}$$

They will be of the form

$$m^{(i)}(a) = \frac{q_i(a)}{(1-a)^{i+1}},$$

where $q_i(a)$ is a degree i polynomial in a with integer coefficients. The first four $q_i(a)$ are $q_0(a) = 1, q_1(a) = 3-a, q_2(a) = 13-10a+3a^2$ and $q_3(a) = 71-89a+55a^2-13a^3$. Thus,

$$\begin{aligned}
f_1(p) = aK_1\Bigg(&\frac{1}{(1-a)} + \frac{3-a}{(1-a)^2}(p-a) + \frac{13-10a+3a^2}{2(1-a)^3}(p-a)^2 \\
&+\frac{71-89a+55a^2-13a^3}{6(1-a)^4}(p-a)^3 + \sum_{i\geq 4}\frac{m^{(i)}(a)}{i!}(p-a)^i\Bigg),
\end{aligned}$$

Note that the equations for f_1 and f_2 in Proposition 2 are not approximations, but exact, in the sense that the infinite series converges. If only a finite number of terms are included, it becomes an approximation. In this, it is like the formula $\pi = \sqrt{\sum_{n=1}^{\infty}\frac{6}{n^2}}$, since the exact value of π cannot be represented with a finite number of rational terms.

The equilibrium densities have two surprising features. First, both $f_1(p)$ and $f_2(q)$ are composed of the same $m^i(\cdot)$ functions, even in an asymmetric equilibrium where the two players use different mixing supports. Second, the upper bounds of the support only enter into the expression for the atomic weights K_1 and K_2. The densities in the 'Nash demand game' actually have these features also. There, the densities were $f_1(p) = \frac{a}{(1-p)^2}$ and $f_2(q) = \frac{1-b}{(1-q)^2}$, both functions of the form $f(x) = \frac{x}{(1-p)^2}$, and the weights were $K_1 = \frac{1-b}{1-a}$ and $K_2 = \frac{a}{b}$.

5. Equilibria Mixing over a Finite Number of Bids

We have so far looked at equilibria mixing over one or more continuous intervals of bids, with particular focus on the symmetric equilibrium in which both player mix over a single interval. We will next look at equilibria mixing over a set of discrete actions. These have been analyzed by Malueg (2010), whose Proposition 2 says:

Proposition (Malueg)**.** *Let A and B be nonempty finite subsets of the open interval (0, 1) such that if and only if element s is in A then* $1-s$ *is in B. There exists a unique equilibrium in which Player 1 mixes over A and Player 2 mixes over B.*

Note that this proposition includes the general definition of balance: the players' bid supports are *balanced* if and only if when element s is in Player 1's bid support, A, $1-s$ is in Player 2's bid support, B.

Malueg proves existence and uniqueness using a sharing rule that generalizes both the "take what you bid" sharing rule of the Nash demand game and the game we have been analyzing so far in which a player receives his bid's proportion of the total bids. He goes on to find the equilibrium under the "take what you bid" rule, but not for our game or other rules, which we will next do.

To start, suppose the two players both mix over just two bids, and they use the same two bids. This is a hawk-dove equilibrium, mathematically the same as the well-known biological model of birds deciding whether to pursue aggressive or pacific strategies. Each player chooses a with probability θ and b with probability $1-\theta$ for $a \leq .5$. These must have $a+b=1$, because otherwise it would be a profitable deviation to raise the lower bid, which would increase the player's share without increasing the probability of breakdown. Thus, the balancing condition applies and we need $b = 1-a$. The mixing probability must make the expected payoff π_0 of each action the same in equilibrium, so

$$\pi_0 = \pi(a) = \theta(.5) + (1-\theta)a = \pi(1-a) = \theta(1-a) + (1-\theta)(0), \tag{38}$$

which solves to $\theta = 2a$ and $\pi_0 = 2a(1-a)$. The players share the pie equally in equilibrium with probability a^2 and bargaining breaks down with probability $(1-a)^2$. Note that there are a continuum of equilibria and they can be pareto-ranked, with higher payoffs if a is closer to .5. In the limiting equilibrium, both players choose $a = 0$ with probability 0 and $1 - a = 1$ with probability 1, and the expected payoff is zero.[3]

This has many of the same attractive features as the symmetric single-interval mixed-strategy equilibrium. A common outcome is a 50-50 split; breakdown can occur; and unequal shares are possible too. It has unattractive features too. Of the three possible outcomes– 50-50 shares, unequal shares, and breakdown— the unequal share is most frequent, then breakdown, and then equal shares.

We have not seen the hawk-dove bargaining equilibrium in print, but it is well known that bargaining is like a hawk-dove game. What is more surprising is that there also exist asymmetric hawk-dove equilibria, something Malueg (2010) points out. Consider the following example. Player 1 chooses a with probability θ and $1-b$ with probability $(1-\theta)$. Player 2 chooses b with probability γ and $1-a$ with probability $(1-\gamma)$. Player 1's expected payoffs must be equal from his two pure strategies, so

$$E\pi_1(a) = \gamma\frac{a}{a+b} + (1-\gamma)(a) = E\pi_1(1-b) = \gamma(1-b) + (1-\gamma)(0) \rightarrow \gamma = \frac{a(a+b)}{a^2-b^2+b} \tag{39}$$

Player 2's expected payoffs must be equal from his two pure strategies, so

$$E\pi_2(b) = \theta\frac{b}{a+b} + (1-\theta)(b) = E\pi_2(1-a) = \theta(1-a) + (1-\theta)(0) \rightarrow \theta = \frac{b(a+b)}{b^2-a^2+a} \tag{40}$$

Thus we might have Player 1 choosing .1 and .4 with probabilities of about .28 and .72, for a payoff of about .11; and Player 2 choosing .6 and .9 with probabilities of about .93 and .07, for a payoff of about .84. Note that this is similar in character to the asymmetric continuous-support equilibrium illustrated in Figure 2 earlier in the paper.

The same 4 points of support are also part of a symmetric equilibrium, in which both players mix using the same probabilities. Malueg's theorem says that there is a unique equilibrium for given supports of players 1 and 2; it does not say that there is a unique equilibrium independent of how the supports are divided between the players (which is false).[4]

There are also equilibria with more than two discrete actions. Suppose there are three bids used for mixing. The lowest of them must be chosen so it and the highest add up to one, to satisfy the balancing condition. The middle bid must equal .5 to satisfy the balancing condition, since it will have no matching bid with which to sum to one. Any equilibrium

[3]The Nash demand sharing rule yields a slightly different result. There, the payoff is $\pi(a) = \theta a + (1-\theta)a = \pi(1-a) = \theta(1-a)$, so $\theta = \frac{a}{1-a}$ and $\pi_0 = a$. This θ is increasing in a, as we found for the Tullock sharing rule. Note that $\theta < 1/2$ for low values of a (for $a < 1/3$ in the Nash demand version and for $a < 1/4$ for the proportional-sharing version), so the players actually choose the higher of their two bids (hawk) with the highest probability, unlike in the standard hawk-dove game.

[4] If the support is $\{.1, .4, .6, .9\}$ for each player, the probabilities can be calculated to be approximately $\{.21, .01, .03, .75\}$.

over $\{a, .5, b\}$ must have $b = 1 - a$ and if symmetric we additionally equate:

$$\begin{aligned} \pi(a) &= \theta_a(.5) + (1-\theta_a-\theta_b)\tfrac{a}{.5+a} + \theta_b a \\ \pi(.5) &= \theta_a \tfrac{.5}{.5+a} + (1-\theta_a-\theta_b)(.5) + \theta_b(0) \\ \pi(b) &= \theta_a b + (1-\theta_a)(0). \end{aligned} \tag{41}$$

This solves to

$$\begin{aligned} \theta_a &= \tfrac{2a+8a^2+8a^3}{1+4a+16a^3-16a^4} \\ \theta_b &= \tfrac{1-2a^2}{1+4a^2} \end{aligned} \tag{42}$$

For $a = .3$ the probabilities are approximately 0.61 at 0.3, 0.10 at 0.5, 0.29 at 0.7. This is the same pattern we have found for our continuous-support mixed-strategy equilibrium—a high atom at the minimum, then low density, but rising monotonically to the maximum bid. This turns out not to be always true for multi-bid mixing equilibria, as Figure 4 shows. Monotonicity of the ω_i for $i > 1$ fails for the simple example of support $\left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right\}$. Here, the bids in the support are unevenly spaced. In a loose sense the third bid, $\frac{2}{3}$, has more space to cover than the fourth bid, and so takes a bigger probability. We will return to this idea later, and in our last proposition we will show that as the finite support becomes more dense, even if it is unevenly spaced, the equilibria will converge to the unique continuous-interval equilibrium we found in Proposition 2 and thus will approach an atom followed by an increasing density.

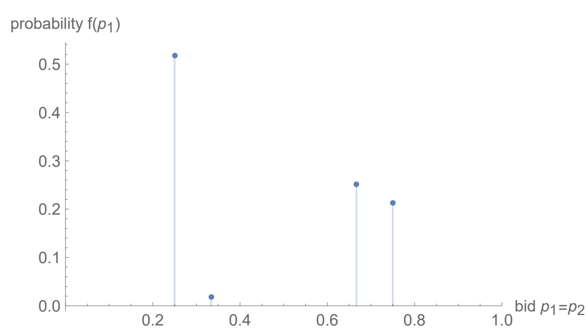

Figure 4. A Nonmonotonic Symmetric Equilibrium $f_1(p)$ Arising from an Unevenly Spaced Support $\left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right\}$

We will now proceed to find the general solution distribution for any equilibrium, symmetric or not, using a balanced discrete support.

Let μ be a general signed probability measure. Define the *support of* μ, denoted $\mathrm{spt}(\mu)$, to be the smallest compact set (with respect to inclusion) on which μ is supported. I.e. $\mathrm{spt}(\mu) = \bigcap_{K:\mu(K^c)=0} K$. Define $S = \mathrm{spt}(\mu)$ For measure μ supported on $S \subset [a, b]$ with $0 \le a = \inf S \le b = \sup S \le 1$, the equation corresponding to the crucial constant-payoff equation (10) is

$$\int_a^{1-q} \frac{q}{q+p} d\mu(p) = \pi_0 \tag{43}$$

for some constant payoff $\pi_0 \in [1-b, 1-a]$ and all $q \in S$.

We will now digress from this paper's focus on the proportional sharing rule, because in our next proposition we would like to extend one of the results in Malueg (2010) which is framed for general payoff functions. Thus, let us temporarily adopt his "Assumption 1," which allows for "take what you bid", proportional sharing, and various other rules.

Assumptions ($\star$). *The payoff function*

$$u_i(p,q) = \begin{cases} v_i(p,q) & \text{if } p+q \le 1 \\ 0 & \text{otherwise} \end{cases}$$

for players $i = 1$ and 2 bidding p and q satisfies the following properties:

(1) $v_i : [0,1] \times [0,1] \to [0,\infty)$ are Borel measurable and continuous on $(0,1] \times (0,1]$.
(2) $v_1(p,0) > v_1(0,0)$ and $v_2(0,q) > v_2(0,0)$ for all $p, q \in (0,1]$.
(3) $v_1(p,q)$ is nondecreasing in p and nonincreasing in q, and strictly increasing in p if $q > 0$.
(4) $v_2(p,q)$ is nonincreasing in p and nondecreasing in q, and strictly increasing in q if $p > 0$.

Remark 4. *The continuity in the first assumption above is not a problem if we wish to consider only restricted supports. Indeed, by the Tietze Extension Theorem any continuous function on a compact set $K \subset [0,1] \times [0,1]$ can be extended to a continuous function on all of $[0,1] \times [0,1]$, and with the same range.*

The expected payoff for each player given a choice of strategy measures (μ_1, μ_2) chosen by each player is then

$$\pi_i(\mu_1, \mu_2) = \int_{[0,1]\times[0,1]} u_i(p,q) d\mu_1(p) d\mu_2(q).$$

At equilibrium measures, the payoff will be not less than the payoff when one measure is a dirac measure. As we have seen, one can solve for the equilibrium measure of one player by requiring constant payoff when the other player chooses any pure strategy. When either μ_1 or μ_2 is a dirac measure at a point in $[0,1]$ and the other measure is understood, then we simplify the notation so that, for instance, the second player's expectation of the payoff if he chooses q is:

$$\pi_2(q) = \int_0^1 u_2(p,q) d\mu_1(p) = \int_a^{1-q} v_2(p,q) d\mu_1(p),$$

where we have assumed the greatest lower bound of the support of μ_1 is a.

We will focus on finding the equilibrium measure for the second player, since the process for the first player is analogous. Set $\mu = \mu_1$ and $v = v_2$, so that the second player's expected payoff $\pi(q) = \pi_2(q)$, for each possible demand q by the second player, given the first player's strategy measure μ is:

$$\pi(q) = \int_a^{1-q} v(p,q)d\mu(p) \tag{44}$$

Here we think of the payoff $\pi(q)$ as a given nonincreasing function on the set $B = 1 - A$ and the problem is to find the measure μ so that equality holds for all $q \in B$. We require that our solution, if any, must be a nonnegative (i.e. unsigned) probability measure. However for our solution technique, it will be useful to observe that even if we allowed any signed measure, any solution measure would still be nonnegative:

Lemma 2. *Provided the expected payoff is nonincreasing, any solution to* (44) *over signed measures must be a nonnegative measure.*

Proof. Let $T = \sup\{t \in A : \mu \geq 0 \text{ on } [a,t] \cap A\}$ and note that by hypothesis $T \geq a$ since $\pi(1 - a) \geq 0$. If $T = b$ the statement holds, and otherwise there is an $\epsilon > 0$ such that $T + \epsilon \in A$.

Recall that the function $v(p,q)$ is nonincreasing in p and nondecreasing in q for all p and q in the interval $[0,1]$. Set $K = \sup_{p \in [T,T+\epsilon]} v(p, 1 - (T+\epsilon)) \geq 0$. Since (44) is satisfied for $1 - q \in [a,T] \cap A$ and μ is nonnegative on $[a,T] \cap A$, then

$$\begin{aligned}
K\mu([T,T+\epsilon]) = K\int_T^{T+\epsilon} d\mu(p) &\geq \int_T^{T+\epsilon} v(p,1-(T+\epsilon))d\mu(p) \\
&\geq \int_a^{T+\epsilon} v(p,1-(T+\epsilon))d\mu(p) - \int_a^T v(p,1-(T+\epsilon))d\mu(p) \\
&= \pi(1-(T+\epsilon)) - \int_a^T v(p,1-(T+\epsilon))d\mu(p) \\
&\geq \pi(1-T) - \int_a^T v(p,1-(T+\epsilon))d\mu(p) \\
&= \int_a^T \left(v(p,1-T) - v(p,1-(T+\epsilon))\right) d\mu(p) \geq 0.
\end{aligned}$$

However, this contradicts the assumption on T, unless $T = b$ and μ is nonnegative on all subsets. □

Lemma 2 will help us prove the following proposition extending Proposition 3 in Malueg (2010) to cover uniqueness as well as existence. The proposition applies when the Players' support is discrete, continuous, or any other compact set, and whether the strategy is pure or mixed.

Proposition 3. *Given the general assumptions ($\star$) on the players' payoffs, for any compact set $A \subset (0,1)$, there exists a unique equilibrium among positive measures with Player 1's support set exactly A. Player 2's support will be exactly $1 - A$.*

Proof. Under precisely these assumptions, Malueg has shown existence of solution measures with support precisely A in Proposition 3 of Malueg (2010). (The argument follows from weak-* convergence of the unique equilibria with finite support that we will consider in the next section.) Moreover, Proposition 1 of Malueg (2010) shows that Player 2's strategy must then be balanced ($B = 1 - A$).

For uniqueness, suppose μ and μ' are both equilibrium measures for player one with support A. We have for all $q \in 1 - A$,

$$0 = \pi(q) - \pi(q) = \int_a^{1-q} v_2(p,q) d(\mu - \mu')(p).$$

However, $\mu - \mu'$ solves (44) for the case of $\pi(q)$ identically 0. By Lemma 2, $\mu - \mu'$ is nonnegative. However, reversing the order, $\mu' - \mu = -(\mu - \mu')$ is another solution measure for the constant 0 payoff which is again nonnegative by Lemma 2. Hence, $\mu = \mu'$. □

Proposition 3 is more radical than it might seem. The term "any compact set $A \in (0,1)$" covers more than just finite sets of points and intervals. It also includes, for instance, the Cantor set, which has measure zero but an uncountable number of points. Thus, there exists a unique equilibrium in which the players mix over Cantor sets suitably chosen to be balanced, having a cumulative distribution which would be a variant of the "Devil's Staircase," which contains no densities or probability atoms, but is continuous everywhere and increasing almost nowhere. A game with players mixing over Cantor sets would not be very useful for modelling real-world behavior, but it shows how extreme are the possibilities for mixed-strategy equilibria.[5]

On the other hand, we would have to keep 0 and 1 out of our Cantor set, because Proposition 3 only covers the case where Player 1's strategy set is $A \subset (0,1)$; that is, it excludes the possibility that 0 or 1 are ever bid by a player. We next address that possibility. Malueg (2010) noted that equilibria that include these extreme bids do not always have balanced strategy sets. He describes certain features of the equilibria for the special case of the Nash demand game, in his Proposition 5. We extend the description to other features and to any payoff function that satisfies assumptions ($\star$), including the Tullock proportional sharing rule. One feature of assumptions ($\star$) that we have not discussed is that they allow for the payoff bidding 0 to be higher than the payoff from $p + q > 1$ and the explosion of the pie, the threat point, which is normalized to 0. The possibility that a player prefers agreement with a zero share to disagreement has interesting implications.[6] That possibility

[5]The standard example of a Cantor set is created by first removing the middle one-third of the [0,1] interval, leaving intervals $A = [0, 1/3]$ and $B = [2/3, 1]$ in the set, then removing the middle one-third of A and of B, then removing the middle one-thirds of the resulting remaining intervals, and continuing the process forever. A "fat Cantor set" can be created by removing the middle interval of width $1/4^n$ from each of the 2^{n-1} remaining intervals instead, which would result in a set with positive measure that is nonetheless nowhere dense, containing no intervals (this particular one is called the Smith-Volterra Cantor set). For an informal explanation, see `https://blogs.scientificamerican.com/roots-of-unity/a-few-of-my-favorite-spaces-fat-cantor-sets/`.

[6]An application where a player would prefer agreement with a zero share to disagreement is a research collaboration in which one partner values the experience even if his share of profit is zero. A third possibility, disallowed by assumptions ($\star$), is that the payoff from agreement at a bid of zero is *less* than the payoff

allows for the following example, in which Player 1's payoff when he bids $p = 0$ and Player 2 bids q is $2\sqrt{1-q}$, more than the 0 from disagreement if $q < 1$.

Example 1. *Let* $v_1(p,q) = \frac{2\sqrt{1-q}}{1-p}$ *and* $v_2(p,q) = \frac{2\sqrt{1-p}}{1-q}$. *These satisfy assumptions* ($\star$). *It is simple to check that the measures* $\mu_1(x) = \mu_2(x) = \frac{dx}{2\sqrt{1-x}}$ *are equilibrium probability measures supported on all of* $[0,1]$, *with no atoms. (Player 1's payoff from the pure strategy of* p *is* $\int_0^{1-p} \frac{2\sqrt{1-q}}{1-p} \frac{1}{2\sqrt{1-q}} dq = \int_0^{1-p} \frac{1}{1-p} dq = 1$.*)*

With slightly more complexity we can obtain the same result of the equilibrium being a density over the entire [0,1] interval even if $v_1(0,1) > 0$, i.e., a player's payoff from bidding 0 is always higher than the disagreement payoff.

Example 2. *Let* $v_1(p,q) = \frac{2(\sqrt{1-q}+1)}{3-p-2\sqrt{p}} = v_2(q,p)$. *These also satisfy assumptions* ($\star$). *Once again, the measures* $\mu_1(x) = \mu_2(x) = \frac{dx}{2\sqrt{1-x}}$ *on all of* $[0,1]$ *with no atoms are equilibrium probability measures. (Player 1's payoff from any pure strategy of* p *is still* $1 = \int_0^{1-p} \frac{2(\sqrt{1-q}+1)}{3-p-2\sqrt{p}} \frac{1}{2\sqrt{1-q}} dq$.*)*

One may easily find similar examples provided $v_1(0,q) > 0$ for some $q > 0$. Nevertheless, as soon as $v_1(0,q) = 0$ for all $q > 0$ (and the corresponding condition holds for v_2), the equilibrium measures simplify. These cases are classified in our next proposition.

Proposition 4. *Given the general assumptions* ($\star$) *on the players' payoffs, assume further that a player's payoff is 0 from the action 0 (the same payoff as for disagreement), so* $v_1(0,q) = 0$ *for all* $q > 0$ *and* $v_2(p,0) = 0$ *for all* $p > 0$. *All equilibrium measures with supports* A *and* B *for players 1 and 2 that include an endpoint* $\{0\}$ *or* $\{1\}$ *are classified as follows.*

(1) If $0 \in A$ *then Player 1 has an atom at 0 and Player 2 always bids 1:* $\mu_1(0) > 0$ *and* $\mu_2 = \delta_1$. *Player 1's expected payoff is* $\pi_1 = 0$ *and Player 2's expected payoff can be any value* $\pi_2 \in (0, v_2(0,1)]$.[7]
(2) If $1 \in A$ *and* $1 \notin B$ *then* $\mu_1 = \delta_1$ *and* $\mu_2(0) > 0$. *Player 2's expected payoff is* $\pi_2 = 0$ *and provided* v_1 *is continuous, Player 1's expected payoff can be any value* $\pi_1 \in (0, v_1(1,0)]$.
(3) If $1 \in A$ *and* $1 \in B$ *then either* $\mu_1 = \mu_2 = \delta_1$, *or* $\mu_1 = \delta_1$ *and* $\mu_2(0) > 0$, *or* $\mu_2 = \delta_1$ *and* $\mu_1(0) > 0$. *For any pair of values* $(\pi_1, \pi_2) \in \{0\} \times [0, v_2(0,1)) \bigcup [0, v_1(1,0)) \times \{0\}$, *there exists an equilibrium with expected payoffs* (π_1, π_2).

All remaining cases (0 *or* 1 *in* B*) are found by switching* A *and* B *and the roles of Player 1 and Player 2. Moreover, any choice of supports* A, B *satisfying the above conditions generate some equilibrium.*

Proof. Case (1): if $0 \in A$, Player 1's expected payoff must be $\pi_1 = \pi_1(0) = \int_0^1 v_1(0,q) d\mu_2(q) = 0$ unless $v_1(0,0) > 0$ and 0 is an atom of μ_2. If $\mu_2(0) > 0$, however, there is some γ small

of 0 from disagreement. No player would ever bid 0 in equilibrium then, and the effect would be similar to contracting the bid space from $[0,1]$ to $[\alpha, 1-\alpha]$ for some $\alpha < .5$.

[7] We interpret the intervals $(a,b]$, $[a,b)$ and (a,b) all to mean $\{b\}$ whenever $a = b$.

enough that $\pi_1(\gamma) = v_1(\gamma,0)\mu_2(0) + \int_0^{1-\gamma} v_1(\gamma,q)d\mu_2(q) > \pi_1(0) = v_1(0,0)\mu_2(0)$, because $v_1(\gamma,0) > v_1(0,0)$ by assumption (2) of ($\star$). Thus, it cannot be true that $\mu_2(0) > 0$, and it must be true that $\pi_1 = 0$.

To see that Player 2 will use $\mu_2(1) = 1$, note that if $\mu_2(1) < 1$ there is an $\epsilon_0 > 0$ such that $\mu_2([0,1-\epsilon_0]) > 0$. By assumptions ($\star$) on v_1 we have

$$\pi_1(\epsilon_0) = \int_0^{1-\epsilon_0} v_1(\epsilon_0,q)d\mu_2(q) > 0.$$

Hence, Player 1's payoff from ϵ_0 is positive, contradicting $\pi_1 = 0$, so it must be that $\mu_2(1) = 1$. Moreover, Player 2's expected payoff at any equilibrium will always be

$$\pi_2 = v_2(0,1)\mu_1(0). \tag{45}$$

Now we consider the existence of equilibria. First, $(\mu_1 = \delta_0, \mu_2 = \delta_1)$ with $\pi_2 = v_2(0,1)$ is an equilibrium since Player 1 cannot gain by moving probability away from $p = 0$, nor Player 2 by moving from $q = 1$. This equilibrium is not unique. Suppose Player 1 bids $p = 0$ with probability α and $p = 1$ with probability $1-\alpha$. Player 2's best responses include $q = 1$ if and only if $\pi_2(1) = \alpha v_2(0,1) + (1-\alpha)(0) \geq \pi_2(0) = \alpha v_2(0,0) + (1-\alpha)v_2(1,0)$, which is true since $v_2(1,0) = 0$ and $v_2(0,1) > v_2(0,0)$ by assumption (2) of ($\star$). These yield any $\pi_2 = \alpha v_2(0,1) \in (0, v_2(0,1))$, but not $\pi_2 = 0$.[8]

The reason why $\pi_2 = 0$ is impossible even in more complicated equilibria is that if $0 \in A$, there exists an ε such that $\mu_1([0,\varepsilon]) > 0$, either because there is an atom at 0 and $\mu_1([0,\varepsilon]) > 0$ or because 0 is arbitrarily close to a bid with positive density. But that means Player 2 could bid $1-\varepsilon$ and get $\pi_2(1-\varepsilon) = \int_0^{\varepsilon} v_2(p,1-\varepsilon)d\mu_1(p) > 0$. Hence, it must be that $\pi_2(1) > 0$. Moreover, since we found in equation (45) that $\pi_2 = v_2(0,1)\mu_1(0) > 0$, it must be that $\mu_1(0) > 0$, as the proposition claims.

Case (2): if $1 \in A$ and $1 \notin B$, then in any equilibrium that might exist, $\sup B < 1$, so there is an $\epsilon_0 > 0$ such that $\mu_2([1-\epsilon_0, 1]) = 0$. Hence, $\pi_1(\epsilon_0) = \int_0^{1-\epsilon_0} v_1(\epsilon_0,q)d\mu_2(q) > 0$ since $v_1(\epsilon_0,q) > v_1(0,q)$. Furthermore, since $1 \in A$, it must be that $\pi_1(1) \geq \pi_1(\epsilon_0)$, so $\pi_1(1) > 0$. Since $u_1(1,q) = 0$ if $q > 0$, however, $\pi_1(1) = v_1(1,0)\mu_2(0)$, so $\mu_2(0) > 0$; there is an atom at $q = 0$. Thus, $0 \in B$, and since $v_2(p,0) = 0$ for $p > 0$, we may apply the argument from Case (1) to obtain that $\mu_1 = \delta_1$, and $\pi_2 = 0$.

The equilibria will also be symmetric to those of Case (1), provided $1 \notin B$. Hence, $(\mu_1 = \delta_1, \mu_2 = \delta_0)$ with $\pi_1 = v_1(1,0)$ is an equilibrium. For other equilibria, suppose Player 2 bids 0 with probability α and $q' \in (0,1)$ with probability $1-\alpha$. Player 1's best responses include $p = 1$ if and only if $\pi_1(1) = \alpha v_1(1,0) + (1-\alpha)(0) \geq \pi_1(1-q') = \alpha v_1(1-q',0) + (1-\alpha)v_1(1-q',q')$, which is true if $\alpha \geq \frac{v_1(1-q',q')}{v_1(1,0)-v_1(1-q',0)+v_1(1-q',q')}$. By continuity we have $\lim_{q'\to 1} v_1(1-q',0) = v_1(0,0) < v_1(1,0)$ and $\lim_{q'\to 1} v_1(1-q',q') = v_1(0,1) = 0$. Hence for q'

[8] Note that there will also exist many other equilibria, with Player 1 mixing over 0 and various disagreement bids instead of just 0 and 1, including equilibria with densities over intervals as well as atoms at 0 and possibly elsewhere.

sufficiently close to 1, the denominator is always uniformly positive and the numerator tends to 0 yielding any $\pi_1 = \alpha v_1(1,0) \in (0, v_1(1,0))$.

Case (3): if $1 \in A$ and $1 \in B$, then $\mu_1 = \mu_2 = \delta_1$ is an equilibrium because each player's payoff is 0 and remains zero if he deviates to any other bid. If both players bid 1 with probability less than 1, let p be another element of A and q be another element of B. If $p = 0$ then we are in Case (1) and $B = 1$, a contradiction. If $q = 0$ we are in the symmetric version of Case (1) and $A = 1$, a contradiction. If $p > 0, q > 0$, then $\pi_1(1) = 0 < \pi_1(1-q)$, so that cannot be true in equilibrium. Thus, the remaining possibility is that one player bids 1 with probability 1 and the other does not.

Without loss of generality, suppose this remaining possibility has $\mu_1 = \delta_1$ and $\mu_2(1) < 1$. Player 2 must therefore have $\mu_2([0, 1 - \epsilon_0])$ for some $\epsilon_0 > 0$. Hence, $\pi_1(\epsilon_0) > 0$, as in Case (2) of this proof and we can continue to its conclusion that $\mu_2(0) > 0$, i.e. $0 \in B$. This puts us in the symmetric version of Case (1) and the expected payoffs are any values $(\pi_1, \pi_2) \in (0, v_1(1,0)) \times \{0\}$. The value $\pi_1 = v_1(1,0)$ is necessarily excluded when $v_1(1,0) > 0$ since $1 \in B$ and thus $\mu_2(0) < 1$. Similarly, for the symmetric case $\mu_2 = \delta_2$ we have $0 \in A$ so the expected payoffs are in $(\pi_1, \pi_2) \in \{0\} \times (0, v_2(0,1))$. The expected payoffs of the equilibrium $(\mu_1 = \delta_1, \mu_2 = \delta_1)$ are $(\pi_1, \pi_2) = (0,0)$ which completes the proof.

□

We will now find an explicit solution for (44) in the compact discrete case, i.e. when the support set A is finite. To consider discrete measure solutions, we need μ to be a finite sum of atomic measures at points of $A = \{p_1, \dots, p_n\}$ with $0 \le a = p_1 < p_2 < \dots < p_n = b \le 1$. After substituting $\mu = \sum_{i=1}^n w_i \delta_{p_i}$, corresponding to weights w_i on Dirac masses at the points p_i, and specifiying that q take values in $\{1-p_1, \dots, 1-p_n\}$ to satisfy the balancing requirement, from (44) we obtain the system,

$$\sum_{i=1}^{j} v(p_i, 1-p_j) w_i = \pi(1-p_j) \quad \text{for } j = 1, \dots, n. \tag{46}$$

Because this system is triangular, the solutions to w_i only depend on $p_1, \dots, p_i$, and are independent of $n \ge i$ since the equations are. For example, for $n \ge 3$ the first three w_i can be solved for explicitly as

$$\begin{aligned}
w_1 &= \frac{\pi(1-p_1)}{v(p_1, 1-p_1)}, \\
w_2 &= \frac{\pi(1-p_2)}{v(p_2, 1-p_2)} - \frac{\pi(1-p_1) v(p_1, 1-p_2)}{v(p_1, 1-p_1) v(p_2, 1-p_2)}, \\
w_3 &= \frac{\pi(1-p_3)}{v(p_3, 1-p_3)} - \frac{\pi(1-p_1) v(p_1, 1-p_3)}{v(p_1, 1-p_1) v(p_3, 1-p_3)} - \frac{\pi(1-p_2) v(p_2, 1-p_3)}{v(p_2, 1-p_2) v(p_3, 1-p_3)} \\
&\quad + \frac{\pi(1-p_1) v(p_1, 1-p_2) v(p_2, 1-p_3)}{v(p_1, 1-p_1) v(p_2, 1-p_2) v(p_3, 1-p_3)}.
\end{aligned}$$

By Proposition 3, there will alway be a unique solution provided $a > 0$ and $b < 1$. (Malueg (2010) solves this in the special case that $v_1(p, q) = p$ and $v_2(p, q) = q$.)

Proposition 5. *For any finite support $A = \{p_1, p_2, \dots, p_n\}$ with $0 < p_1 < p_2 < \cdots < p_n < 1$, provided $v(p_j, 1-p_j) \neq 0$ for $j = 1, \dots, n$, there is a unique solution to (44) given by the measure $\mu = \sum_{j=1}^n w_j \delta_{p_j}$, where the weights $w_j > 0$ are given explicitly by:*

$$w_j = \frac{\pi(1-p_j)}{v(p_j, 1-p_j)} + \sum_{r=1}^{j-1} (-1)^r \sum_{1 \le i_1 < i_2 < \cdots < i_r < j} \frac{\pi(1-p_{i_1}) v(p_{i_1}, 1-p_{i_2}) \cdots v(p_{i_{r-1}}, 1-p_{i_r}) v(p_{i_r}, 1-p_j)}{v(p_{i_1}, 1-p_{i_1}) v(p_{i_2}, 1-p_{i_2}) \cdots v(p_{i_r}, 1-p_{i_r}) v(p_j, 1-p_j)}. \tag{47}$$

Proof. We first note that (46) corresponds to a lower-triangular linear system which has a unique solution if and only if its determinant, $\prod_{j=1}^n v(p_j, 1-p_j)$, is nonzero. This condition is equivalent to the condition given in the proposition that $v(p_j, 1-p_j) \neq 0$ for $j = 1, \dots, n$. It remains to identify this unique solution.

Substituting $\sigma_i = v(p_i, 1-p_i) w_i$ into (46) and rearranging terms we find,

$$\sigma_j = \pi(1-p_j) - \sum_{i=1}^{j-1} \sigma_i \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)}. \tag{48}$$

We may rewrite formula (47) for σ_j as,

$$\sigma_j = \pi(1-p_j) + \sum_{r=1}^{j-1} (-1)^r \sum_{1 \le i_1 < i_2 < \cdots < i_r < i_{r+1} = j} \pi(1-p_{i_1}) \prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})}. \tag{49}$$

Now we verify this formula for σ_j by substituting the corresponding formula for σ_i into the right hand side of (48). After distributing sums we obtain,

$$\begin{aligned}
\sigma_j &= \pi(1-p_j) - \sum_{i=1}^{j-1} \sigma_i \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} \\
&= \pi(1-p_j) - \sum_{i=1}^{j-1} \left(\pi(1-p_i) + \sum_{r=1}^{i-1} (-1)^r \sum_{1 \le i_1 < i_2 < \cdots < i_r < i_{r+1} = i} \pi(1-p_{i_1}) \prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})} \right) \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} \\
&= \pi(1-p_j) - \sum_{i=1}^{j-1} \pi(1-p_i) \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} + \\
&\qquad \sum_{i=1}^{j-1} \sum_{r=1}^{i-1} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \cdots < i_r < i_{r+1} = i} \pi(1-p_{i_1}) \left(\prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})} \right) \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)}.
\end{aligned}$$

After switching the order of sums in i and r this becomes,

$$\begin{aligned}
&= \pi(1-p_j) - \sum_{i=1}^{j-1} \pi(1-p_i) \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} + \\
&\qquad \sum_{r=1}^{j-2} \sum_{i=r+1}^{j-1} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \cdots < i_r < i_{r+1} = i} \pi(1-p_{i_1}) \left(\prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})} \right) \frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)}.
\end{aligned}$$

Since the index i satisfies $r < i < j$, we may interpret i as a new index i_{r+1} in the second sum, and setting $j = i_{r+2}$ we obtain,

$$= \pi(1-p_j) - \sum_{i=1}^{j-1} \pi(1-p_i)\frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} + \\ \sum_{r=1}^{j-2}(-1)^{r+1} \sum_{1\le i_1<i_2<\cdots<i_r<i_{r+1}<i_{r+2}=j} \pi(1-p_{i_1}) \prod_{\ell=1}^{r+1} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})}.$$

Re-indexing $r+1 \to r$, this quantity becomes,

$$= \pi(1-p_j) - \sum_{i=1}^{j-1} \pi(1-p_i)\frac{v(p_i, 1-p_j)}{v(p_i, 1-p_i)} + \\ \sum_{r=2}^{j-1}(-1)^{r} \sum_{1\le i_1<i_2<\cdots<i_r<i_{r+1}=j} \pi(1-p_{i_1}) \prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})}.$$

Lastly, we recognize $-\sum_{i=1}^{j-1} \pi(1-p_i)\frac{v(p_i,1-p_j)}{v(p_i,1-p_i)}$ as the $r = 1$ case of the second sum. Combining these we obtain,

$$= \pi(1-p_j) + \sum_{r=1}^{j-1}(-1)^{r} \sum_{1\le i_1<i_2<\cdots<i_r<i_{r+1}=j} \pi(1-p_{i_1}) \prod_{\ell=1}^{r} \frac{v(p_{i_\ell}, 1-p_{i_{\ell+1}})}{v(p_{i_\ell}, 1-p_{i_\ell})}$$

which is the original formula for σ_j. This shows that this formula yields the unique solution for ((48)), and hence the solution of the w_j for 44 completing the proof.

□

Remark 5. *When there are $k \ge 1$ of the $v(p_j, 1-p_j)$ which vanish, then either there are no solutions or there is a k-dimensional affine subspace of solutions, depending on whether the other values of $v(p_i, 1-p_j)$ and $\pi(1-p_j)$ make the linear system (46) inconsistent or consistent. This can be handled using standard linear algebraic methods; we omit the details.*

Corollary 2.1. *For any finite support $A = \{p_1, p_2, \ldots, p_n\}$ with $0 < p_1 < p_2 < \cdots < p_n < 1$, provided $v(p_j, 1-p_j) \neq 0$ for $j = 1, \ldots, n$, the unique equilibrium is the probability measure $\mu = \sum_{j=1}^{n} w_j \delta_{p_j}$, where the weights $w_j > 0$ are given explicitly by:*

$w_j = \pi_0 \omega_j$, *where*

$$\omega_j = \frac{1}{v(p_j, 1-p_j)} + \sum_{r=1}^{j-1} (-1)^r \sum_{1\le i_1<i_2<\cdots<i_r<j} \frac{v(p_{i_1}, 1-p_{i_2})\cdots v(p_{i_{r-1}}, 1-p_{i_r}) v(p_{i_r}, 1-p_j)}{v(p_{i_1}, 1-p_{i_1}) v(p_{i_2}, 1-p_{i_2})\cdots v(p_{i_r}, 1-p_{i_r}) v(p_j, 1-p_j)}.$$

and the constant payoff is $\pi_0 = \dfrac{1}{\sum_j \omega_j}$.

Thus for any increasing finite set of points $0 < p_1 < p_2 < \cdots < p_n < 1$ we obtain a unique probability measure μ supported on this set that provides an equilibrium.

We now return to the proportional Tullock sharing rule. Since $v(p_j, 1-p_j) = 1-p_j > 0$ provided $p_j \neq 1$, Corollary 2.1 implies:

Corollary 2.2. *For $v(p,q) = \frac{q}{q+p}$, the weights $w_i = \pi_0 \omega_j$ simplify:*

$$\omega_j = \frac{1}{1-p_j} + \sum_{r=1}^{j-1} (-1)^r \sum_{1 \le i_1 < i_2 < \dots < i_r < j} \frac{1}{(1-p_{i_1})(1+p_{i_1}-p_{i_2})(1+p_{i_2}-p_{i_3}) \cdots (1+p_{i_{r-1}}-p_{i_r})(1+p_{i_r}-p_j)} > 0$$

and the payoff is $\pi_0 = \left(\sum_i \omega_i\right)^{-1}$.

Remark 6. *Even in the Tullock case above, the expression for π_0 does not simplify to any closed form rational expression of universally bounded length.*

Using the proportional Tullock sharing rule, for $n \ge 3$ the first three ω_i become:

$$\omega_1 = \frac{1}{1-p_1},$$
$$\omega_2 = \frac{1}{1-p_2} - \frac{1}{(1-p_1)(1-p_2+p_1)},$$
$$\omega_3 = \frac{1}{1-p_3} - \frac{1}{(1-p_1)(1-p_3+p_1)} - \frac{1}{(1-p_2)(1-p_3+p_2)} + \frac{1}{(1-p_1)(1-p_2+p_1)(1-p_3+p_2)}.$$

The only other possible exceptions not covered by Proposition 5 are when $a = p_1 = 0$ or $b = p_n = 1$. We will treat these cases for the general setting in the next proposition which will demonstrate convergence of equilibrium as the discrete support becomes dense in an arbitrary compact subset of $[0,1]$. Recall that Figure 4 illustrated an equilibrium over a finite unevenly spaced support that did not have the pattern we found for the equilibria over an interval— an atom at the bottom of each interval followed by a convexly increasing density. Is this a conseqence of the sparsity of the support? Perhaps if the number of the points in the support increased enough, the discrete equilibria would converge to the continuous equilibrium. This conjecture turns out to be true, once we formalize "increased enough" and "converge" to make it precise.

The uniqueness of the solution to (43) for discrete measures, together with the density of Dirac measures in the space of all Borel probability measures (these are always Radon measures) on $[0,1]$ with respect to the weak-* topology, implies that given any compact support set $K \subset [0,1]$, finite or not, there is a unique measure supported on K solving (43). (Note that since the integrand in (43) is continuous, weak-* limits of solutions will be solutions.) Moreover we have already established uniqueness of any solutions with a given support since the integrand is positive, so all weak-* limit solutions are unique no matter how they are approached. We summarize this in the following proposition.[9]

[9] Recall that the Hausdorff metric d_H on the space $\mathcal{X}$ of compact subsets of a metric space (X,d) is defined for any $A, B \in \mathcal{X}$ by

$$d_H(A,B) = \inf\{\epsilon : N_\epsilon(A) \supset B \text{ and } A \subset N_\epsilon(B)\}$$

where $N_\epsilon(A) = \bigcup_{x \in A} B(x,\epsilon)$ is the ϵ-neighborhood of A.

Proposition 6. *Given the general assumptions ($\star$) on the players' payoffs, for any compact subset $K \subset (0,1)$ let μ_K be the unique solution probability measure to (44) guaranteed by Proposition 3. If K_i is any sequence of compact sets converging to K in the Hausdorff metric,*[10] *then μ_{K_i} weak-* converges to μ_K.*

Proof. Note that we may assume $K_i \subset (0,1)$ since for all sufficiently large indices i, $d_H(K_i, K) < \min\{\inf K, 1 - \sup K\}$. Since the weak-* topology on the space of probability measures on $[0,1]$ is compact by the Banach-Aloaglu Theorem, the sequence of measures μ_{K_i} has a convergent subsequence, say limiting to a probability measure μ. Since $v(p,q)$ is continuous on $(0,1] \times (0,1]$, we have that for each q, $u(p,q)$ is continuous on $[0,1]$ except perhaps at the bounds of integration. By an immediate consequence of the definition of weak-* convergence,

$$\int_{[0,1]} u(p,q) d\mu(p) = \lim_i \int_{[0,1]} u(p,q) d\mu_{K_i}(p) = \lim_i \int_{\inf K_i}^{1-q} v(p,q) d\mu_{K_i}(p).$$

However, for each $q \in K_i$ the latter integral is $\int_{\inf K_i}^{1-q} v(p,q) d\mu_{K_i}(p) = \pi(q)$. Moreover, by the continuity assumption (1) of ($\star$) on v, the expected payoff π is continuous on $(0,1]$ which implies that for any sequence $q_i \in K_i$ limiting to $q \in K$, $\pi(q_i)$ converges to $\pi(q)$. Hence, for each $q \in K$, we have

$$\int_{\inf K}^{1-q} v(p,q) d\mu(p) = \int_{[0,1]} u(p,q) d\mu(p) = \pi(q).$$

The uniqueness of solution measures from Proposition 3 then implies that $\mu = \mu_K$. $\square$

Corollary 2.3. *As the number of bids in a discrete support becomes dense in an interval $[a,b] \subset (0,1)$, the mixing probabilities for the equilibrium with the discrete support converge to the mixing atom and density of Proposition 2 for the equilibrium on $[a,b]$.*

Thus, although the mixing distribution for an equilibrium mixing over a discrete set of bids may appear qualitatively different from that mixing over a continuous interval of bids, as the number of discrete bids increases and fills up the interval, the discrete equilibrium will come to look more and more like the continuous equilibrium. Similarly, we could even take a sequence of Cantor sets Hausdorff converging to $[a,b]$ and their equilibria would also converge.

6. Concluding Remarks

"Splitting a Pie" is the simplest of bargaining games, but it generates surprisingly complex equilibria. In this paper we have explored those equilibria when players bid over compact supports, with particular attention to the case when a player's share is his bid proportional to the sum of the bids. We have seen that any "balanced" pair of compact supports for the two players, discrete or continuous or neither (e.g. Cantor sets), can support an equilibrium, mixed or pure, and have derived explicit formulae for the probabilities and

[10] The space of compact subsets of $[0,1]$ is compact in the topology induced by the Hausdorff metric, and sequences in a compact space with at most one accumulation point are convergent.

densities for the two cases when a player's support is an interval and for when it is a finite set of bids. The derivations were technical, and the equilibria can be quite baroque, so although the mathematical properties of these mixed strategies are fascinating (to some of us at least) it is worth returning to the motivation.

The situation being modelled is that of two players choosing how tough to be with each other in sharing something of value. In the pure-strategy equilibria, they begin with an expectation of how tough each other will be, and their behavior confirms that expectation. In the mixed-strategy equilibria, they begin with an expectation of a range of toughness that the other player might choose, and their behavior, while different from realization to realization, does not contradict that expectation, and with time and repetition would confirm it. To model mixed strategies, one must also make some assumption as to how the pie is shared when neither player is very tough so the bids—the toughness levels— add up to less than one. Our preferred assumption is the Tullock Rule that the shares are proportional to the bids and that the shares add up to one.

We found, as Malueg and others have noted, that mixed-strategy equilibria do exist and that although we must start with the players having a common expectation of the supports of the bids, they can then deduce the mixing distributions that will be chosen if they are both rational maximizers. If both players mix over the same interval, the most common choice for each player will be to play the minimum level of toughness in the support, which means that a common outcome will be for both to choose equal toughness and receive equal shares. Also common will be disagreement, when each player guesses wrong as to what the other player will choose and wishes, ex post, that he had not been so tough. Also common though, will be a continuous range of equilibrium shares, not from 0% to 100%, but over a range depending on their expectations of the minimum and maximum toughness of their rival.

We think these are good features to find for a model of complete information, but the solution is intricate because each player's share depends continuously on both how tough he is and how tough his rival is. The Nash demand game, with its "take what you bid" sharing rule, is similar in its qualitative features and far simpler to solve. Its downside is that players who agree usually agree only over part of the pie and mysteriously discard the rest. Still simpler and without that downside is the symmetric hawk-dove equilibrium, in which each player mixes over being tough and mild using the same probability, and which is simple to analyze even under a proportional sharing rule. In the end, for applied work the hawk-dove equilibrum is what may be most useful, but we hope that this paper has shown that its qualitative features are not so very different from a model that uses a continuum of toughness levels instead of just two.

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