SPLITTING A PIE: MIXED STRATEGIES IN BARGAINING UNDER COMPLETE INFORMATION

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Abstract. We study the mixed-strategy equilibria for the bargaining game in which two players simultaneously bid p and q for a share of a pie and receive shares proportional to their bids, or zero if the bids sum to more than 100%. Of particular interest is the equilibrium in which each player's support is a single interval. We show that this consists of convex increasing densities $f_1(p)$ on $[a,b]$ and $f_2(q)$ on $[1-b, 1-a]$ together with atoms of probability at $p = a$ and $q = 1-b$. These are unique for any given $0 < a < b < 1$. The two single outcomes with highest probability are breakdown and a $\left\{\frac{a}{a+1-b}, \frac{1-b}{a+1-b}\right\}$ split. We provide formulas for the probability weights of the atoms and exact power series representations for the densities f_1 and f_2.

1. Introduction

A fundamental problem in game theory is how to model bargaining between two players who must agree on shares of surplus or obtain zero payoff. In the classic bargaining problem "Splitting a Pie" (also known as "Dividing the Dollar"), two players bid simultaneously for shares of the pie. If their bids add up to more than one, both get a breakdown payoff of zero. The game has a continuum of pure strategy equilibria, for example, (0.00, 1.00) and (.30,.70). A multitude of other models of bargaining exist, of which the two best known are the Nash bargaining solution of Nash (1950) and the alternating offer model of Rubinstein (1982). Nash takes the approach of cooperative game theory and finds axioms which guarantee a 50-50 split. Rubinstein takes the approach of a noncooperative alternating-offer model and finds close to a 50-50 split, with a small advantage to whichever player makes the first offer. In both models, the players always agree in equilibrium. Other models incorporate incomplete information, in which case failure to agree can occur in equilibrium when a player refuses to back down because he thinks, wrongly, that the other player's payoff function will cause him to agree to accept his proposal.

Muthoo (1999) is the standard book-length treatment of bargaining under complete information and the Ausubel, Cramton & Deneckere (2002) *Handbook of Game Theory* chapter treats bargaining under incomplete information. Since Nash there has been much work on this game and its relation to the Nash bargaining solution, notably Binmore, Rubinstein & Wolinsky (1986), Binmore (1987), and Carlsson (1991). More recently, Anbarci & Boyd

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(2011) relate the game to cooperative game theory more broadly and Bastianello & LiCalzi (2019) use a different axiomatic approach generalizing Nash by the introduction of a hypothetical mediator and probabilistic acceptances. Andersson, Argenton & Weibull (2018) continue with the idea of using a different kind of uncertainty than incomplete information, and a number of authors have further explored Rubinstein's 1982 model of offers over time under complete information— see Britz, Herings & Predtetchinski (2010), Kawamori (2014), and Friedenburg (2019).

In this paper we return to the classic "Splitting a Pie" and investigate its mixed-strategy equilibria. Although examples of mixed-strategy equilibria appear as early as Roth (1985), they do not seem to have been studied systematically until Malueg (2010), who showed that a huge variety exist. In particular, he showed that for any "balanced" pair of supports, such that for each bid in a player's support there is a bid in the other player's support that sums to one, there exists an equilibrium. He does this for general utility functions and general sharing rules for what happens if the two players' bids add up to less than 1.

The easiest sharing rule is the "take what you bid" rule used in the "Nash Demand Game" of Nash (1953): if the bids are .20 and .30, Player 1 receives 20%, Player 2 receives 30%, and the other 50% is abandoned. Nash himself used the "split the difference" sharing rule that if the bids were .20 and .30, the shares would be 45% (= .2 + .5/2) and 55% (= .3+ .5/2).

We believe the most natural specification is proportional sharing (also called "the Tullock rule"): if the bids are .20 and .30, the shares are 40% (= .2/(.2+.3)) and 60% (= .3/(.2+.3)). Proportional sharing is suitable if the model is to be a metaphor for players deciding how "tough" to be in negotiations. If both choose to be too tough, they fail to reach agreement, which is represented by complete breakdown and payoffs of zero. If they are less tough, it is their relative toughness that matters, and they split the pie in proportion to toughness. Proportional sharing has desirable properties that "take what you bid" lacks: (a) if agreement is reached, one player's share is greater if the other player is less tough, and (b) if the players come to an agreement, they share the entire pie rather than leaving some to waste.

Other sharing rules such as "split the difference" also satisfy (a) and (b), and since "toughness" is an artificial construct without natural units, the differences between sharing rules that do allocate the entire pie upon agreement with interaction between the players' shares are less important. Of course, all these sharing rules maintain the assumption that there is a discontinuity when bids sum to more than 1, in which case bargaining breaks down and both players receive zero. That assumption is what prevents the players from each bidding an infinite amount; in the bargaining game, unlike, for example, rentseeking, bidding higher is costless in terms of effort or expenditure, so there needs to be some reason why bidding higher is not always better.

Unfortunately, under proportional sharing the mathematics is difficult because a player's share depends not just on his own bid but on the other player's. Nonetheless, Malueg (2010) proved existence for general sharing rules, though he restricted himself to rules like "take what you bid" for uniqueness and describing the equilibrium.

We focus on the proportional sharing rule, and we completely characterize a particularly interesting class of equilibria from an economic point of view, the "interval equilibria," in which each player mixes over a single interval. An example is the symmetric equilibrium in which both players mix over [.30, .70]. We show that there is a unique mixing distribution for this support, consisting of a probability atom at .30 and a convexly increasing density over (.30, .70]. The resulting combination of a modal 50-50 split, occasional breakdown, and a large range of other possible shares is consistent with real-world behavior. As we will see, there are also asymmetric interval equilibria.

Section 2 lays out the model and discusses two simple approaches to mixed-strategy equilibria: the Hawk-Dove equilibria in which each player mixes over two bids, and interval equilibria in the "Nash Demand Game." Section 3 analyzes interval equilibria for the game with proportional sharing. We first characterize interval equilibria, both symmetric and asymmetric, using a relatively nontechnical approach that provides intuition (Proposition 1). We turn to a more technical methodology to prove uniqueness and derive the exact atom size and density function (Proposition 2).

2. The Model and Two Easy Special Cases

Players 1 and 2 simultaneously choose bids p and q in the interval $[0,1]$. If their bids add up to more than 100%, then we say *breakdown* occurs, and both get zero. Otherwise, they split a pie of size 1 according to a sharing rule $\{v_1, v_2\}$. Thus, Player i's payoff function is (for $i = 1, 2$)

$$u_i(p,q) = \begin{cases} v_i(p,q) & \text{if } p+q \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Throughout this paper we assume that the players' strategies are Borel probability measures (μ_1, μ_2) with supports (A, B) with A and B subsets of $[0,1]$.[1] Thus, the expected payoff for player $i = 1, 2$ is:

$$\pi_i(\mu_1,\mu_2) = \int_{[0,1]\times[0,1]} u_i(p,q)\,d\mu_1(p)\,d\mu_2(q).$$

At equilibrium measures (μ_1, μ_2), the expected payoff of player i will be equal to his expected payoff when μ_i is replaced by a single atom in the support of μ_i, so one can solve for his equilibrium measure by requiring a constant payoff across the pure strategies he mixes among. When either μ_1 or μ_2 is an atom at a point in $[0,1]$ and the other measure

[1] For a nonnegative Borel measure μ, its *support* $\operatorname{supp}(\mu)$ is defined to be the smallest closed set whose complement has 0 μ-measure, i.e. $\operatorname{supp}(\mu) = \bigcap_{\{K=\overline{K}:\mu(K^c)=0\}} K$. For a general signed measure μ its support is $\operatorname{supp}(\mu^+)\bigcup\operatorname{supp}(\mu_-)$ where $\mu = \mu_+ - \mu_-$ for the unique nonnegative measures μ_+ and μ_- given by the Hahn decomposition theorem.

is understood, we simplify the notation so that, for instance, Player 2's expectation of the payoff if he bids q is:

$$\pi_2(q) = \int_0^1 u_2(p,q)d\mu_1(p) = \int_a^{1-q} v_2(p,q)d\mu_1(p), \tag{1}$$

where we have assumed the greatest lower bound of the support of μ_1 is $a = \inf A$.

Starting with sets $A \subset [0,1]$ and $B \subset [0,1]$, we think of the expected payoff $\pi_2(q)$ as a function on the set B, and the problem is to find all signed measures μ_1 supported on A so that equality (1) holds for some constant $\pi_2^* = \pi_2(q)$ for all $q \in B$. (And similarly for μ_2 on B.) Since equation (1) is invariant under scaling the measure μ_1 and payoff $\pi_2(q)$ by any constant $r \in \mathbf{R}$, without loss of generality we may assume $\|\mu\| = 1$. In other words, we wish to find signed measures μ such that when Player 1's strategy measure μ_1 equals μ, there is a constant π_2^* such that:

$$\pi_2^* = \pi_2(q) = \int_a^{1-q} v_2(p,q)d\mu_1(p) \quad \text{and} \quad \|\mu_1\| = 1, \tag{2}$$

and similarly for μ_2.

For his general existence result, Malueg (2010) makes the following assumptions on the sharing rules.

Assumptions $\star$. *For players $i = 1$ and 2 bidding p and q the sharing rule $\{v_1, v_2\}$ satisfies the following properties:*

(1) $v_i : [0,1] \times [0,1] \to [0,\infty)$ are Borel measurable and continuous on $(0,1] \times (0,1]$.
(2) $v_1(p,0) > v_1(0,0)$ and $v_2(0,q) > v_2(0,0)$ for all $p, q \in (0,1]$.
(3) $v_1(p,q)$ is nondecreasing in p and nonincreasing in q, and strictly increasing in p if $q > 0$.
(4) $v_2(p,q)$ is nonincreasing in p and nondecreasing in q, and strictly increasing in q if $p > 0$.

Assumption ($\star$1) is a set of technical assumptions ensuring payoffs are well defined and that certain approximation arguments hold. Assumption ($\star$2) guarantees that simultaneous bids of 0 are not pure strategy equilibria, and Assumptions ($\star$3)-($\star$4) guarantee that while bids sum to less than one, increasing one player's bid helps that player while not helping the other.

As discussed in Section 1, proportional sharing— players receiving shares proportional to their bargaining intensity— is our favored bargaining rule.

Definition 1. *The* **proportional sharing rule***, also called the Tullock Rule, is the sharing rule $v_1 = p/(p+q)$ and $v_2 = q/(p+q)$ unless $p = q = 0$, in which case $v_1 = v_2 = .5$.*

It is easy to verify that the proportional sharing rule satisfies assumptions ($\star$). Much of our analysis will also be looking at one intuitive category of equilibrium: interval equilibria.

Definition 2. *An* **interval equilibrium** *is a Nash equilibrium in which Player 1 bids by mixing over the support* $A = [a, b]$ *with* $b > a$ *and Player 2 bids by mixing over the support* $B = [c, d]$ *with* $d > c$.

In particular, in an interval equilibrium every nonempty open subinterval of $[a, b]$ and $[c, d]$ has positive probability for their corresponding measures. A key feature of any equilibrium will be "balanced supports":

Definition 3. *The player's bid supports are* **balanced** *if and only if element* s *being in Player 1's support* A *implies that* $1 - s$ *is in Player 2's support* B*, and vice versa.*

A balanced support for an interval equilibrium can be either symmetric, e.g. both players mixing over $[.2, .8]$, or asymmetric, e.g. Player 1 mixing over $[.2, .4]$ and Player 2 over $[.6, .8]$.

Splitting a Pie has a continuum of pure strategy Nash equilibria: every pair of bids with $p + q = 1$. There are also many mixed-strategy equilibria, some with bidding over a finite set of points, some with bidding over a continuum. Before we proceed to characterizing interval equilibria, it will be helpful to understand two special cases: Hawk-Dove equilibria for Splitting a Pie, and interval equilibria for the Nash Demand Game.

2.1. **Hawk-Dove Equilibria for Splitting a Pie.**

The simplest mixed strategy for Splitting a Pie has the two players each mixing over two bids. This is a Hawk-Dove equilibrium, mathematically the same as the well-known biological model of birds deciding whether to pursue aggressive or timid strategies.

In a symmetric Hawk-Dove equilibrium, each player chooses a with probability $\mu(a)$ and $1 - a$ with probability $\mu(1 - a) = 1 - \mu(a)$ for $a < .5$. The two bids must add up to 1, because otherwise it would be a profitable deviation for one player to choose a bigger number for his lower bid, increasing his share without any greater likelihood of breakdown. This is an immediate example of the importance of "balanced supports." Let us assume that the sharing rule is proportional sharing. The mixing probabilities must make each action's expected payoff the same in equilibrium, so for $i = 1, 2$,

$$\pi_i(a) = \mu(a)(.5) + (1 - \mu(a))a = \pi_i(1 - a) = \mu(a)(1 - a) + (1 - \mu(a))(0), \tag{3}$$

which solves to $\mu(a) = 2a$ and $\pi_1 = \pi_2 = 2a(1 - a)$.

The players share the pie equally in equilibrium with probability $4a^2$, and bargaining breaks down with probability $(1 - 2a)^2$. Note that there is a continuum of equilibria, since any value of a in (0,.5) can support an equilibrium, and they can be pareto-ranked, with higher payoffs if a is closer to .5. In the case when $a = 0$, both players choose $a = 0$ with probability 0 and $1 - a = 1$ with probability 1, and the expected payoff is zero. Observe that the players actually choose the higher of their two bids (hawk) with the highest probability $(1 - \mu(a) > 1/2)$ if $a < 1/4$, in contrast to the result in biology's Hawk-Dove game.

There also exist asymmetric Hawk-Dove equilibria, as Malueg (2010) points out. Player 1 chooses a with probability $\mu_1(a)$ and b with probability $(1-\mu_1(a))$, for $a < b < .5$. Player 2 chooses $1-b$ with probability $\mu_2(1-b)$ and $1-a$ with probability $\mu_2(1-a) = (1-\mu_2(1-b))$. Player 1's expected payoffs from his two pure strategies must be equal for him to be willing to mix between them, so

$$\pi_1(a) = \mu_2(1-b)\left(\frac{a}{a+1-b}\right)+(1-\mu_2(1-b))\left(\frac{a}{a+1-a}\right) = \pi_1(b) = \mu_2(1-b)\left(\frac{b}{b+1-b}\right)+(1-\mu_2(1-b))\cdot 0,$$

and $\mu_2(1-b) = \frac{a(a+1-b)}{a^2-b^2+b}$ with payoff $\pi_1 = \frac{ab(a+1-b)}{a^2-b^2+b}$.

Similarly, Player 2's expected payoffs from his two pure strategies must be equal, so

$$\pi_2(1-b) = \mu_1(a)\left(\frac{1-b}{a+1-b}\right)+(1-\mu_1(a))\left(\frac{1-b}{b+1-b}\right) = \pi_2(1-a) = \mu_1(a)\left(\frac{1-a}{a+1-a}\right)+(1-\mu_1(a))\cdot 0,$$

and $\mu_1(a) = \frac{(1-b)(a+1-b)}{(1-b)^2-a^2+a}$ with payoff $\pi_2 = \frac{(1-a)(1-b)(a+1-b)}{(1-b)^2-a^2+a}$.

Thus we might have Player 1 choosing .10 and .40 with probabilities of .28 and .72 for an expected payoff of .11; and Player 2 choosing .60 and .90 with probabilities of .93 and .07 for an expected payoff of .84 (rounding to two digits).

The Hawk-Dove equilibrium, where the probability mass must be concentrated at the two endpoints, is of interest since it is a stepping stone to the interval equilibrium. While it cannot tell us about the densities, it does illustrate why supports must be balanced and how asymmetric equilibria can arise.

2.2. **The Nash Demand Game.** In the "Nash Demand Game," players use the "take what you bid" sharing rule. The players bid p and q, and they receive p and q as payoffs if $p+q \le 1$ but zero if $p+q > 1$ (Nash (1953)). As discussed in Section 1, this means that if both players are mild in their bargaining strategy, it will happen that $p+q < 1$ and in their agreement they will specify that some of the pie goes to neither of them, a strange outcome for successful bargaining. The sharing rule also has the feature that unless there is breakdown, each player's payoff depends only on his own bid, not the other player's. This feature will make it easy to solve for equilibrium.

In one equilibrium of the Nash Demand Game, each player bids a with probability K and then mixes using density $f = f_1 = f_2$ over the interval $[a, 1-a]$. Player 2 can guarantee a payoff of a by bidding a, since $p + a \le 1$ for any bid p Player 1 might play. Player 2 will have a payoff of $K(1-a)$ from bidding $q = (1-a)$, since Player 1 bids $p = a$ with probability K and otherwise Player 1 bids $p > a$, whence $p + (1-a) > 1$ and both players receive 0. Since Player 2 is only willing to mix between bids if they have equal expected payoffs, this implies $\pi_2(a) = a = \pi_2(1-a) = K(1-a)$ and we can conclude that $K = \frac{a}{1-a}$.

For bids between a and $1-a$, Player 2's expected payoff is

$$\pi_2(q) = Kq + \int_a^{1-q} qf(p)dp, \tag{4}$$

which we can rewrite using F as the cumulative distribution for f and our knowledge that $K = \frac{a}{1-a}$, and combine with the requirement that $\pi_2(q) = \pi(a) = a$ to yield

$$\pi_2(q) = \left(\frac{a}{1-a}\right)q + qF(1-q) = a. \tag{5}$$

Using the change-of-variables $p = 1-q$, this becomes $\left(\frac{a}{1-a}\right)(1-p) + F(p)(1-p) = a$, which solves to $F(p) = \frac{a}{1-p} - \frac{a}{1-a}$, which can be differentiated to yield the equilibrium mixing density, $f(p) = \frac{a}{(1-p)^2}$. Thus, each player chooses to bid $x \in (a, 1-a]$ with density $f(x) = \frac{a}{1-x^2}$ and $x = a$ with probability $K = \frac{a}{1-a}$.

The Nash Demand Game is easy to solve because each player's payoff function depends on the other player's bid only if breakdown occurs. If the bids add up to less than one, a player's payoff is entirely independent of what the other player does. This is what allows us to move smoothly from equation (4)'s $\int_a^{1-q} qf(p)dp$ to equation (5)'s $qF(1-q)$. If Player 2's share of the pie depended on Player 1's bid instead of being just $v_2(q,p) = q$, equation (4) would include the expression $\int_a^{1-q} v_2(q,p)f(p)dp$ and it would no longer be straightforward to extract $f(p)$ from the integral. We will do that extraction in Section 3 for the proportional sharing rule.

3. Interval Equilibria under Proportional Sharing

We will now consider interval equilibria for Splitting a Pie, e.g., both players mix over [.2, .8], or Player 1 mixes over [.2, .4] and Player 2 over [.6, .8]. Under proportional sharing we characterize interval equilibria (Proposition 1) and find an explicit formula for their mixing distributions (Proposition 2).

Proposition 1. *Any interval equilibrium consists of exactly one probability atom of size K_1 at a and density f_1 on $[a,b] \subset (0,1)$ for Player 1, and probability atom K_2 at $1-b$ and density f_2 on $[1-b, 1-a]$ for Player 2. The densities start strictly positive with $f_1(a) = \left(\frac{a}{1-a}\right)K_1$ and $f_2(1-b) = \left(\frac{1-b}{b}\right)K_2$ and are convex increasing with positive derivatives of every order.*

Proof. (Included here rather than the appendix because it is relatively simple and helpful for intuition.)

(1) The lower and upper bounds are strictly between zero and one: $0 < a < b < 1$ *for Player 1's strategy* p *and* $0 < c < d < 1$ *for Player 2's strategy* q.

In an interval equilibrium, Player 2 will place positive probability on bids in some proper subinterval $(c', d') \subset [c, d]$. If Player 1 plays $p = 0$, his payoff will be 0 if Player 2 bids anything but $q = 0$, and will be positive (at 1/2) if Player 2 bids $q = 0$. If Player 1 bids $1 - d'$, on the other hand, his payoff will be positive if Player 2 bids in $[c', d']$ and 1 if Player 2 bids $q = 0$. Hence, Player 1 should never bid $p = 0$ and the lower bound of his support is $a > 0$ (with $c > 0$ by similar reasoning).

Since Player 2 puts zero probability on $[0, c)$, if Player 1 bids $p = 1$ there is always breakdown and a payoff of 0. On the other hand, if Player 1 bids $p = 1 - c$ he always receives a positive payoff, and thus he should never choose $p = 1$, showing that $b < 1$ (and $d < 1$ by parallel reasoning).

(2) The supports are balanced: Player 2's support is $[c, d] = [1 - b, 1 - a]$.

Suppose $c < 1 - b$. Player 2 would deviate because a bid of $q \in [c, 1 - b)$ would earn a lower share than $q = 1 - b$ and he could therefore move all the mass in $[c, 1 - b)$ to $1 - b$ without risking breakdown. Suppose $c > 1 - b$. In that case there is breakdown whenever Player 1 bids p in the interval $(1 - c, b]$. Player 1 can avoid that zero payoff by reducing b so that $c = 1 - b$ and reassigning the probability in any way. Thus, $c = 1 - b$.

Suppose $d < 1 - a$. Player 1 would deviate because a bid of $p \in [a, 1 - d)$ would earn a lower share than $p = 1 - d$ and he could therefore move all the mass in $[a, 1 - d)$ to $1 - d$ without risking breakdown. Suppose $d > 1 - a$. In that case there is breakdown whenever Player 2 bids q in the interval $(1 - a, d]$. Player 2 can avoid that zero payoff by reducing d so that $d = 1 - a$ and reassigning the probability in any way. Thus, $d = 1 - a$.

(3) The mixing distributions have probability atoms $K_1 > 0$ *and* $K_2 > 0$ *at* a *and* $1 - b$, *and only there.*

Player 1's expected payoff is positive because Player 2 will not bid more than $q = 1 - a$, and if Player 1 bids $p = a$ breakdown will not result and his expected payoff will be at least a. Consider Player 1's pure strategy of bidding $p = b$, the upper end of his support. This will cause breakdown unless Player 2 bids $q = 1 - b$, the lower end of his support. But if there is no probability atom for Player 2 at $1 - b$, Player 1's expected payoff from bidding b is zero. That is impossible in equilibrium, since zero is less than Player 1's positive payoff from bidding a. Hence, in equilibrium, Player 2 must have a probability atom at $1 - b$, the low end of his support.

Suppose Player 2 has an atom of size K at some other point x in $(1 - b, 1 - a]$. This would induce Player 1 to deviate. Player 1's pure strategy of bidding $1 - x$ must have positive expected payoff, since that share is positive and breakdown will not occur unless $q > x$, which has probability less than 1. Player 1's pure strategy of bidding $p = 1 - x + \epsilon$ for sufficiently small ϵ, however, will have a lower expected payoff than from $1 - x$ because the expected loss from the increased probability of breakdown will be at least $(1 - x)K$ but the expected gain from having an ϵ higher share tends to 0 as ϵ tends to 0. Note that this argument fails to apply if $x = 1 - b$, because then a pure-strategy bid of $1 - x + \epsilon$ is not in Player 1's support and does not require a payoff equal to that of $p = 1 - x$.

A parallel argument shows that Player 1 must have an atom at $p = a$ and only at $p = a$.

(4) The equilibrium strategies apart from the atoms consist of densities f_1 and f_2 that start at the positive levels $f_1(a) = \left(\frac{a}{1-a}\right)K_1$ and $f_2(1-b) = \left(\frac{1-b}{b}\right)K_2$.

Player 2's expected payoff from the pure strategy of bidding q is

$$\pi_2(q) = \left(\frac{q}{q+a}\right)K_1 + \int_a^{1-q}\left(\frac{q}{p+q}\right)d\mu_1(p) \tag{6}$$

for $q \in [1-b, 1-a]$. Since the payoffs from Player 2's pure-strategy best responses are constant for all q in the support, the derivative of payoff (6) identically equals zero, and by the generalized Leibniz rule we may differentiate under the integral sign:

$$0 \equiv \frac{d\pi_2(q)}{dq} = \left(\frac{a}{(q+a)^2}\right)K_1 + \int_a^{1-q}\left(\frac{p}{(p+q)^2}\right)d\mu_1(p) - \frac{d}{dx}_{x=1-q}\int_a^{x}\left(\frac{q}{p+q}\right)d\mu_1(p). \tag{7}$$

Since the left hand side is 0 and μ_1 is nonatomic in $(a, b]$, the first two terms on the right hand side are continuous in q. Hence the derivative $\frac{d}{dx}_{x=1-q}\int_a^x\left(\frac{q}{p+q}\right)d\mu_1(p)$ must exist and be continuous for all $q \in [1-b, 1-a]$. Since $\frac{p}{q+p}$ is smooth in p and q, the Borel measure μ_1 must be in the Lebesgue class with continuous density f_1, i.e. $d\mu_1(p) = f_1(p)dp$ and $\frac{d}{dx}_{x=1-q}\int_a^x\left(\frac{q}{p+q}\right)d\mu_1(p) = qf_1(1-q)$. Setting $x \equiv 1-q$ for $x \in [a, b]$ and solving for $f_1(x)$, we see that for it to be the equilibrium density, it must be true that:

$$f_1(x) = \left(\frac{1}{1-x}\right)\left[\left(\frac{a}{(1-x+a)^2}\right)K_1 + \int_a^x\left(\frac{p}{(1-x+p)^2}\right)f_1(p)dp\right]. \tag{8}$$

By setting $x = a$ we can see that the starting density is $f_1(a) = \left(\frac{a}{1-a}\right)K_1$. We can derive Player 2's density similarly by starting with $\pi_1(p)$, differentiating with respect to p, solving for $f_2(x)$, and setting $x = 1-b$ to yield $f_2(1-b) = \left(\frac{1-b}{b}\right)K_2$.

(5) The densities $f_1(p)$ and $f_2(q)$ have strictly positive derivatives of all orders, e.g., $f_1' > 0, f_1'' > 0, f_1''' > 0, \ldots$

Noting that the right hand side of (8) is differentiable, we obtain that f_1 is differentiable, and we can differentiate equation (8) to obtain

$$\begin{aligned} f_1'(x) = &\left(\frac{1}{(1-x)^2(1-x+a)^2} + \frac{2}{(1-x)(1-x+a)^3}\right)aK_1 + \frac{xf_1(x)}{(1-x)} \\ &+ \int_a^x\left(\frac{p}{(1-x)^2(1-x+p)^2} + \frac{2p}{(1-x)(1-x+p)^3}\right)f_1(p)dp \end{aligned} \tag{9}$$

The right hand side of (9) is differentiable for the same reason as in (8), and so f_1' is differentiable and we obtain,

$$
\begin{aligned}
f_1''(x) = &\left(\frac{2}{(1-x)^3(1-x+a)^2} + \frac{4}{(1-x)^2(1-x+a)^3} + \frac{6}{(1-x)(1-x+a)^4} \right) aK_1 \\
&+ \left(\frac{1}{1-x} + \frac{2x}{1-x} + \frac{2x}{(1-x)^2} \right) f_1(x) + \frac{x f_1'(x)}{1-x} \\
&+ \int_a^x \left(\frac{2p}{(1-x)^3(1-x+p)^2} + \frac{4p}{(1-x)^2(1-x+p)^3} + \frac{6p}{(1-x)(1-x+p)^4} \right) f_1(p)\, dp.
\end{aligned} \tag{10}
$$

Note that $f_1' > 0$ and $f_1'' > 0$, since $0 < x < 1$. Continuing inductively, we obtain that derivatives of every order for f_1 since $f_1^{(k)}$ consists of rational functions and integrals of rational functions in x times lower order derivatives of f_1 as in expressions (9) and (10).

We claim that for each k, $f_1^{(k)}$ is positive. At the k-th stage of differentiation, the rational function coefficients for aK_1 or $f_1^{(\ell)}$ with $0 \le \ell < k$ appearing in the expression for $f_1^{(k)}$, when kept unsimplified, have summands whose denominators all have the form $(1-x)^m(1-x+a)^n$ for nonnegative integers m and n, with the numerators being positive integers or positive integer multiples of x. Similarly, for the coefficient f_1 in the integrand, the summands are rational functions whose denominators have the form $(1-x)^m(1-x+p)^n$ for nonnegative integers m and n and whose numerators are positive integer multiples of p. By the inductive hypothesis, the lower order derivatives of f_1 are positive and so each of these terms is positive for $0 < a \le x < 1$. The positive sign of the derivatives of $f_2(q)$ can be shown similarly. □

We have not yet demonstrated existence or uniqueness or found solutions for the atoms and the mixing densities, but it will be appropriate to discuss the intuition here and explain the difficulties that will be involved in finding the distributions. Figure 1 shows what we will find when the densities are computed using the formula from Proposition 2 below. It shows symmetric equilibria for four different supports: $[.1, .9]$, $[.2, .8]$, $[.3, .7]$, and $[.4, .6]$. As a gets smaller, the density begins flatter and ends steeper and the atom is smaller.

The derivative expression (7), which we expand here, is helpful for intuition about the tradeoffs the players are making. It must hold for every q in Player 2's support.

$$
0 = \frac{d\pi_2(q)}{dq} = \frac{aK_1}{(q+a)^2} + \int_a^{1-q} \left(\frac{p}{(p+q)^2} \right) f_1(p)dp - q f_1(1-q). \tag{11}
$$

The first two terms of (11) are the advantages of using a higher bid q. Term 1 represents the marginal payoff from Player 2's share increasing when Player 1 chooses the atom bid of a, which never causes breakdown. Term 2 represents Player 2's marginal payoff as the result of his share increasing when he raises q if Player 1 bids between a and q so raising q does not cause breakdown. Term 3 represents Player 1's disadvantage from bidding higher. It is the

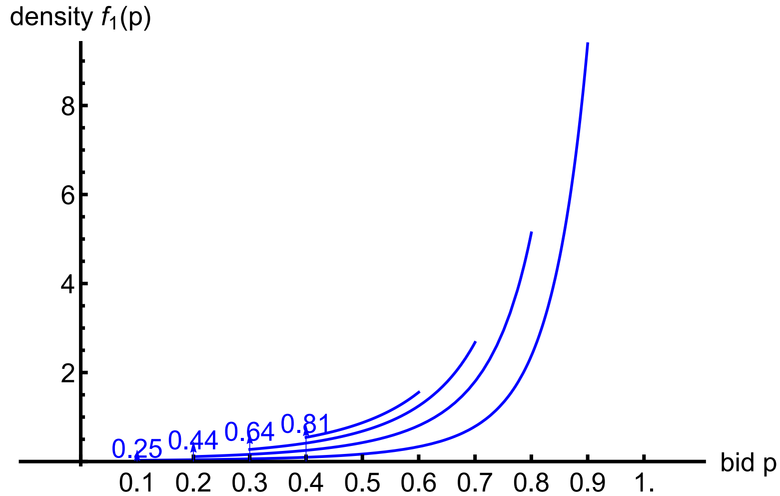

FIGURE 1. THE SYMMETRIC-EQUILIBRIUM MIXING DENSITY $f_1(p)$ AND ATOM K_1 FOR $a = .10, .20, .30, .40$

increase in the probability of breakdown, and the resulting loss of the share q. The crucial level of p is $p = 1 - q$, since that is at the edge of breakdown. The probability density for Player 1 choosing that bid is $f_1(1 - q)$, and since the two player's bids add up to 1 at that bid, Player 2's lost pie share is q if he edges past breakdown.

To see why the equilibrium density is increasing, think of how the marginal benefits and costs change for Player 2 as q rises. Imagine if the f_1 density were uniform instead of increasing. Both benefits, terms 1 and 2, in equation (11) would be larger for small q, for the reason that with small q, increasing the bid a given amount from that low level (and thus starting from a small share) has a bigger effect on the share. For term 2, the integral, there is an additional reason why the benefit would be bigger unless the density increases: at small q, term 2 includes a larger probability range $[a, 1 - q]$ over which the shares add up to less than one and breakdown is avoided. If we start with a large q, near the top of the support, then increasing q yields a greater share, but obtained with small probability; most of the expected payoff from a high q is coming from the atom of probability with which Player 1 bids $p = a$. This gives the second reason $f_1(p)$ must be rising— so that a small q's wider interval of p's that avoid breakdown is offset by smaller values of the density f_1 over that wide interval.

Term 3 in (11) provides another reason why f_1 must increase: the rise in the marginal cost of breakdown, $qf_1(1 - q)$. The density takes the argument $1 - q$ because $p = 1 - q$ is the threshold for breakdown, and hence $f_1(1 - q)$ is the rate of increase of the probability of breakdown as q increases. If f_1 were uniform, the marginal cost would increase with q, because a constant rate of increase in the probability of breakdown causes Player 2 more harm when his share is bigger. Thus, f_1 must be rising, to make the marginal cost of $qf_1(1-q)$ smaller for the critical $1 - q$ threshold when q is large.

There also exist asymmetric interval equilibria, as shown in Figure 2 (again, using the explicit formula from Proposition 2 below). Player 1 bids an atom of approximately .94 at $a = .2$ and mixes over $[.2, .4]$, while Player 2 bids an atom of approximately .33 at $a = .6$ and mixes over $[.6, .8]$. The supports are balanced because $.2 + .8$ and $.4 + .6$ equal 1. Player 2 always bids more and has a higher expected payoff, and the modal outcome will be a share of .25 for Player 1 and .75 for Player 2 as a result of bids of $p = .2$ and $q = .6$. Each player's mixing density is rising convexly, as Proposition 1 requires.

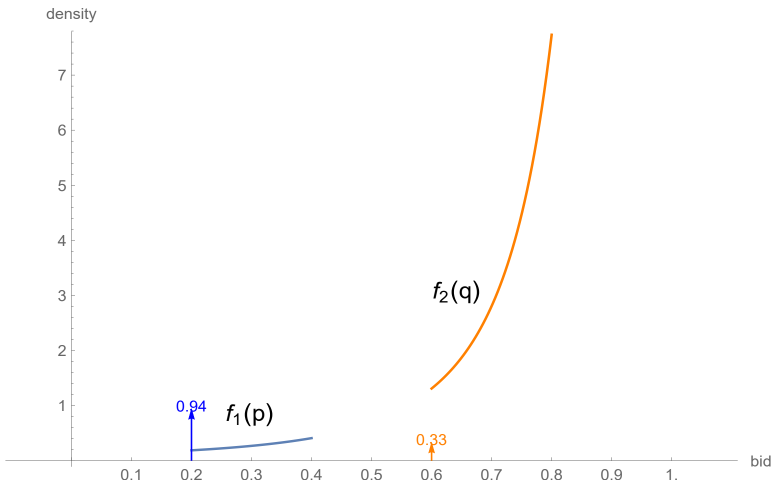

FIGURE 2. THE ASYMMETRIC-EQUILIBRIUM MIXING DENSITIES $f_1(p)$ OVER $[.2,.4]$ AND $f_2(q)$ OVER $[.6,.8]$

We will now proceed to find a solution for the mixing density and atom for an interval equilibrium. Proposition 1 tells us that the mixing distribution for Player 1 is a measure μ_1 on $[a, b]$ consisting of a single atom at a together with a positive measure in the Lebesgue class,

$$\mu_1(p) = K_1 \delta_a(p) + f_1(p)dp,$$

where $K_1 = \frac{\pi_1}{1-a}$, with expected payoff π_1, and f_1 is some smooth function with all positive derivatives and subject to the normalization that μ_1 is a probability measure. The fundamental equation that characterizes the equilibrium then says that Player 1's f_1 and K_1 must be such that Player 2's expected payoff from any q in his support equals that from playing $q = 1 - a$:

$$\pi_2(q) = K_1 \left(\frac{q}{q + a} \right) + \int_a^{1-q} \left(\frac{q}{q + p} \right) f_1(p)dp = K_1(1 - a). \tag{12}$$

The purpose of the following proposition is to solve equation (12) for f_1 and K_1.

Proposition 2. *Splitting a Pie with the proportional sharing rule has unique interval equilibrium mixing strategies among Borel measures for given supports. Given any $0 < a < b < 1$, Players 1 and 2 use real-analytic densities $f_1(p)$ on $[a,b]$ and $f_2(q)$ on $[1-b, 1-a]$ with atoms of probability K_1 at a and K_2 at $1-b$ such that:*

$$f_1(p) = aK_1 \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i, \quad K_1 = \frac{1}{1 + a\sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1}}, \tag{13}$$

and

$$f_2(q) = (1-b)K_2 \sum_{i\geq 0} \frac{m^{(i)}(1-b)}{i!}(q-(1-b))^i, \quad K_2 = \frac{1}{1 + (1-b)\sum_{i\geq 0} \frac{m^{(i)}(1-b)}{(i+1)!}(b-a)^{i+1}} \tag{14}$$

with $m^{(i)}(\cdot)$ for both densities recursively expressed as:

$$m^{(i)}(x) = \frac{(i+1)!}{1-x} + \sum_{j=0}^{i-1}\left(\frac{(j+1)(i-1-j)! + (i-j)!x}{1-x}\right)m^{(j)}(x). \tag{15}$$

For the proof, see Appendix A.

If we expand out the first few terms (see the end of Appendix A for calculations), f_1 evaluates to:

$$\begin{aligned} f_1(p) =& aK_1\left(\frac{1}{(1-a)} + \left(\frac{3-a}{(1-a)^2}\right)(p-a) + \left(\frac{13-10a+3a^2}{2(1-a)^3}\right)(p-a)^2 + \right. \\ & \left. \left(\frac{71-89a+55a^2-13a^3}{6(1-a)^4}\right)(p-a)^3 + \sum_{i\geq 4}\left(\frac{m^{(i)}(a)}{i!}\right)(p-a)^i\right). \end{aligned} \tag{16}$$

The equations for f_1 and f_2 in Proposition 2 are not approximations, but exact: the infinite series converges. Showing this is the hardest part of the proof. If only a finite number of terms are included, it becomes an approximation. In this, it is like the formula $\pi = \sqrt{\sum_{n=1}^{\infty}\frac{6}{n^2}}$, since the exact value of π can be represented by an infinite number of rational terms.

The equilibrium densities have two surprising features. First, both $f_1(p)$ and $f_2(q)$ are composed of the same $m^{(i)}(\cdot)$ functions, even in an asymmetric equilibrium where the two players use different mixing supports. Second, the upper bounds of the support only enter into the expression for the atom weights K_1 and K_2 and thus the overall scaling of f_1 and f_2.

4. Concluding Remarks

"Splitting a Pie" is the simplest of bargaining games but it generates surprisingly complex equilibria. We have explored those equilibria when players bid over interval supports with proportional sharing. We have seen that any "balanced" pair of intervals for the two players supports a unique equilibrium, and have derived explicit formulae for the Borel probability measures of the equilibria. These equilibria will always consist of a single probability atom at the lowest bid in the player's interval plus an increasing probability density over the rest of the interval that begins strictly positive and is convex with positive derivatives of all orders. Although the qualitative features of the equilibria are simple, the derivations were technical, so let us now return to the economic motivation for the model.

The situation being modeled is that of two players choosing how tough to be with each other in sharing something of value. In the pure-strategy equilibria, they begin with an expectation of how tough each other will be, and their behavior confirms that expectation. In the mixed-strategy equilibria, they begin with an expectation of a range of toughness that the other player might choose, and their behavior, while different from realization to realization, does not contradict that expectation. Indeed, time and repetition would confirm it. To model mixed strategies, one must also make some assumption as to how the pie is shared when both players bid conservatively so that the bids—the toughness levels— add up to less than one. Our preferred assumption is that the shares are proportional to the bids and that the shares add up to one.

We found, as Malueg and others have, that mixed-strategy equilibria do exist. Although we must start with the players having a common expectation of the interval supports of the bids — the standard common-knowledge assumption behind Nash equilibrium — the players can then deduce the unique mixing distributions that will be chosen in equilibrium. If both players mix over the same interval, the most common choice for each player will be to bid the minimum level of toughness in the support, which means that a common outcome in a symmetric equilibrium will be for both to choose equal toughness and receive equal shares. Also common will be breakdown, when each player guesses wrong as to what the other player will choose and wishes, ex post, that he had not been so tough. The remaining case will be a continuous range of equilibrium shares, not from 0% to 100%, but over a range depending on their expectations of the minimum and maximum toughness of their rival. If players use unequal support intervals, this asymmetric equilibrium models the case where one player tends to choose tougher bids and so will have a higher expected share.

We think these are good features to find for a model of bargaining, but the solution is intricate because each player's share depends continuously on both how tough he is and how tough his rival is. The Nash Demand Game, with its "take what you bid" sharing rule, is similar in its qualitative features and far simpler to solve. Its downside is that players who agree, usually agree only over part of the pie and discard the rest. Without that downside, but still simple to analyze even under the proportional sharing rule, is the Hawk-Dove equilibrium in which each player mixes over a single tough and a single mild bid. The interval equilibria we have described turn out to have qualitative features in some ways like those of Hawk-Dove. Both types of equilibria have as common realizations the two outcomes of (a) breakdown and (b) a split depending on the players' expectation of their relative toughnesses. The difference is that with interval equilibria there is also a continuous range of other splits. Thus, the lesson for applied work may be that the simpler Hawk-Dove equilibria have results not dissimilar from interval equilibria.

Appendix A. Proof of Proposition 2.

Proposition 2. *Splitting a Pie with the proportional sharing rule has unique interval equilibrium mixing strategies among Borel measures for given supports. Given any $0 < a < b < 1$, Players 1 and 2 use real-analytic densities $f_1(p)$ on $[a, b]$ and $f_2(q)$ on $[1-b, 1-a]$ with atoms of probability K_1 at a and K_2 at $1-b$ such that:*

$$f_1(p) = aK_1 \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i, \quad K_1 = \frac{1}{1 + a\sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1}}, \tag{17}$$

and

$$f_2(q) = (1-b)K_2 \sum_{i\geq 0} \frac{m^{(i)}(1-b)}{i!}(q-(1-b))^i, \quad K_2 = \frac{1}{1 + (1-b)\sum_{i\geq 0} \frac{m^{(i)}(1-b)}{(i+1)!}(b-a)^{i+1}} \tag{18}$$

with $m^{(i)}(\cdot)$ for both densities recursively expressed as:

$$m^{(i)}(x) = \frac{(i+1)!}{1-x} + \sum_{j=0}^{i-1} \left(\frac{(j+1)(i-1-j)! + (i-j)!x}{1-x} \right) m^{(j)}(x). \tag{19}$$

Proof. It will later reduce clutter to define $m(p) \equiv \frac{1}{aK_1} f_1(p)$ and rewrite the crucial equation (12) that describes equilibrium as

$$K_1 \left(\frac{q}{q+a} \right) + \int_a^{1-q} \left(\frac{q}{q+p} \right) aK_1 m(p) dp = K_1(1-a). \tag{20}$$

Our aim will be to find the derivatives of m and use them to construct a power series representation of the function. Moving the first term of the previous equation (the contribution from the point mass) to the right-hand side and dividing by aK_1 yields

$$\int_a^{1-q} \left(\frac{q}{q+p} \right) m(p) dp = \left(\frac{1-a}{a} - \frac{q}{a(q+a)} \right) = \frac{1-q-a}{q+a} \tag{21}$$

for all $q \in [0, 1-a]$.

By Proposition 1, m is infinitely differentiable. Differentiating both sides of (21) with respect to q yields, since the equation is true for all q in the interval $[0, 1-a]$,

$$-qm(1-q) + \int_a^{1-q} \left(\frac{1}{q+p} - \frac{q}{(q+p)^2} \right) m(p)\, dp = -\frac{1-q-a}{(q+a)^2} - \frac{1}{q+a} \tag{22}$$

which simplifies to

$$qm(1-q) - \int_a^{1-q} \left(\frac{p}{(q+p)^2} \right) m(p)\, dp = \frac{1}{(q+a)^2}. \tag{23}$$

We obtain, for example by setting $1-q = a$, that $m(a) = \frac{1}{1-a}$, which gives the $f_1(a) = \frac{aK_1}{1-a}$ we found in Proposition 1. Again taking the first derivative with respect to q of both sides of equation (23), and multiplying through by -1, we obtain

$$qm'(1-q)-(2-q)m(1-q)-2\int_a^{1-q}\left(\frac{p}{(q+p)^3}\right)m(p)\,dp=2(q+a)^{-3}.$$

Recursively define $r_i(q)$ by

$$r_0(q)=qm(1-q)\quad\text{and}\quad r_i(q)=-r'_{i-1}(q)-i!\,(1-q)\,m(1-q)\quad\text{for}\quad i>0. \tag{24}$$

Our objective is to obtain expressions for the derivatives of $m^{(i)}(1-q)$ that we can use to construct the f_1 density. The linear recursion relation (24) can be solved to get $r_n(q)$ in terms of all the $m^{(i)}$ from $i=0$ to n instead of in terms of $r'_{n-1}(q)$, yielding

$$r_n(q)=qm^{(n)}(1-q)-\sum_{i=0}^{n-1}\left((i+1)(n-1-i)!+(n-i)!(1-q)\right)m^{(i)}(1-q). \tag{25}$$

Taking the n-th derivative of equation (23) in $1-q$ and multiplying by (-1) we obtain

$$r_n(q)+(n+1)!\int_a^{1-q}\left(\frac{p}{(q+p)^{n+2}}\right)m(p)\,dp=(n+1)!(q+a)^{-(n+2)}. \tag{26}$$

Evaluating (26) at $q=1-a$ gives $r_n(1-a)=(n+1)!$ since the integral vanishes. We can substitute $r_n(1-a)=(n+1)!$ into (25) in evaluating $r_n(q)$ at $q=1-a$ to get

$$(n+1)!=(1-a)m^{(n)}(a)-\sum_{i=0}^{n-1}((i+1)(n-1-i)!+(n-i)!a)m^{(i)}(a), \tag{27}$$

which after rearranging terms becomes

$$m^{(n)}(a)=\frac{(n+1)!}{1-a}+\sum_{i=0}^{n-1}\left(\frac{(i+1)(n-1-i)!+(n-i)!a}{1-a}\right)m^{(i)}(a). \tag{28}$$

Equation (28) is a recursive formula yielding the derivative $m^{(n)}(a)$ in terms of $m^{(i)}(a)$ for $i<n$, and hence ultimately in terms of a. This immediately implies that all of the $m^{(i)}(a)$ are positive, since $m^{(0)}(a)=m(a)$ is positive. Hence, the mixing density rises and is convex at a. We will able to use this to find a power series solution for $m(p)$ if we can find an appropriate bound for the sum of its derivatives.

Assuming that bound, we can also find the formula for the size of the atom, K_1. Since μ_1 is a probability measure, integrating $d\mu_1(p)=K_1\delta_a(p)+f_1(p)dp=K_1\delta_a+aK_1m(p)dp$ over the full support $[a,b]$ gives us 1. Therefore $K_1=\frac{1}{1+a\int_a^b m(p)dp}$. We can calculate the integral in the denominator in terms of the power series expansion for $m(p)$:

$$\int_a^b m(p)dp = \int_a^b \sum_{i\geq 0} \left(\frac{m^{(i)}(a)}{i!}\right)(p-a)^i dp = \sum_{i\geq 0}\left(\frac{m^{(i)}(a)}{i!}\right)\int_a^b (p-a)^i dp$$
$$= \sum_{i\geq 0}\left(\frac{m^{(i)}(a)}{i!}\right)\left(\frac{(b-a)^{i+1}}{i+1}\right) = \sum_{i\geq 0}\left(\frac{m^{(i)}(a)}{(i+1)!}\right)(b-a)^{i+1}. \tag{29}$$

It remains to establish the key bounds, which we do in Lemma 1 below. Corollary 1.2 of Lemma 1 will then establish the main statement of Proposition 2, namely the existence, uniqueness and real-analyticity of the power series solution whose coefficients are given by (19). The statements and formulas for f_2 and K_2 follow immediately from the symmetry forced by the balance requirement.

□

Lemma 1. *The derivatives $m^{(n)}(a)$ arising in the formal power series for $m(p)$ at $v = a$ satisfy the double-sided bounds,*

$$\frac{n!}{(1-a)^n} < m^{(n)}(a) < \frac{(n+3)!(n+1)}{(1-a)^{n+1}}. \tag{30}$$

Proof. We will proceed inductively for both bounds. We treat the upper bound first. First observe that the proposition is true for the base case, the original function's value $m^{(0)}(a)$:

$$m^{(0)}(a) = m(a) = \frac{1}{1-a} < \frac{(0+3)!(0+1)}{(1-a)^{0+1}}.$$

We will show that if the proposition holds for all derivatives lower than n, then it holds for $m^{(n)}(a)$ too. Thus, the inductive step that we assume is that

$$m^{(i)}(a) < \frac{(i+3)!(i+1)}{(1-a)^{i+1}} \text{ for all } 0 \leq i \leq n-1. \tag{31}$$

Equation (28) gave us an expression for $m^{(n)}(a)$ in terms of a sum of the lower derivatives $m^{(i)}(a)$ multiplied by $\frac{(i+1)(n-1-i)!+(n-i)!a}{1-a}$. Let us multiply both sides of (31) by that coefficient and rearrange:

$$
\begin{aligned}
\frac{(i+1)(n-1-i)!+(n-i)!a}{1-a}m^{(i)}(a) &< \left(\frac{(i+1)(n-1-i)!+(n-i)!a}{(1-a)^{i+2}}\right)(i+3)!(i+1)\\
&= \frac{(n-i)!(\frac{i+1}{n-i}+a)(i+3)!(i+1)}{(1-a)^{i+2}}\\
&= \frac{(i+1)(n-i)!(i+3)!(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{(1-a)^{n+1}}\\
&= (n+3)!\left(\frac{(i+1)(n+3-(i+3))!(i+3)!(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{(n+3)!(1-a)^{n+1}}\right)\\
&= (n+3)!\left(\frac{(i+1)(\frac{i+1}{n-i}+a)(1-a)^{n-1-i}}{\binom{n+3}{i+3}(1-a)^{n+1}}\right)\\
&\leq (n+3)!\left(\frac{(i+1)(i+2)}{\binom{n+3}{i+3}(1-a)^{n+1}}\right).
\end{aligned}
$$

This last inequality holds for all positive n and $i < n$, and establishes a bound for each of the terms in the sum of derivatives making up $m^{(n)}(a)$ in (28). To check the bound for $m^{(n)}(a)$, we sum the right hand side bounds for $i = 0, \dots, n-1$, and add $\frac{(n+1)!}{1-a}$ from (28) to get

$$
\begin{aligned}
m^{(n)}(a) &= \frac{(n+1)!}{1-a} + \sum_{i=0}^{n-1} \left(\frac{((i+1)(n-1-i)! + (n-i)!a)}{1-a} \right) m^{(i)}(a) && (32)\\
&< \frac{(n+1)!}{1-a} + \sum_{i=0}^{n-1} (n+3)! \left(\frac{(i+1)(i+2)}{\binom{n+3}{i+3}(1-a)^{n+1}} \right) && (33)\\
&\le \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{(1-a)^n}{(n+3)(n+2)(n+1)} + \sum_{i=0}^{n-1} (i+1) \binom{n+3}{i+3}^{-1} \right) && (34)\\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{(1-a)^n}{(n+3)(n+2)(n+1)} + \frac{6}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} \right. && (35)\\
&\qquad \left. + \frac{2(n-1)}{(n+3)(n+2)} + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + \sum_{i=1}^{n-4} (i+1) \binom{n+3}{i+3}^{-1} \right)\\
&\le \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{1}{(n+3)(n+2)(n+1)} + \frac{6}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} \right. && (36)\\
&\qquad \left. + \frac{2(n-1)}{(n+3)(n+2)} + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + (n-3) \sum_{i=1}^{n-4} \binom{n+3}{4}^{-1} \right)\\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(\frac{7}{(n+3)(n+2)(n+1)} + \frac{n}{n+3} + \frac{2(n-1)}{(n+3)(n+2)} \right. && (37)\\
&\qquad \left. + \frac{6(n-2)}{(n+3)(n+2)(n+1)} + \frac{24(n-3)(n-4)}{(n+3)(n+2)(n+1)n} \right)\\
&= \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \left(1 + \frac{-n^3 + 21n^2 - 181n + 288}{(n+3)(n+2)(n+1)n} \right) && (38)\\
&\le \frac{(n+3)!(n+1)}{(1-a)^{n+1}}. && (39)
\end{aligned}
$$

Inequality (34) results from pulling out the $\frac{(n+3)!(n+1)}{(1-a)^{n+1}}$ factor and noting that $i+2 \le n+1$ for $i \le n-1$. Equality (35) results from expanding the $i = 0, n-1, n-2, n-3$ terms of the sum. Inequality (36) results from noting that $(1-a)^n \le 1$ and also in the range $1 \le i \le n-4$ we have both $i+1 \le n-3$ and $\binom{n+3}{i+3} = \binom{n+3}{n-i} \ge \binom{n+3}{4}$. Equality (37) simply records the fact that $(n-3)(n-4)\binom{n+3}{4}^{-1} = \frac{24(n-3)(n-4)}{(n+3)(n+2)(n+1)n}$, and the ensuing algebra gives equality (38). Inequality (39) follows from the fact that the numerator polynomial $-n^3+21n^2-181n+288$ has a leading negative term and a single real root in the interval $(2,3)$. Hence it is negative for all integers $n \ge 3$. The decomposition of the sum we used also holds only when $n \ge 4$. We have checked the $n = 0$ base case already, so it remains to check the inductive step for $n = 1, 2,$ and 3.

Recall that $a \in (0,1)$. For $n = 1$ we have

$$\begin{aligned} m^{(1)}(a) &< \frac{(1+3)!(1+1)}{(1-a)^{1+1}}\left(\frac{(1-a)^1}{(1+3)(1+2)(1+1)} + \sum_{i=0}^{1-1}(i+1)\binom{1+3}{i+3}^{-1}\right) \\ &= \frac{(1+3)!(1+1)}{(1-a)^{1+1}}\left(\frac{(1-a)}{24} + \frac{1}{4}\right) \\ &< \frac{(1+3)!(1+1)}{(1-a)^{1+1}}. \end{aligned}$$

For $n = 2$ we have from (33),

$$\begin{aligned} m^{(2)}(a) &< \frac{(2+3)!(2+1)}{(1-a)^{2+1}}\left(\frac{(1-a)^2}{(2+3)(2+2)(2+1)} + \sum_{i=0}^{2-1}(i+1)\binom{2+3}{i+3}^{-1}\right) \\ &= \frac{(2+3)!(2+1)}{(1-a)^{2+1}}\left(\frac{(1-a)^2}{60} + \frac{1}{2}\right) \\ &< \frac{(2+3)!(2+1)}{(1-a)^{2+1}}. \end{aligned}$$

Finally, for $n = 3$ we have

$$\begin{aligned} m^{(3)}(a) &< \frac{(3+3)!(3+1)}{(1-a)^{3+1}}\left(\frac{(1-a)^3}{(3+3)(3+2)(3+1)} + \sum_{i=0}^{3-1}(i+1)\binom{3+3}{i+3}^{-1}\right) \\ &= \frac{(3+3)!(3+1)}{(1-a)^{3+1}}\left(\frac{(1-a)^3}{120} + \frac{41}{60}\right) \\ &< \frac{(3+3)!(3+1)}{(1-a)^{3+1}}. \end{aligned}$$

Thus we obtain the upper bound of Lemma 1, equation (30):

$$m^{(n)}(a) < \frac{(n+3)!(n+1)}{(1-a)^{n+1}} \quad \text{for all} \quad n \geq 0. \tag{40}$$

For the lower bound we proceed similarly. First, we consider the general case assuming the inductive hypothesis.

$$\begin{aligned} m^{(n)}(a) &= \frac{(n+1)!}{1-a} + \sum_{i=0}^{n-1} \frac{(i+1)(n-1-i)! + (n-i)!a}{1-a} m^{(i)}(a) \\ &\geq \frac{(n+1)!}{1-a} + \sum_{i=0}^{n-1} \frac{(i+1)(n-1-i)! + (n-i)!a}{1-a} \cdot \frac{i!}{(1-a)^i} \\ &= \frac{(n+1)!}{1-a} + \frac{n! + a(n-1)!}{(1-a)^n} + \sum_{i=0}^{n-2} \frac{((i+1)(n-1-i)! + (n-i)!a)i!}{(1-a)^{i+1}} \\ &> \frac{n!}{(1-a)^n}, \end{aligned}$$

where we have simply pulled the $n-1$ term out of the sum in the third line. Note that we will need to check just the base case, $n = 0$, as the above calculation holds for all $n \geq 1$.

For the $n = 0$ case we check,

$$m^{(0)}(a) = m(a) = \frac{1}{(1-a)} > \frac{(0)!}{(1-a)^0} = 1, \tag{41}$$

as desired. This completes the lower bound and the proof of Lemma 1. ☐

Lemma 1 has two important corollaries we are using to prove Proposition 2:

Corollary 1.1. *The power series,*

$$\sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i, \tag{42}$$

converges uniformly for all $p \in (-1+2a, 1)$. *Moreover, it has a pole at* $p = 1$.

Proof. By Lemma 1, whenever $-(1-a) < p-a < 1-a$ or equivalently, $-1+2a < p < 1$, we have

$$\left| \frac{m^{(i)}(a)}{i!}(p-a)^i \right| < \frac{(i+3)(i+2)(i+1)^2}{1-a} \epsilon^i,$$

where $\epsilon = \left|\frac{p-a}{1-a}\right| < 1$. In particular, this is the summand of an absolutely (and exponentially) convergent power series.

On the other hand, the lower bound of Lemma 1 shows that

$$\frac{m^{(i)}(a)}{i!}(1-a)^i > 1,$$

and hence the series diverges at $p = 1$. ☐

Corollary 1.2. *The function m defined on the interval $[a, b]$ for any $0 < a < b < 1$ given by*

$$m(p) = \sum_{i \geq 0} \frac{m^{(i)}(a)}{i!} (p - a)^i,$$

where the $m^{(i)}(a)$ are computed from (28) *is real-analytic on $[a, b]$ and yields the unique measurable function solving* (21).

Proof. First observe that $[a, b] \subset (-1 + 2a, 1)$. Since the series converges on the domain of definition by Lemma 1, it is real-analytic there. (Indeed it converges and is real-analytic on a larger open interval.)

We have established earlier in the proof of Proposition 2 that any solution to (21) is a real-analytic function $m(p)$ satisfying (19). Any such function $m(p)$ is uniquely determined by the values of $m^{(i)}(x)$ at $x = a$ which may be computed from the recursion (28).

□

Corollary 1.1 having established convergence and Corollary 1.2 uniqueness, the proof of Proposition 2 is complete.

A.1. **Calculating the $\mathbf{m^{(i)}(a)}$.** The terms $m^{(i)}(a)$ in Proposition 2 are calculated recursively. To obtain formula (16), we start with $m^{(0)}(a) = 1/(1 - a)$, and then using (19) we continue,

$$\begin{aligned} m^{(1)}(a) &= \frac{(1+1)!}{1-a} + \frac{(0+1)(1-1-0)! + (1-0)!a}{1-a} \cdot \frac{1}{1-a} \\ &= \frac{3-a}{(1-a)^2}, \end{aligned}$$

$$\begin{aligned} m^{(2)}(a) &= \frac{(2+1)!}{1-a} + \frac{(0+1)(2-1-0)! + (2-0)!a}{1-a} \cdot \frac{1}{1-a} + \frac{(1+1)(2-1-1)! + (2-1)!a}{1-a} \cdot \frac{3-a}{(1-a)^2} \\ &= \frac{13 - 10a + 3a^2}{(1-a)^3}, \text{ and} \end{aligned}$$

$$\begin{aligned} m^{(3)}(a) &= \frac{(3+1)!}{1-a} + \frac{(0+1)(3-1-0)! + (3-0)!a}{1-a} \cdot \frac{1}{1-a} \\ &\quad + \frac{(1+1)(3-1-1)! + (3-1)!a}{1-a} \cdot \frac{3-a}{(1-a)^2} + \frac{(2+1)(3-1-2)! + (3-2)!a}{1-a} \cdot \frac{13-10a+3a^2}{(1-a)^3} \\ &= \frac{71 - 89a + 55a^2 - 13a^3}{(1-a)^4}. \end{aligned}$$

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