The Parking Lot Problem

Maria Arbatskaya, Kaushik Mukhopadhaya, and Eric Rasmusen

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Abstract

If property rights are not assigned over individual goods such as parking spaces, competition for them can eat up the entire surplus. We find that when drivers are homogeneous, there is a discontinuity in social welfare between "enough" and "not enough." Building slightly too small a parking lot is worse than building much too small a parking lot, since both have zero net benefit and larger lots cost more to build. More generally, the welfare losses from undercapacity and overcapacity are asymmetric, and parking lots should be "overbuilt." That is, the optimal parking lot size can be well in excess of mean demand. Uncertainty over the number of drivers, which is detrimental in the first-best, actually increases social welfare if the parking lot size is too small.

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Maria Arbatskaya, Department of Economics, Emory University, Atlanta, GA 30322-2240. Phone: (404) 727 2770. Fax: (404) 727 4639. Email: marbats@emory.edu.

Kaushik Mukhopadhaya, Department of Economics, Emory University, Atlanta, GA 30322-2240. Phone: (404) 712 8253. Fax: (404) 727 4639. Email: kmukhop@emory.edu.

Eric Rasmusen, Dan R. and Catherine M. Dalton Professor, Department of Business Economics and Public Policy, Kelley School of Business, Indiana University, BU 456, 1309 E 10th Street, Bloomington, Indiana, 47405- 1701. Phone: (812) 855 9219. Fax: (812) 855 3354. Erasmuse@indiana.edu. http://www.rasmusen.org.

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1 Introduction

The problem of the commons shows up in many forms. In the classic version, citizens of the village graze too many animals in the communal green, wrecking its value for all. Overuse reduces the value of the property. The problem is that no single person owns the green. Dividing it up into private plots would solve the overgrazing, but would also eliminate economies of scale, and so villages turn to other solutions such as formal or informal regulation.

The version of the commons problem of this paper is different. A parking lot is divided into separate plots and access lanes in which parking is forbidden, an organizational form that solves the "overgrazing" problem and maximizes the benefit of the property. The problem we will address arises when the plots are demarked, but it is impractical to make each plot separately owned, controlled entirely by its owner and used either by himself or by someone who pays him a fee during the time he is not there. In this version, inefficiency will arise not from behavior that reduces the benefit of the property, but from behavior that creates costs as it determines the allocation of the property.

The context of parking is familiar to all of us from personal experience. Finding a parking space at all is a perennial source of frustration on college campuses, and tough competition for the limited number of parking spaces compels drivers to arrive well in advance of their preferred time. "Survival of the earliest" is the rule. On the other hand, shopping malls display an apparently useless excess of parking spaces. In contrast to universities, malls seem to go to ridiculous extremes, with acres of parking lots that are never full except at Christmas. Oddly enough, it is the non-profits that seem to spend too little on parking lots and the for-profits that seem to spend too much.

Parking studies recognize the importance of planning for a sufficient "cushion" in excess of necessary spaces. However, these studies are concerned mostly with the smooth flow of vehicles in and out of the parking area, or unforeseen circumstances such as disruptive repair. A 1987 study by *Walker Parking Consultants* (commissioned by Indianapolis to develop a

Regional Center Parking Plan) reports: "Thus, a supply of parking operates at peak efficiency when occupancy is 85% to 90%. When occupancy exceeds this level, there may be delays and frustration in finding a space. The parking supply may be perceived as inadequate even though there are spaces available in the system" (http://www.bts.gov/NTL/DOCS/rc.html).

In this paper we suggest that an equally important concern is strategic behavior by drivers. If the parking lot is built even slightly too small, rent-seeking competition by drivers can dissipate all of the rents from parking there, eliminating every trace of the lot's social usefulness and making its construction entirely wasteful. Contrary to first-best engineering concerns, a parking lot should be large enough to forestall wasteful competition. It can easily be socially optimal to have parking lots that on average are half empty, as we will show below by example. Shopping malls seem to realize this better than nonprofit institutions.

It is natural to view empty spaces as a reason to reduce the amount of available parking. Based on the finding that the parking supply count exceeded demand count by 30%, a Seattle report recommends reducing the parking supply, though the authors admit that *"A major policy- related issue is how much allowance to provide over the design-level demand in setting the size of a given parking facility."*[1] In a comprehensive book on parking, Shoup (2005) advocates against free and plentiful parking, which has various social costs including construction and maintenance costs, road congestion and environmental pollution. He argues that minimum off-street parking requirements for new construction should be abolished. Instead, he recommends charging for curb parking, at a price at which 15 percent of spaces are vacant.

In our paper, we will take the fact that parking is unpriced (or underpriced) as given, and we will argue that if parking is to be free, it ought to be plentiful. Strategic behavior of drivers must be considered when deciding on parking lot size. Having 30% of the spaces empty, on average, may well indicate too *small* a parking lot size, given uncertainty in the demand for parking. We will show that there is an asymmetry in the welfare effects of over- and under-supply of parking. Extra parking spaces increase costs gradually, but even

[1] See the *1991 Parking Utilization Study* undertaken by the Research and Market Strategy Division of the Transit Department in the Municipality of Metropolitan Seattle.

a minor shortage can result in a discontinuous and huge social loss, as rent-seeking eats up the entire value of whatever parking is available. Moreover, the welfare loss can be more severe with a small shortage than with a large shortage, for which there are fewer rents to dissipate.

In our model, drivers face a trade-off between the disutility of arriving earlier than their preferred time and the increased probability of securing a space in the parking lot. Since the cost of arriving early is incurred regardless of whether the driver is successful in finding a parking space, the contest is a multi-unit all-pay auction. Given the dynamic nature of the players' decisions about *when* to arrive at the lot, their strategies can be quite complex. Nevertheless, we are able to show that full (or almost full) rent dissipation occurs in any equilibrium when the size of the lot is too small. When the number of drivers is known to the planners, the optimal size of the parking lot is equal to the number of drivers. When the number is not known, the lot should be made so large that on average a large proportion of the spots will be unoccupied. The analysis indicates that perhaps universities have it wrong, while the malls have it right.

2 The Literature

The problem of managing congested facilities is a long-standing one in economics, a problem usually analyzed taking the size of the facility as given and studying the effects of various allocation systems. Queueing models with tolls, such as Naor (1969), assume an exogenous stationary customer arrival process and random service times. Customers benefit from the service, but they incur a constant cost per unit of time from queueing. In equilibrium, a consumer joins a queue if its length is below a threshold level. The last consumer in the queue is just indifferent between joining and staying out. The equilibrium outcome is inefficient, as a result of the negative externality that a customer imposes on those arriving later. Nevertheless, rent dissipation through queueing is not full because of the randomness in the arrival and departure processes. Naor shows how tolls can remedy the problem.

The large literature on traffic congestion focuses on the effects of different pricing systems. Arnott et al. (1993), for example, compare alternative toll regimes in a model with a single traffic bottleneck and identical commuters incurring linear time inconvenience costs. Part of this literature deals with capacity, looking especially at the interaction between different transportation modes: road capacity (and congestion) and rail capacity. Arnott and Yan (2000) survey this literature, mainly dating from the 1970's and add their own analysis of the problem. One of the elements in the capacity decision is what we will look at in the parking context: that more capacity will attract more users. Thus, Arnott and Small (2001) note that traditional approaches to dealing with congestion can be counterproductive; increased road capacity can attract drivers from alternative routes until the road is as congested as before. Recent papers by Anderson and de Palma (2004) and Arnott and Rowse (1999) study parking congestion and its pricing in a model with drivers cruising for parking on two distinct spatial structures: a line with side streets leading to a common desired destination and a circular road with uniformly distributed destinations, respectively. The optimal pricing of parking internalizes the congestion externality in parking. Arnott (2001) provides a good overview of the literature and offers comments on its future direction.

A difference between the congestion literature and the present paper is that we take a game-theoretic approach rather than the more common aggregate approach. Moreover, the road congestion cost rises gradually, whereas the rent-seeking cost rises sharply when the number of drivers just exceeds the parking lot size. Vickrey's seminal 1969 article on traffic bottlenecks is a good illustration of this. In his example, if capacity is below 120, congestion causes delays, while if it is above 120 there is no delay. As a result, some drivers arrive at the bottleneck early to avoid the delay. Vickrey notes a "sharp discontinuity" in the amount of delay at the level of capacity just sufficient to accommodate the traffic, and points out that optimal investments in capacity extension differ in the first-best and the second-best situations. In the first- best, using the optimal price structure (a toll fee that leads to efficient facility use), the benefits of capacity extension are not as "capricious" as in the second-best, when access cannot be restricted by fees. In the second-best, "Expansion inadequate to take care of the entire traffic demand...may turn out to be hardly worthwhile." In Vickrey's model, however, any capacity extension does reduce delays and is beneficial to travelers. We will show that an insufficient increase in capacity might not have any benefit whatsoever to offset construction costs, so that the facility is a pure waste.

Underpricing has also been studied in the context of shopping. The strategic incentives of people to adjust their purchase schedules are analyzed by Deacon and Sonstelie (1991). Consumers choose the size of purchases to minimize the total cost of shopping for an underpriced good, which includes shopping and storage costs. The waiting time in a queue increases until the market clears. A price ceiling makes consumers no better off, though suppliers are worse off, which thus generates a deadweight loss in rationing by waiting. (See also Deacon and Sonstelie [1985, 1989] and Deacon [1994].)

In the theory of waiting lists proposed by Lindsay and Feigenbaum (1984), market clearing occurs as a result of depreciation of product value over time. Since delivery of a service in the future can be less valuable than immediate delivery, potential consumers are discouraged from putting their names on a waiting list when it is too long. By this argument the authors explain the persistence of long waiting lists for non-emergency in-patient care at Britain's National Health Service and the fruitlessness of short-term measures aimed at a substantial reduction of the waiting list. (See, however, the critique by Cullis and Jones [1986].)

In our model, there is no waiting list, congestion, or waiting in line. Rather, rent dissipation comes in the form of costly schedule adjustment by the travelers. In an effort to secure a parking spot, they arrive well in advance of their preferred time and dissipate nearly all rents whenever the number of drivers in known to exceed the number of parking spaces.

3 The Parking Game

3.1 The Model

A set of players – the drivers – is indexed by $I \equiv \{1, ..., N\}$. The drivers are workers who must show up for work no later than time $t = T$. Each driver demands one space in the parking lot, and his value for it is $v > 0$; e.g., if he cannot find a spot, he must park somewhere further away and walk in at cost v.[2] Let $K > 0$ be the size of the parking lot, and let $c < v$ be the cost of increasing capacity by one space. If $N \leq K$, each driver is guaranteed a parking space, and all drivers choose to arrive at their preferred time, T. If $N > K$, however, the drivers compete for spaces, and they may wish to arrive early to secure spots. Assume that a driver who arrives earlier than time T incurs a cost of $w > 0$ per unit of time, so his cost of arrival at time t is $L(t) = (T - t)w$. We will model time as discrete, with the interesting case being what happens as the time interval $\Delta > 0$ shrinks to zero. Thus, a time grid includes times $t = k\Delta \in [0, T]$ where $k \in \{0, ..., T/\Delta\}$.

What matters to a player deciding whether to rush to the parking lot at time t is the inconvenience of arrival that early and the number of parking spaces still unoccupied. We assume that the size of the parking lot, K, is common knowledge and compare two alternative assumptions on whether a driver knows the number of spaces still open in the parking lot at time t. When a player making his arrival decision at time t observes all arrivals prior to t, this is called *full observability*; when he must decide without knowing if the parking lot is full, this is called *unobservability*. For any time t, define N_t as the number of unarrived drivers and K_t as the number of unoccupied parking spaces; $N_t = N - K + K_t$. At time t, the remaining drivers simultaneously decide whether to arrive at the parking lot at that instant. Under full observability, the decision can be conditioned on the number of parking spaces still available, K_t. Once a parking spot is taken by the player, the spot remains occupied until the end of the game. If player i arrives at t he obtains a spot with probability

[2]The players in our model are identical; we will later show what happens when players are heterogeneous. Also, we assume that the value v is independent of the number of drivers unable to secure a parking space. More realistically, it would increase, as they would have to walk from more and more distant lots. That could be modelled as a sequence of limited-capacity lots of the kind we model here.

$p_{i,t} = \min\{K_t/n_t, 1\}$, where $n_t \leq N_t$ denotes the number of players arriving at exactly time t.

If player i arrives at time t, given that $K_t \in \{0, ..., K\}$ parking spaces are unoccupied at t and $n_t \in \{0, ..., N_t\}$ players arrive at t, his payoff is

$$u_{i,t} = p_{i,t}v - L(t). \tag{1}$$

A player who arrives at time T and finds no parking space must park in a less convenient lot, obtaining payoff zero. This is the same payoff structure as in an all-pay auction with multiple prizes. Everyone bids, a bidder pays what he bids, and the prizes are distributed to the top bidders, with a coin toss breaking ties.

Let us define the *indifference arrival time*, t^*, as the time at which the arrival cost, $L(t^*)$, equals the prize value, v. It follows that

$$t^* \equiv T - v/w. \tag{2}$$

A player who arrives at t^* and finds a parking space receives a payoff of zero since the disutility of the early arrival equals the value of the parking space. When looking for equilibria, we can restrict our attention to $t \geq t^*$ since arriving before t^* yields a negative payoff and is strictly dominated by arriving at T.

Similarly, define t_p^* as the time at which a player receives a zero payoff from participating in a lottery with probability p of winning a parking space; t_p^* is determined from equation $pv - L(t_p^*) = 0$. It follows that

$$t_p^* \equiv T - pv/w. \tag{3}$$

Finally, let $\underline{t}$ and $\bar{t}$ denote the earliest and the latest time any player arrives with a positive probability along the equilibrium path.

As we are interested in characterizing equilibria that exist for any time grid, even a very fine one, we will assume that time periods are short.

Definition 1. *A time grid is fine if*

$$\Delta < \frac{v}{Nw}. \tag{4}$$

When a time grid is fine, there are more than $N \geq K+1$ periods between t^* and T since $T - t^* = v/w > N\Delta$. No more than K players can profitably enter at time $t = t^* + \Delta$ when the time grid is fine. Otherwise, even at the highest odds of obtaining a space, $K/(K+1)$, the payoff of a player is negative at $t = t^* + \Delta$ for a fine time grid. When a player is not guaranteed to obtain a parking spot, the player would choose to arrive one period earlier to secure a parking space, because for a fine time grid the cost of one-period earlier arrival, $w\Delta$, is justified by the gain associated with the higher probability of obtaining parking.

3.2 Pure-Strategy Equilibria under Full Observability

Under full observability, player i's strategy specifies the probability of his arrival, $\sigma_{i,t} \in [0,1]$ at any time $t = k\Delta \in [t^*, T]$ and history $K_t \in \{0, ..., K\}$. In the first-best outcome, all N players arrive at T, and K of them get to park in the parking lot: $\sigma_{i,t} = 0$ for $t < T$ and $\sigma_{i,T} = 1$. However, this cannot be part of an equilibrium strategy profile when the time grid is fine, since each player would prefer to deviate by arriving just before T and increase his probability of finding a spot to 100%. Although it will turn out that the details of the equilibrium strategies are more various and intricate than one might expect in such a simple game, the equilibrium payoffs converge to zero as time becomes closer to continuous. Thus, the parking lot will have no benefit for the drivers.

The mathematical structure of the parking game is similar to preemption games and games of timing such as those of Fudenberg and Tirole (1985) and Simon and Stinchcombe (1989), though those games usually have just one winner rather than the many successful drivers we have here. It is also similar to a Dutch auction, in which the price gradually falls, observed by all bidder, and the auction ends when a bidder agrees to pay the price fearing that someone else will claim the prize first. The complexity in the parking game will arise because if some driver fails to arrive at the time specified in the equilibrium, we must determine the reactions of all the other drivers to the unforeseen opportunity to claim his spot.

The Simplest Case: Two Drivers

We start by illustrating the logic of what happens in the simplest case: two drivers who compete for one parking space. The equilibrium arrival schedule of Claim 1 below, illustrated in Figure 1, specifies contingencies under which players arrive provided they have not arrived earlier.

[**Figure 1**]

Claim 1. *Consider the parking game on a fine time grid under full observability when there are two drivers and one parking spot ($N = 2$ and $K = 1$). The following arrival strategies constitute a subgame-perfect pure-strategy equilibrium: contingent on the parking lot not being full at time t, (i) at $t = t^* + k\Delta \in [t^*, t^*_{1/2})$, player 1 arrives if $k = 0, 2, 4, ...$ and player 2 arrives if $k = 1, 3, 5, ...$; (ii) at $t \in [t^*_{1/2}, T)$, both players arrive.*

The proof of Claim 1 is in the Appendix.

When Claim 1's strategies are played out, player 1 arrives at t^*, player 2 arrives at T, and both players obtain zero payoffs. Thus, the parking lot yields zero social benefit; and if it is costly to construct or the land has other uses, the lot has a negative social value.

Most of the description of the equilibrium is needed to specify what the players do off the equilibrium path, of one of them were to deviate from equilibrium. In particular, suppose player 1 fails to arrive at t^*. Player 2 then enters at $t^* + \Delta$ and obtains a positive payoff. Player 1 waits until T rather than trying to enter at $t^* + \Delta$, because he knows that Player 2 will be entering at $t^* + \Delta$, and a fifty percent probability of getting a spot is not enough to justify showing up at $t^* + \Delta$. A player's arrival at any $t \in [t^*, t^*_{1/2})$ is sustained by the threat of arrival by the competitor at $t + \Delta$. For $t \in [t^*_{1/2}, T)$, arrival at t weakly dominates any other time-t decision of a player.

The equilibrium is not unique. There is another pure-strategy equilibrium in which players alternate their out-of-equilibrium threats of arrival starting from time $t^* + \Delta$. The two equilibria yield similar outcomes in which player 1 arrives at t^* or $t^* + \Delta$ and player 2 arrives at T, and they are the only equilibria in the game, up to the identity of the players.

The General Case: N Drivers

Let us now describe the pure-strategy subgame-perfect Nash equilibria for the game with N players. Denote by $\widehat{n}_t$ the critical number of players for whom entry yields a nonnegative payoff at time t, given that K_t spaces remain unclaimed by time t. The *free-entry* number of players, $\widehat{n}_t$, for given K_t and $t \geq t^*$, is determined as the largest n_t that satisfies the positive entry payoff condition $u_{i,t} = (K_t/n_t)\, v - L(t) \geq 0$. By definition, $u_{i,t^*} = v - L(t^*) = 0$, and therefore for a given K_t there exists a unique

$$\widehat{n}_t \equiv \max \{n_t \in Z_+ : u_{i,t} \geq 0\} . \tag{5}$$

Claim 2 describes all pure-strategy subgame-perfect equilibrium for the parking game on a fine time grid.

Claim 2. *Consider the parking game on a fine time grid under full observability and with more drivers than parking spaces ($N > K$).*

A. *In any pure-strategy equilibrium to the game, the parking lot is full after $t^* + \Delta$. The equilibrium outcome is for $K_0 \in \{0, ..., K\}$ players arrive at t^*; $K - K_0$ players to arrive at $t^* + \Delta$, and $N - K$ players to arrive at T.*

B. *All the existing pure-strategy equilibria have the following arrival schedule: (i) at $t = t^*$, $K_0 \in \{0, ..., K_t\}$ players arrive; (ii) at $t = t^* + \Delta$, K_t players arrive; if player $i \in I$ deviated by not arriving at t^*, the set of arriving players excludes player i; (iii) at $t \in [t^* + 2\Delta, t^*_{1/2})$, if player $i \in I$ deviated by not arriving at $t - \Delta$, one other player arrives; (iv) at $t \in [t^*_{1/2}, T)$, if player $i \in I$ deviated by not arriving at $t - \Delta$, then $\min \{\widehat{n}_t, N_t\} \geq 2$ players arrive at t; (v) at $t = T$: all players arrive who have not yet arrived.*

Claim 2 shows that there are multiple equilibria under full observability, even if we restrict our attention to pure-strategy equilibria. Some players might arrive at the indifference arrival time t^*. This yields them zero payoffs and has the same effect on the remaining players as if the parking lot size K had shrunk and the game were started at $t^* + \Delta$. There are many sizes of this initial shrinkage that support equilibria. For a fine time grid, the equilibrium outcome in any pure-strategy equilibrium is for $K_0 \in \{0, ..., K\}$ parking spaces to be filled at t^* and the rest to be filled at $t^* + \Delta$. For example, there is an equilibrium outcome in which one group of K players arrive at time t^* and they park in the lot, while a second group of $N - K$ players arrive at T and park elsewhere. Both groups receive the same payoff of zero. Another polar equilibrium outcome is for no drivers to arrive at t^* and for K players to arrive at $t^* + \Delta$. In all pure-strategy equilibria, the parking lot fills up no later than $t^* + \Delta$ and no players arrive between $t^* + \Delta$ and T.

An important corollary of Claim 2 concerns the extent of rent dissipation in the parking game under observability.

Corollary. *As the time grid becomes infinitely fine, drivers dissipate all the rents from parking in any pure-strategy equilibrium of the parking game under full observability when there are more drivers than parking spaces.*

In the limit, as $\Delta \to 0$, players dissipate the entire value of the parking lot: each driver has an expected payoff of zero and the total sum of all the losses incurred by drivers is equal to the total value of parking, vK. The parking lot is still costly, however, so the social payoff is negative and equal to the cost of building the parking lot.

We will not attempt to fully describe the mixed-strategy equilibria due to their complexity. However, in Section 4 we discuss the extent of rent dissipation in any equilibrium to the parking game, including those in mixed strategies.

3.3 Nonexistence of Pure-Strategy Equilibria under Unobservability

Recall that "unobservability" means that a player does not know how full the parking lot is when he chooses the time at which to arrive. Claim 2 said that under full observability some players arrive at $t \leq t^* + \Delta$ and others arrive at T, both having nearly zero expected payoffs. Under unobservability, Claim 2 does not apply because one of the players who is supposed to arrive at $t \leq t^* + \Delta$ could deviate and arrive at $T - \Delta$ instead, reducing his early-arrival cost. The other players would not observe that he had failed to arrive as scheduled at t, so they would be unable to respond by taking "his" spot before $T - \Delta$.[3]

Claim 3. *There does not exist a pure-strategy Nash equilibrium under unobservability when the time grid is fine and there are more drivers than parking spaces.*

Proof of Claim 3. Denote by t' the time when the parking lot becomes full in a pure-strategy equilibrium. If at t' not all arriving drivers obtain parking, then each of them would find it profitable to deviate by arriving a period earlier. If parking is guaranteed at $t^{'}$, then there are drivers who arrive at T because arriving between t' and T yields a negative payoff. Each of these drivers benefits from arriving shortly before t' unless $t' \leq t^* + \Delta$. A fine time grid implies that only K drivers would arrive at $t' = t^*$ or $t' = t^* + \Delta$. Then, each of them can arrive later and still obtain the parking space. Hence, no pure- strategy equilibrium exists in this case either. Q.E.D.

This parking game is a particular form of a variety of rent-seeking game most intensely studied in the context of multi-prize auctions. The arrival times are like bids, and the empty parking spaces are like prizes. The reason for the nonexistence of a pure-strategy Nash equilibrium under unobservability when the time grid is fine is essentially the same as

[3]Of the equilibria in Claim 2, one survives, but only when the time grid is *not* fine. The following is an equilibrium for the parking game under unobservability for $N > K$ and the time grid that is not fine, (i) At $t = t^* + \Delta$, $\min(\widehat{n}_t, N_t) > K$ players arrive; (ii) At T, all players arrive who have not yet done so. If any player deviated to arrive at t^*, he would not improve on his equilibrium payoff. If any player deviated to arrive after $t^* + \Delta$ but before T, he would not find a parking spot and would receive a negative payoff. This establishes that the specified strategies form an equilibrium.

in a multi-prize all-pay auction with continuous strategy space (see Barut and Kovenock, 1998, Clark and Riis, 1998, or the general characterization by Baye et al. [1996]). Auction models focus on bidding rules rather than capacity choice, but as in the parking game, the question is how a limited number of goods are allocated to a large number of players.[4] In a pure-strategy equilibrium, everyone knows which bids win and which ones lose. Bidders who know they would lose would bid zero. But if they did that, winning bidders would bid no more than slightly above zero, in which case the losers would want to deviate to bids slightly higher than that and win instead.

In a pure-strategy equilibrium of this parking game, players who did not obtain parking would arrive at the preferred time, so other players would have no incentive to arrive more than slightly earlier. This, however, cannot be an equilibrium because the players arriving at the preferred time would deviate and arrive even earlier for a guaranteed empty space.

Mixed-strategy equilibria under unobservability are complicated to describe fully. In the two-driver game there is, for example, an alternating equilibrium in which for $k \in \{0, 1, 2...\}$ and $t \leq T$ player 1 arrives at $t = t_0 + 2k\Delta$ with probability $(v/(2w\Delta) - k)^{-1}$ and player 2 arrives at $t = t_0 + (2k+1)\Delta$ with the same probability. In the case of $w = 1$, $v = 10$, $T = 10$, and $\Delta = 1$, the probability of player 1 arriving at $t = 1, 3, 5, 7, 9$ and player 2 arriving at $t = 2, 4, 6, 8, 10$ is 1/5, 1/4, 1/3, 1/2, and 1. After $t = T - \Delta = 9$ the parking lot is full with probability one. What we can more easily (and usefully) characterize, however, is the nature of the payoffs that result from the equilibrium mixed strategies. In the next section we will show that almost complete rent dissipation occurs in any equilibrium.

4 Full Rent Dissipation

We first establish that in any equilibrium, players arriving at $\underline{t}$ and $\bar{t}$ (the earliest and the latest times any player arrives with a positive probability along the equilibrium path) obtain

[4] Holt and Sherman (1982) even use auction theory to analyze waiting times in a queuing model. They use auction theory's revenue equivalence theorem to show that various rules for determining the length of the wait all lead to the same amount of waiting time.

nearly zero payoffs whenever there are more drivers than parking spaces. Then, we will show that all the players in between must almost fully dissipate the value from parking. Our propositions will apply under either full observability or unobservability.

Claim 4. *Consider the parking game on a fine time grid with more drivers than parking spaces. In any subgame-perfect equilibrium, the probability that the parking lot is full at time* $\bar{t} \leq T$ *is one under full observability and approaches one as the time grid becomes infinitely fine under unobservability. The equilibrium payoffs of players who arrive at* $\bar{t}$ *with a positive probability are zero under full observability and tend to zero under unobservability as the time grid becomes infinitely fine. Any player who arrives at* $\underline{t}$ *has a payoff that tends to zero as the time grid becomes infinitely fine.*

The proof is given in the Appendix. Claim 4 says that the parking lot is almost surely full at $\bar{t}$, and the payoffs of players who arrive at $\bar{t}$ and $\underline{t}$ are close to zero. It is nearly impossible to find parking at $\bar{t}$, because otherwise a player arriving at $\bar{t}$ could deviate by arriving at $\bar{t} - \Delta$ (while following the same strategy before $\bar{t} - \Delta$) and increase his odds of obtaining parking. An incentive to deviate exists whenever there is a parking space available at $\bar{t} - \Delta$ (under full observability) or when there is a nontrivial probability that parking is available at $\bar{t} - \Delta$ (under unobservability). Since there is almost no chance that a parking spot is available at $\bar{t}$, players' payoffs at $\bar{t}$ are nearly zero. It also follows that no parking spaces are available at time $T \geq \bar{t}$ with a probability approaching one.

Corollary. *The parking lot is almost surely full by time* T*.*

Moreover, we can easily prove that the earliest and the latest time anyone arrives with a positive probability in an equilibrium approaches t^* and T, respectively, as the time grid becomes infinitely fine.[5] We next show that players tend to dissipate all the rents from parking in any equilibrium of the parking game.

[5]First, we show that $\underline{t} \to t^*$ as $\Delta \to 0$. If $\underline{t} \leq t^* + \Delta$, then $\underline{t} \to t^*$. Suppose that $\underline{t} > t^* + \Delta$. At $\underline{t}$, all parking spaces are available, and the probability of obtaining a space at $\underline{t}$ converges to one because otherwise

Proposition 1. *If there are more drivers than parking spots ($N > K$), then under either full observability or unobservability, players fully dissipate rents from the parking lot in any equilibrium as the time grid becomes infinitely fine.*

Proof of Proposition 1. The proof is by contradiction. Suppose player i is earning a positive payoff bounded away from zero. Let $t \in (\underline{t}, \bar{t})$ be the earliest moment the player arrives with a positive probability; $t > \underline{t}$. Consider a player (player j) who arrives at $\bar{t}$ (with pure strategy) and earns an almost zero payoff (by Claim 4). If player j follows his equilibrium strategy until $t - \Delta$ and arrives with probability one at $t - \Delta$, he would obtain a positive payoff bounded away from zero. The costs of arriving at t and $t - \Delta$ differ by Δw, a difference that becomes negligible as $\Delta \to 0$. It suffices to prove that player j's odds of obtaining parking at $t - \Delta$ are no worse than player i's odds of obtaining parking at t. Note that by the definition of time t, player i has not attempted to arrive before t. Player j may have been mixing before t. Therefore, player j has the expected payoff from arriving at $t - \Delta$ that is no less than that of player i. Since there is a profitable deviation, there cannot exist an equilibrium with a player obtaining a positive payoff at a time t between $\underline{t}$ and $\bar{t}$. Hence, all players earn almost zero payoffs. Q.E.D.

That the driver payoffs approach zero if the parking lot is even slightly too small is the important finding for deciding on the optimal size of the parking lot. The nature of the equilibrium strategies that lead to this unhappy outcome is not only less determinate, but less useful.

5 Welfare and Parking Lot Size

Sections 3 and 4 described and analyzed the rent-seeking competition among drivers for a fixed number of parking spaces. We next characterize welfare more fully by accounting for the cost of building the parking lot. The drivers have a value $v > 0$ for each of the parking

a player assigned to arrive at $\underline{t}$ would arrive at $\underline{t} - \Delta$. Since the payoff at $\underline{t}$ approaches zero, $\underline{t} \to t^*$ in this case as well. Second, since it is nearly impossible to obtain parking at $\bar{t}$ in an equilibrium and players receive nonnegative payoffs, the cost of arriving at $\bar{t}$ must be zero in the limit as well. Hence, $\bar{t} \to T$ as $\Delta \to 0$.

spaces in use. Let us denote the cost of providing K parking spaces by an increasing function $C(K)$. The welfare from a parking lot of size K is

$$W(K) = \begin{cases} vN - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right) - C(K) & \text{if} \quad K \geq N \\ vK - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right) - C(K) & \text{if} \quad K < N, \end{cases} \tag{6}$$

where $E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right)$ denotes the expected equilibrium cost to drivers of arriving early for given K and N. Welfare can be decomposed into three parts: the *total value* of the parking lot (which is either vN or vK, depending on whether $K \geq N$ or not), the *rent-seeking loss*, and the *construction cost* $C(K)$. The next two subsections deal with the problem of finding the optimal parking lot size in cases of a certain and an uncertain number of drivers for an infinitely fine time grid.

5.1 Optimal Capacity Under Certainty

Section 4 showed that in competition for a limited number of parking spaces, drivers tend to dissipate all the rents from parking. In any equilibrium, they are indifferent between arriving early enough to secure a spot and not parking in the parking lot at all. The rent-seeking loss from earlier arrivals equals zero when $K \geq N$ and is vK when $K < N$. Hence, welfare is

$$W(K) = \begin{cases} vN - C(K) & \text{if} \quad K \geq N \\ \\ -C(K) & \text{if} \quad K < N \end{cases} . \tag{7}$$

Full rent dissipation occurs if the number of players exceeds the size of the lot. To avoid wasteful rent-seeking activity, parking lots should be designed to accommodate 100% of the people who want parking. When the number of such people, N, is known with certainty, the parking lot should have N parking spaces; having slightly too few spaces is disastrous, a result summarized in Proposition 2 and illustrated in Figure 2.

Proposition 2. *The optimal size of the parking lot under certainty equals the number of users: $K^* = N$. All smaller sizes have negative welfare, with the minimum welfare* *being at $K = N - 1$. All greater sizes have the same total value from parking equal to vN, but increasingly high construction costs. This is true under both full observability and unobservability.*

Proof of Proposition 2. If $K < N$, then by Proposition 1, in any equilibrium almost all rents are dissipated and $W(K) = -C(K)$. Thus, the minimum welfare for $K \in \{0, 1, 2, ..., N-1\}$ is at $K = N-1$. If $K \geq N$, each player is guaranteed a parking space and arrives at the preferred time T; therefore $W(K) = vN - C(K)$ and welfare falls gradually in K. It follows that welfare is maximized at $K = N$ as long as it is at all socially beneficial to build a parking lot, i.e. when $vN \geq C(K)$. Q.E.D.

According to Proposition 2, the size of the parking lot should be set at $K = N$, as long as it is at all socially beneficial to build a parking lot. This result is clear from the shape of the welfare function in Figure 2. There is a sharp discontinuity in social welfare between "enough" and "not enough, " with the minimum social welfare being at slightly too small a parking lot.

[**Figure 2**]

Instead of finding the optimal capacity for a given number of drivers (fixing N and choosing K), we could consider the problem of choosing a number of permits for a parking lot of a given size (fixing K and choosing N). This interpretation is appropriate when capacity is a chosen in the long run, while access to the parking lot can be regulated in the short run. Proposition 2 implies that the optimal number of permits is equal to the lot's capacity. The next subsection deals with the case of uncertain number of drivers. We will see that uncertainty over the number of drivers actually increases social welfare if the parking lot size is too small.

5.2 Optimal Capacity under Uncertainty

Assume that planners must decide on the size of the parking lot before they know the number of drivers, N, whereas drivers know both N and the parking lot capacity, K. Thus, drivers know N but planners only know that it is drawn from a known probability distribution, $f(N)$, with support $\{0, 1, ..., \overline{N}\}$.[6] What is the optimal parking lot size? We will next show that maximization of the expected welfare would require a much bigger parking lot than the expected number of drivers because the loss function in Figure 2 is asymmetric.

When there is competition for parking spots, $K < N$, the welfare from the parking lot is negative, $W = -C(K)$. When the size of the lot is large enough, $K \geq N$, the welfare is $W = vN - C(K)$. Therefore, for $K \leq \overline{N}$, the expected welfare is

$$EW(K) = \sum_{N=0}^{K} (vN)f(N) - C(K) = vE(N \mid N \leq K) - C(K). \tag{8}$$

The size of the parking lot should be increased from $K-1$ to K as long as the change in the expected welfare is nonnegative: $EW(K) - EW(K-1) \geq 0$. Thus, the optimal K is the biggest K such that $K \leq \bar{N}$ and

$$vKf(K) - (C(K) - C(K-1)) \geq 0. \tag{9}$$

The expression on the left-hand side of inequality (9) has intuitive meaning. Drivers benefit from the Kth parking space only if $N = K$ exactly. If $N < K$ the marginal space is unused, and if $N > K$ the benefit is dissipated by rent-seeking so it does not matter whether there are K or $K-1$ spaces. Since the Kth space matters only if $N = K$, the change in the expected welfare is the probability that there are exactly K drivers multiplied by the benefit from eliminating rent-seeking behavior, vK, net of the change in construction costs. Whether the marginal benefit of a parking space is decreasing or increasing depends on the relative strength of two effects. On the one hand, at larger parking lot sizes, it is more important to have a sufficiently big parking lot because there are more people who could get benefit from it. On the other hand, it could be less likely that larger parking lots are filled up. When the first effect dominates and $Kf(K)$ increases in K, the marginal benefit increases with K too. For a constant-marginal-cost technology, $C(K) = cK$, this implies a corner solution to the problem: the parking lot should accommodate all potential drivers, even if that is much greater than the expectation of the number of drivers, or not be built at all.

[6] We do not consider here the case where drivers are equally uncertain about the number of people seeking parking. We conjecture that this case would result in a lower optimal capacity.

To illustrate, consider a discrete uniform distribution for the number of drivers on the support $\{0, ..., \overline{N}\}$. For the uniform distribution, $Kf(K)$ does increase in K, so we have a corner solution. Under the uniform distribution, it is equally likely at any capacity that the parking demand will be barely met. At a larger capacity benefits accrue to more people, so at larger capacity levels, the benefit from building an additional space is higher, and planners should design the parking lot for the "peak demand," accommodating all potential drivers.[7] Figure 3 shows the expected welfare for the uncertainty case and welfare for the certainty case at different capacity levels when $c = 1$, $w = 1$, $v = 5$, and $\overline{N} = 100$.

[**Figure 3**]

At the optimal capacity level, only 50% of parking spaces are occupied on average. Compare the consequences of a limited capacity for certain and uncertain N. Uncertainty over the number of drivers actually increases social welfare if the parking lot size is too small. While the optimal capacity under uncertainty is $K^* = 100$, welfare is positive even if $K = 40$. This is a big difference from the case without uncertainty, where welfare would be negative and large in magnitude at $K = 40$. The reason is that under uncertainty, even with very few parking spaces, it may happen that very few people need to park, and so there is no wasteful rent-seeking and the parking spaces are valuable.

Thus, uncertainty over the number of drivers actually increases social welfare if the parking lot size is too small. Although under uncertainty the welfare is positive when the parking lot is slightly too small, under certainty it would be near its minimum and negative. Uncertainty creates the possibility that $N < K$, so costly rent-seeking is less necessary. This advantage of uncertainty is somewhat paradoxical. In the first-best, when people are allocated to parking spots, welfare would be highest under certainty since no parking spot would ever go unused and no driver would ever fail to find a parking spot. The same is true when drivers are not strategically changing their arrival schedules in an attempt to secure a spot. Uncertainty would then make some excess capacity optimal, but would reduce welfare.

[7] Designing for peak demand is not always the optimal choice. For example, in a case of the binomial distribution of N, if each of 100 drivers is in need of parking with probability 0.5 and $c/v = 1/5$, the optimal parking size is $K^* = 58$. Only about 8 out of 58 spaces are empty, on average. This corresponds to about 86% utilization level.

If people are strategic, however, the consequences of mistaken policy lead to quite different outcomes. Having a slightly too small parking lot would be disastrous under certainty, with zero total value from parking, but under uncertainty the total value would still be positive. The problem of rent-seeking certainly does not disappear under uncertainty, however, and optimal capacity is much larger under uncertainty than under certainty. Importantly, empty parking spaces are not an indication that the parking lot is too big. In a stochastic model, there will usually be many empty spaces in a lot of optimal size. Planners should not be tempted to reduce capacity (or increase the number of parking permits), although some observers may decry the wastefulness of building parking lots that are too large (or of cruelly limiting the number of permits despite the presence of unused parking spaces).

Finally, we present the analysis of the parking problem when the number of drivers is large and calculus methods can be employed. Consider the production technology with a constant marginal cost, $C(K) = cK$ where $c < v$. The expected welfare from a parking lot of size K is then

$$EW(K) = v \int_0^K Nf(N)dN - cK. \tag{10}$$

The optimal size of the parking lot is the solution to the first-order condition

$$\frac{\partial EW(K)}{\partial K} = vKf(K) - c = 0, \tag{11}$$

which can be written as

$$Kf(K) = \frac{c}{v}. \tag{11'}$$

Expansion of the parking lot improves welfare if the probability that exactly K drivers compete for K parking spaces times the size of the parking lot exceeds the relative cost of building a parking space. We can use (**??**) to find the optimal level of K as an internal solution if the maximand in (10) is concave. Unfortunately, that seems unlikely. The second-order condition, $\partial^2 EW(K)/\partial K^2 < 0$, can be written as $Kf'(K)+f(K) < 0$, or $Kf'/f < -1$. To guarantee the existence of the interior solution, the density must be declining and elastic in the relevant range of K. Otherwise, a parking lot should be built for the peak demand (the highest N possible).[8]

6 Heterogeneous Drivers

Above, we analyzed the extreme case in which all players have the same parking value v, early arrival cost w, and preferred arrival time T. What happens if players differ in these parameters? It turns out that the extreme result of full rent dissipation disappears but rents are still partially dissipated, and parking lots should be built large because the welfare loss from having capacity of a wrong size is still asymmetric: the welfare loss from a slightly too small capacity is much larger than from a slightly too large capacity. We will establish these results in the framework of the parking game under full observability, for which the pure-strategy subgame-perfect equilibrium continues to exist under driver heterogeneity.

The effects of heterogeneity in parking value, v, or early arrival cost, w, are the same. Suppose drivers have heterogenous values for parking. Let N be the number of drivers with positive values and let v_i denote the value of i'th-highest-valuing driver, so $v_1 \geq v_2 \geq ... \geq v_N > 0$. For example, in the case of linear demand, $v_i = a - bi$ for some constants a and b. Assume that there are also drivers who have zero value for parking and hence arrive at their preferred time, T. As before, the cost of constructing a parking space is c, and the cost per unit time of early arrival is w.

In the first-best case, both the parking lot capacity and access to parking can be regulated. The first-best policy is to choose capacity to accommodate all players with values higher (or weakly higher) than $v_i = c$ and to give only them access to the parking lot.

[8] For example, consider a family of continuous density functions $f(N) = \alpha N^{-\beta}$ defined on the support $[\underline{N}, \overline{N}]$, where $\alpha, \beta > 0$, and $[\underline{N}, \overline{N}] \subset [0, \infty)$ are such that $\int_{\underline{N}}^{\overline{N}} \alpha N^{-\beta} dN = 1$. The first-order condition implies the optimal capacity $K^* = \left(\frac{1}{\alpha}\frac{c}{v}\right)^{-1/(\beta-1)}$, and the second-order condition is satisfied if and only if $\beta > 1$. For $\beta \leq 1$, the parking lot should have $\overline{N}$ spaces, if it is built. The mean utilization of the parking lot of optimal size can be measured as a ratio of the mean number of parking spots taken, $E(X)$, to the optimal capacity size, K^*. If $N < K$, then all N drivers find parking; if $N \geq K$, then K out of N drivers find parking. Hence, $E(X) = \int_{\underline{N}}^{K^*} N f(N) dN + \int_{K^*}^{\overline{N}} K^* f(N) dN$. For example, 50% of the parking spots will remain unoccupied on average when $c/v = 0.5$, $\beta = 1.5$, $K^* = 4$, and $E(X) = 2$, and the 50%-utilized parking lot is socially optimal.

When access to parking cannot be restricted, however, planners should design a larger parking lot. Intuitively, the second-best policy is to set a higher than the first-best capacity in order to reduce rent-seeking losses that occur when access to parking cannot be restricted. The simple policy prescription to equate the marginal benefit to the marginal construction cost is not valid when drivers can engage in costly schedule adjustments. In the second- best world, the optimally designed parking lot provides parking even to drivers who value it less than the construction cost. In many cases, including that of linear demand for parking, the second-best policy is to choose $K = N$ and guarantee parking to all players with positive values.

To establish these surprising results, we need to derive the welfare for a parking lot of size K when drivers are heterogeneous. When a parking lot has N or more spaces, all players with $v > 0$ arrive at T and find a parking space. There is no rent-seeking loss since nobody arrives before T and there is no "unserved demand loss" because all players with $v > 0$ find spaces. For parking lots of size $K > N$, welfare falls due to "excess construction cost " at a rate equal to the marginal cost of construction. For example, if $K = N+1$ instead of $K = N$ spaces are built, construction cost is higher by c because one extra space is constructed.

When a parking lot has fewer than N spots, in any subgame-perfect pure-strategy equilibrium to the parking game under observability, rent-seeking will occur in the equilibrium. If $K < N$, the equilibrium outcome to the parking game on an infinitely fine time grid dictates that the K players with the highest values arrive at $t^*_{K+1} \equiv T - v_{K+1}/w$ and find a space, and players with values v_{K+1} or less arrive at T. The player with value v_{K+1} will not deviate because his benefit from arriving at t^*_{K+1} would be equal to his cost. The strategies that sustain this equilibrium outcome are similar to those given in Claim 2, and arguments similar to those given in Proposition 1 imply that the rent-seeking losses are equal to $K \cdot v_{K+1}$, because the K players with the highest values must incur disutility v_{K+1} of arriving at t^*_{K+1} to deter players with value v_{K+1} or less from arriving early. There is always partial rent dissipation when the parking capacity cannot accommodate all drivers with positive value for parking, that is, when $v_{K+1} > 0$.

The welfare from a parking lot of size $K \geq N$ is equal to the total value minus construction costs. For any $K < N$, the welfare also includes rent-seeking losses. Thus, welfare is

$$W(K) = \begin{cases} \sum_{i=1}^{N} v_i - cK & \text{if} \quad K \geq N \\ \\ \sum_{i=1}^{K} v_i - K \cdot v_{K+1} - cK & \text{if} \quad K < N, \end{cases} \tag{12}$$

and the change in welfare from adding an extra parking spot for $K \leq N$ is

$$W(K) - W(K-1) = K\left(v_K - v_{K+1}\right) - c. \tag{13}$$

Expansion of capacity is warranted whenever $v_K - v_{K+1} \geq c/K$, or the value from parking does not decline at K faster than c/K. For linear demand, there is a corner solution: a planner designs the parking lot to accommodate all N players with positive values, as long as it is worthwhile to build the parking lot at all, that is, as long as $W(N) = \sum_{i=1}^{N} v_i - cN \geq 0$.

When K and N are large we could treat them as continuous variables and use calculus. Using $v = v(x)$ to denote the demand for parking, welfare can be written as

$$W(K) = \begin{cases} \int_0^N v(x)dx - cK & \text{if} \quad K \geq N \\ \\ \int_0^K v(x)dx - K \cdot v(K) - cK & \text{if} \quad K < N. \end{cases} \tag{14}$$

Taking the derivative, we obtain $\partial W(K)/\partial K = -K \cdot v'(K) - c$ for $K < N$. An interior solution exists when v is sufficiently convex to ensure that $v'(K) + K \cdot v''(K) > 0$. In other words, allowing all drivers with positive values to park is not optimal when $K \cdot v''(K)/\left(-v'(K)\right) > 1$, that is, when the slope of the inverse demand is elastic.

[**Figure 4**]

Figure 4 depicts the welfare from parking lots of various sizes for a linear demand. It illustrates how welfare can be decomposed into total value, rent-seeking loss, and construction cost. In this example, we assume that $N = 50$ drivers have positive values from 10 to 0.2 (player 1 has the highest value, $v_1 = 10$, player 2 has $v_2 = 9.8$, and so forth till $v_{50} = 0.2$), and some players have value $v = 0$. The discrete demand for parking can be written as $v_i = (51 - i)/5$; $i = 1, ..., 50$. As before, we assume that $c = 1$ and $w = 1$. As is apparent in

Figure 4, the linear demand model has partial but not complete rent dissipation (rent-seeking losses are lower than the total value). The biggest difference is that minimum welfare is at $K = 4$ or $K = 5$ rather than $K = N - 1$. It remains true, however, that the biggest drop in welfare results from going from the optimum, $K = N$, to a slightly smaller lot, $K = N - 1$. What is most important for planning purposes is that (i) as in the homogeneous-player model, the loss function from choosing the wrong capacity is asymmetric near the optimum, and choosing too small capacity is worse than choosing too large;[9] and (ii) as in the case of uncertain N, the conclusion that parking lots should be overbuilt holds.

To see this, think about the first-best capacity when the planner can restrict access to parking. In the first-best, the planner does not need to worry about rent-seeking losses; they will equal zero. All he needs to look at are the marginal construction cost of $c = 1$ and the marginal benefit of v. Thus, he will choose either $K = 45$ or $K = 46$, which both yield the same welfare since $v_{46} = c = 1$. If the social planner cannot regulate access to parking, however, then $K = 45$ is not optimal. All $N = 50$ players with positive values will attempt to park, even though it is inefficient for those with $v < 1$ to do so, and it will be necessary for 45 players to arrive early to avoid losing spaces to the 5 low-valuing players. The second-best capacity is $K = N = 50$.

Another way to look at the problem is that if capacity is chosen at the first-best optimum of $K = 45$ there will be a discontinuous loss from allowing more than $N = 45$ players access to the parking lot. Consider the problem of choosing the number of parking permits, n, for a lot of a given size, K, which arises when capacity is chosen in the long run while permits are allocated in the short run. Assume that permits are rationed efficiently to the highest-value players. Clearly, the optimal number of permits cannot be less than capacity. For any $n > K$, the welfare is $K \cdot v_{K+1}$ lower than at $n = K$. Hence, the welfare loss from having too few or too many parking permits is not symmetric. Welfare jumps down by $K \cdot v_{K+1}$

[9] If $K = N = 50$, the welfare is the sum of drivers' values for parking net of construction cost, $W(N) = 205$. If $K > N$, the welfare is reduced linearly by $c = 1$ for each extra space constructed. On the other hand, if $K < N$, construction costs are saved, but there is a loss from unserved demand and rent-seeking. For instance, if $K = N - 1$, the welfare loss relative to $K = N$ is 9 since 1 is saved in construction, 0.2 is lost due to unserved demand, and 9.8 is lost in rent-seeking. Similarly, if $K = N - 2$, the welfare loss is 17.8, and for $K = N - 3$ it is 26.4. This is in contrast with the welfare loss from overprovision, which is 1 for each extra unit constructed.

when one too many parking permit is issued, but welfare falls gradually when too few drivers are served. At the first-best capacity level $K = 45$, welfare drops due to rent-seeking loss by $45 \cdot 5 = 380$ when 46 permits (or more) are allocated. In contrast, welfare drops by 1.2 when $n = 44$ and one driver with value 1.2 is unserved. There is a discontinuity in welfare at the optimal number of permits, $n = K$; oversupply of permits is more dangerous than undersupply.

7 Concluding Remarks

We have constructed various versions of model of strategic parking. Suppose 1,001 drivers want to arrive at the same time and have the same costs of arriving early, and each derives a benefit of $250 from parking in a particular lot during the year. If the cost of a parking space is $200 per year, it is obvious that 1,001 spaces should be built, for a net payoff of $50,050 per year. What is not so obvious, and what has been the theme of this paper, is that if 1,000 spaces are built instead, the net payoff is not $50,000, but -$200,000. Competition in the form of early arrival for the scarce spots eats up the entire benefit of the parking lot. Of course, the implications of a shortage are not as dramatic when, for example, the number of drivers is uncertain, but the extreme case shows the nature of the problem.

Strategic incentives are an essential element in planning capacity for an underpriced good – as important, or perhaps more important, than the obvious decision-theory problem of predicting uncertain demand and engineering problem of predicting capacity cost. If for some reason direct pricing is impractical, and the planner is aware that, with some probability, demand for the good will exceed the supply, he should realize that the damage from such situations in not limited to a few people being left unable to find a parking space. People's actions to forestall being shut out vastly increase the social loss. This idea may extend to other settings as well.[10] What is special about this model compared to congestion

[10] The parking problem calls to mind matching games such as those studied by Roth and Xing (1994) and Avery et al. (2001). In Avery et al., for example, federal judges must each select one clerk from graduating law students, and students can work for no more than one judge. What has happened in recent years is that clerks and judges pair up earlier and earlier, rather towards the end of the student's last year of law school. A judge who waits too long to hire would not be able to find any good clerks available, so judges hire clerks

models is that there is a sharp break in players' equilibrium payoffs when the number of people seeking to use the facility equals its capacity.

early even though a lot more is known about an older student's quality. The top students are analogous to the parking spaces in our model, and the possibility of mistakenly hiring an incompetent student is the cost of arriving early.

Appendix: Proofs

Proof of Claim 1. To prove that the listed strategies are part of a subgame-perfect Nash equilibrium we must show that there are no profitable deviations for any player at any point in time, given the strategy of his rival.

Both players have equilibrium payoffs of zero. Time t^* is such that player 1 is indifferent between arriving at t^* and not parking in the lot at all. Arriving earlier than t^* (at $t < t^*$) yields a negative payoff, $v - L(t) < 0$. If player 1 deviates at t^* by delaying his arrival, player 2 arrives at $t^* + \Delta$. Arriving at $t^* + \Delta$, player 1 obtains $v/2 - L(t^* + \Delta) < 0$. Hence, player 1 cannot do any better than to arrive at T and receive a zero payoff.

Similarly, a player assigned to arrive at $t \in (t^*, t^*_{1/2})$ will do so, because otherwise the rival arrives next period. A one-period delay in player i's arrival would save the player the cost of arriving one period earlier, $w\Delta$, but would reduce his odds of obtaining parking from 1 to 1/2. For a fine grid, player i would prefer a secure space at t to a competition for the parking space at $t + \Delta$ with fifty-fifty odds. The parking lot is full in later periods. A player assigned to arrive at $t \in [t^*_{1/2}, T)$ will do so when the parking lot is not full at t, because arrival at these times results in a nonnegative payoff even when the player's rival is certain to arrive at t.

No player who is not assigned to arrive at $t \in [t^*, t^*_{1/2})$ would arrive at that time because this would yield a negative payoff to the player. Q.E.D.

Proof of Claim 2. Consider a pure-strategy subgame- perfect Nash equilibrium. No player can arrive earlier than t^*. In part A, we need to show that the parking lot is full after $t^* + \Delta$ at the latest. Let t' denote the time when the parking lot becomes full and suppose that $t' \in [t^* + 2\Delta, T]$. If the odds of obtaining a parking space at t' are less than one for player i who is supposed to arrive at t', the player would instead arrive one period earlier. If a parking space is guaranteed at t', then a player who is supposed to arrive at T would instead arrive at $t' - \Delta$. Such players exist, since $N > K$ and the number of arrivals prior to T is equal to the number of parking spaces. Hence, $t' \leq t^* + \Delta$.

The proof of part B is similar to that of Claim 1. Arriving earlier than t^* never benefits a player. If player i deviates from his equilibrium strategy by not arriving at t^*, player $j \neq i$ arrives at $t^* + \Delta$, and player i receives a payoff of zero at best. For $t \in (t^*, t^*_{1/2})$, it is always better for a player to obtain a parking spot at t then to contest one remaining parking space next period or wait till later periods. At $t \in [t^*_{1/2}, T)$, a delay by the player assigned to arrive at t implies that the player does not obtain parking, as the parking lot is filled at time t. The condition on the fineness of the time grid implies $\Delta < \frac{v}{(K+1)w}$; this ensures that if $K_0 = 1$ and the single player assigned to arrive at t^* deviates by arriving at $t^* + \Delta$, there are enough other players to compete for the parking spaces at $t^* + \Delta$ and make his deviation unattractive. Q.E.D.

Claim 4. *Consider the parking game on a fine time grid with more drivers than parking spaces. In any subgame-perfect equilibrium, the probability that the parking lot is full at time $\bar{t} \leq T$ (a) is one under full observability and (b) approaches one as the time grid becomes infinitely fine under unobservability. The equilibrium payoffs of players who arrive at $\bar{t}$ with a positive probability (c) are zero under full observability and (d) tend to zero under unobservability as the time grid becomes infinitely fine. Any player who arrives at $\underline{t}$ has a payoff that (e) tends to zero as the time grid becomes infinitely fine.*

Proof of Claim 4. Recall that $\underline{t}$ and $\bar{t}$ are defined as the earliest and the latest time any player arrives with a positive probability along an equilibrium path. Consider a subgame-perfect equilibrium of the parking game, under either full observability or unobservability. By definition of $\bar{t}$, some player arrives at $\bar{t}$ with a positive probability along an equilibrium path and thus does not have a pure strategy of arriving before $\bar{t}$. Denote the player as player i.

First, we prove statements a) and c) of Claim 4 under full observability. Any unarrived player i who at $\bar{t} - \Delta$ observes $K_{\bar{t}-\Delta} > 0$ parking spaces available, would choose to arrive immediately at $\bar{t} - \Delta$ rather than wait to arrive at $\bar{t}$. This is true regardless of the number of other players, n, who happen to arrive at $\bar{t} - \Delta$. Intuitively, the earlier arrival increases player i's odds of obtaining parking by a positive amount bounded away from zero and any such increase in odds justifies the additional cost of the earlier arrival, $w\Delta$, when the grid is fine.

More formally, denote by $p_{i,t}$ player i's odds of obtaining parking at t conditional on player i's arrival at t. Consider two cases which could occur along the equilibrium path depending on realizations of $K_{\bar{t}-\Delta} > 0$ and $n \geq 0$. If $0 \leq n < K_{\bar{t}-\Delta}$, parking lot becomes full only at $\bar{t}$; $p_{i,\bar{t}} = K_{\bar{t}}/(N - (K - K_{\bar{t}})) < K/N < 1$ and $p_{i,\bar{t}-\Delta} = 1$, and therefore $p_{i,\bar{t}-\Delta} - p_{i,\bar{t}} > (1 - K/N) \geq 1/N$. If $n \geq K_{\bar{t}-\Delta} > 0$, parking lot becomes full at $\bar{t} - \Delta$; $p_{i,\bar{t}} = 0$ and $p_{i,\bar{t}-\Delta} = K_{\bar{t}-\Delta}/(n+1) > 0$, and therefore $p_{i,\bar{t}-\Delta} - p_{i,\bar{t}} = K_{\bar{t}-\Delta}/(n+1) > 1/N > 0$. By definition of a fine grid, $w\Delta < \frac{v}{N}$. This implies that the benefit of earlier arrival $\left(p_{i,\bar{t}-\Delta} - p_{i,\bar{t}}\right) v > \frac{v}{N}$ is higher than the cost, $w\Delta$, regardless of $K_{\bar{t}-\Delta} > 0$ and $n \geq 0$.

We proved that any unarrived player i who observes $K_{\bar{t}-\Delta} > 0$ arrives immediately at $\bar{t} - \Delta$. But then no parking is available at $\bar{t}$ since all N players arrive before $\bar{t}$. By definition of $\bar{t}$, some player must arrive at $\bar{t}$ with a positive probability along an equilibrium path. Hence, with the positive probability $K_{\bar{t}-\Delta} = 0$ is realized in the equilibrium. Any unarrived player who observes $K_{\bar{t}-\Delta} = 0$ parking spaces available at $\bar{t} - \Delta$, chooses to arrive at T. Hence, with a positive probability some players arrive at T, and by definition of $\bar{t}$, $\bar{t} = T$. Moreover, no parking is available at $\bar{t} = T$ in the case of $K_{\bar{t}-\Delta} > 0$ as well. To summarize, we proved that under full observability $\bar{t} = T$ and no parking is available at $\bar{t}$. Any player arriving at $\bar{t} = T$ along an equilibrium path has a zero probability of obtaining a parking space and, therefore, a zero equilibrium payoff.

Second, we prove statements b) and d) of Claim 4 under unobservability. Player i who arrives at $\bar{t}$ with a positive probability along an equilibrium path must be unwilling to deviate to a pure strategy of arriving at $\bar{t} - \Delta$ if his strategy (which may be mixed) has not led him to arrive before that. Player i's deviation is not profitable if $u_{i,\bar{t}-\Delta} - u_{i,t} = E_{\sigma_{-i}}\left[p_{i,\bar{t}-\Delta} - p_{i,\bar{t}}\right] \cdot v - w\Delta \leq 0$, where the expected value is based on the equilibrium mixed strategies of other players (which determine $K_{\bar{t}-\Delta}$ and n). The analysis under full observability reveals that if $K_{\bar{t}-\Delta} = 0$, $p_{i,\bar{t}-\Delta} - p_{i,\bar{t}} = 0$, whereas if $K_{\bar{t}-\Delta} > 0$, $\left(p_{i,\bar{t}-\Delta} - p_{i,\bar{t}}\right) v > \frac{v}{N}$ for any $n \geq 0$. Under unobservability, the expected gain to player i from the earlier arrival is therefore larger than $\left[\Pr\left(K_{\bar{t}-\Delta} = 0\right) \cdot 0 + \Pr\left(K_{\bar{t}-\Delta} > 0\right) \cdot \frac{v}{N}\right]$. The expression in the brackets has to be lower than the expected loss of $w\Delta$ for player i not to deviate from the equilibrium strategy. Since the expected loss of $w\Delta$ approaches zero as the time grid becomes infinitely fine, the expected gain has to be converging to zero as well. This is only possible if $\Pr(K_{\bar{t}-\Delta} > 0)$ approaches zero as Δ shrinks to zero. Under unobservability we find that almost certainly no parking is available to player i arriving at $\bar{t}$, which is part (b) of Claim 4. Player i's payoff must tend to zero too, which is part (d) of Claim 4, because his probability of finding a parking space tends to zero yet his payoff cannot be negative (since he could always just arrive at T).

Third, we prove statement e) under both full observability and unobservability. Suppose a player arriving at $\underline{t}$ with a positive probability were to obtain a positive payoff bounded away from zero. This would imply that the player with a nearly zero payoff at $\bar{t}$ could increase his payoff by arriving at $\underline{t} - \Delta$. Hence, the payoff of the player arriving at $\underline{t}$ must approach zero as Δ shrinks to zero, which is part (e) of Claim 4. Thus, we have proved all five statements of Claim 4. Q.E.D.

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Figure 1. Equilibrium with Two Drivers and One Space (N=2, K=1) under Full Observability

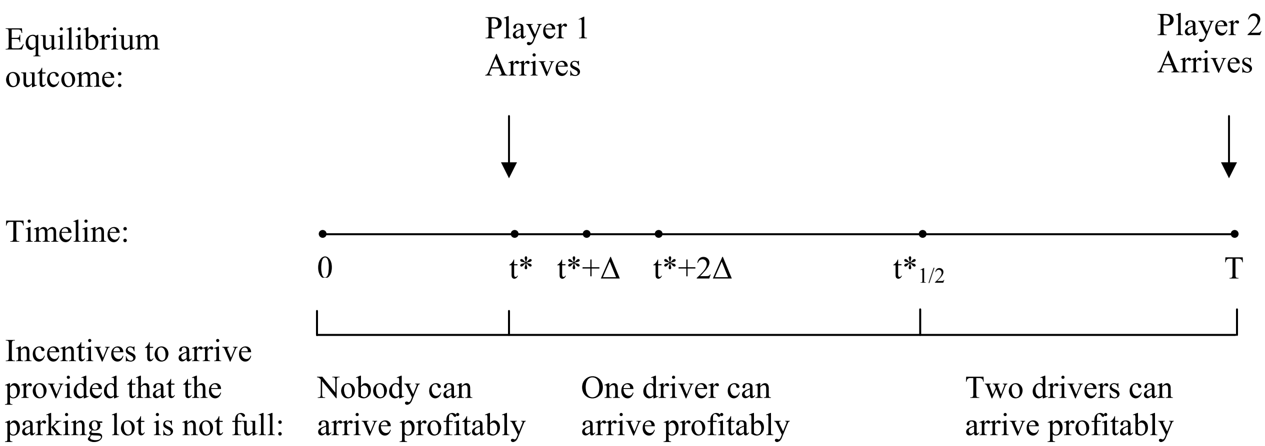

Figure 2. Welfare from a Parking Lot of Size K when the Number of Drivers Is N=50 with No Uncertainty; c=1, w=1, and v=5

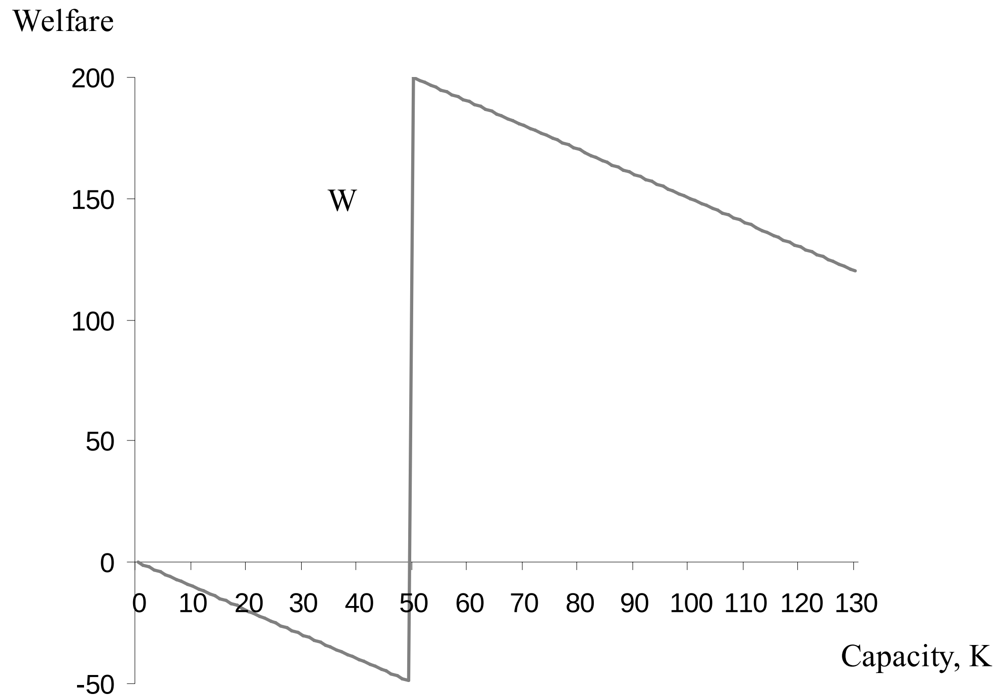

Figure 3. Welfare from a Parking Lot of Size K when the Number of Drivers Is N=50 and N Is Uniformly Distributed on {0,1,…,100}; c=1, w=1, and v=5

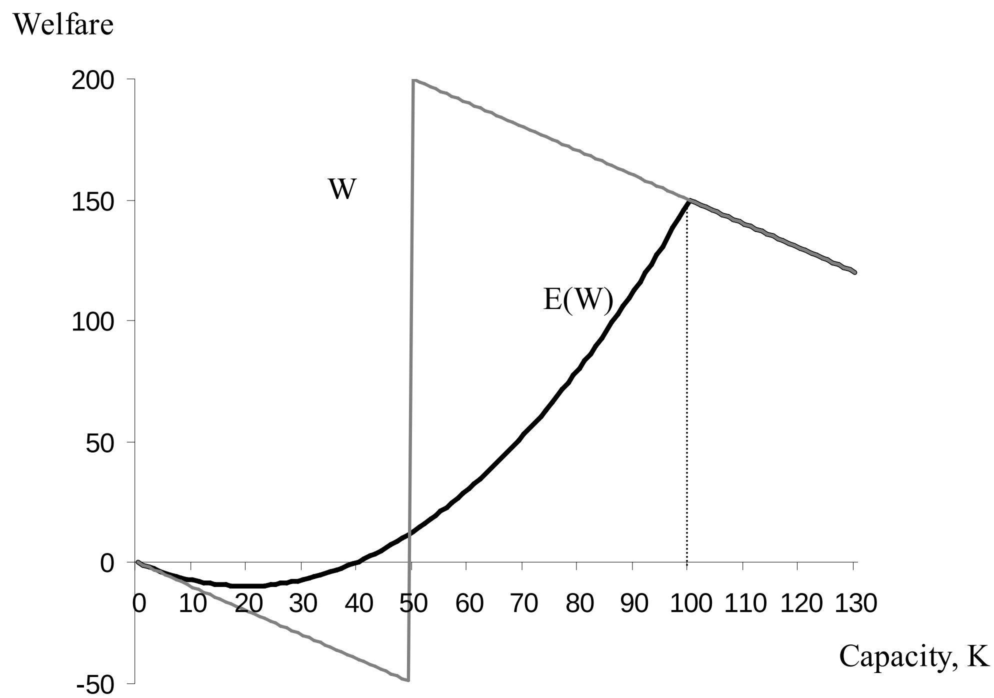

Figure 4: Decomposition of Welfare from a Parking Lot of Size K when Demand for Parking Is Linear $v_i=(51 - i)/5$, $i=1,\ldots,50$; c=1 and w=1