Deciding Whether to Buy a Public Good for Two Players Who Know Each Others' Valuations: A Simple, Almost Efficient, Budget-Balancing, Breakdown Mechanism

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July 20, 2019

Abstract

Consider two players with values for a public good v_1 and v_2 known to both but not to the decisionmaker, who knows only the cost of 100. It is natural that both would lobby the decisionmaker, and that he would decide whether to build the project and how much each should pay as a continuous function of the intensity of their stated desires. This will not attain the efficient result (build if and only if $v_1 + v_2 \geq 100$) but it has a unique Nash equilibrium, with a fair cost allocation. The decisionmaker can choose a "building probability function" to come arbitrarily close to the efficient result. We compare this with a variety of well-known mechanisms that do attain efficiency but have other drawbacks.

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This paper: http://www.rasmusen.org/papers/publicgoods.pdf.

Keywords: mechanism, dominant-strategy mechanism, Nash-implementable, lobbying, public goods, Groves mechanism, Clarke tax.
JEL Code: xxx

We thank Valerio Dotti for inspiring this note, and John Maxwell and the Indiana Law and Economics Lunch for helpful comments.

1. Introduction

This paper takes a new approach to the classic mechanism-design problem of deciding whether to invest in an undivisible public good when the decisionmaker does not know the values put on the project by two players with quasilinear utility but they do know each others' values ("complete information" in the terminology of Palfrey [199x]). The problem is usually illustrated with an example such as a public official deciding whether to build a bridge, but since the model makes most sense for a small group of people who know each other well, I invite the reader to think of a father deciding whether to install a swimming pool in the back yard, and of the players as his two sons, whose "taxes" will be the share each takes in cleaning and maintaining the pool if it is installed.

Moulin (1981) and Jackson & Moulin (1992) have described two mechanisms for this problem that are efficient, fair, and budget-balanced, so in one sense this is a solved problem. The mechanism we will describe is simpler and more natural, even though it will be only epsilon-efficient.

1. The Model

A project costs 100. It benefits players 1 and 2 by v_1 and v_2. The players observe each others' values. They send messages m_1 and m_2 to the "decider", with $m \in (0, \overline{m}]$ for some sufficiently large $\overline{m}$. The decider decides whether to build the project or not, and how much to tax each player, t_1 and $t_2 = 100 - t_1$. He does not observe v_1 and v_2. The players' utility functions are quasilinear, so if the project is built, $\pi_1 = v_1 - t_2$ and $\pi_2 = v_2 - t_2$ and the payoffs are zero otherwise.

This paper's analysis of the problem will be to look at what happens if the decider builds the project with probability $p(m_1, m_2)$ and taxes the two players amounts $t_1(m_1, m_2)$ and $t_2 = 100 - t_1$, so the budget is balanced. After laying out what would happen, we will look at other mechanisms, including the leadership auction mechanism of

Moulin (1981) and the three-stage mechanism of Jackson & Moulin (1992), which are both efficient and "fair".

Let's be a little more precise. The decider asks the players to simultaneously submit intensities m_1 and m_2 that are greater than 0 and cannot exceeds some large bound $\overline{m}$ (so that we avoid situations in which players wish to send infinite messages). If either submits the intensity $VETO$, he cancels the project. Otherwise, he is committed to build the project with probability $p(m_1, m_2)$ with taxes $t_1(m_1, m_2)$ and $t_2 = 100 - t_1$, which means taxes will be budget-balancing, exactly covering the cost of 100.

The problem is to find p and t_1 functions with the right properties. The functions cannot depend on v_1 and v_2, except indirectly through the players' choices of m_1 and m_2. The probability p should be bounded between 0 and 1, and the tax t_1 between 0 and 100. We want efficiency: that the equilibrium probability will be close to 1 when $v_1 + v_2 \geq 100$ and close to 0 when $v_1 + v_2 < 100$. We want a unique solution to exist when the players maximize their payoffs. We want a strong Nash equilibrium— that is we do not want any player to be indifferent about playing his equilibrium strategy, at least on the equilbirium path. We want the taxes to be fair, meaning that t_1 is increasing in v_1 for a given value of v_2, and similarly for t_2— for example, $t_1 = \frac{v_1}{v_1+v_2}100$, which is proportional taxation.[1] We do not require that $m_1 = v_1$ and $m_2 = v_2$, a "direct mechanism", though very likely functions that meet our requirements will result in v_1 and v_w being deducible from m_1^* and m_2^*.

Player 1's maximizes his payoff by choice of m_1

$$\pi_1 = p(m_1, m_2)\left(v_1 - t_1(m_1, m_2)\right),$$

[1]Note that this is not the meaning of "fair" in Varian (1974), Pazner & Schmeidler (1978), and Crawford (1979), which try to avoid ordinal utility by saying that a fair allocation is such that no player would want to trade his observable outcome, a goods and cash allocation, with another player. In the current context it would mean trading tax levels, and it would be satisfied if and only if $t_1 = t_2 = 50$. We would define that as an *unfair* outcome if, for example, $v_1 = 200$ and $v_2 = 100$.

which has the first order condition

$$\frac{d\pi_1}{dm_1} = \frac{dp}{dm_1}(v_1 - t_1) - p\frac{dt_1}{dm_1} = 0$$

and the second order condition

$$\frac{d^2\pi_1}{dm_1^2} = \frac{d^2p}{dm_1^2}(v_1 - t_1) - \frac{dp}{dm_1}\frac{dt_1}{dm_1} - \frac{dp}{dm_1}\frac{dt_1}{dm_1} - p\frac{d^2t_1}{dm_1^2} < 0,$$

We will see that if β is small, p^* goes to 1– complete dissipation. Cite Baye and Kovenock. We can probably get existence and uniqueness very generally.

The problem in coming up with functions is that we want m_1 and m_2 to be strategic substitutes for enough of the domain of the function so the reactino curves cross and we get an equilibrium. That means we want m_1 to decrease in m_2, even though p increases in both, so we want the cross-partial to be negative, $\frac{d^2p}{dm_1dm_2} < 0$. which makes the marginal benefit of bigger m_1 fall in m_2.

Let's think about the inefficiency of this mechanism. Most obvious is that with positive probability the project is not built, because $p(m_1^*, m_2^*) < 1$ for any v_1, v_2. Second, the project might not even get to the stage of $p < 1$. Recall that either player can veto the project by bidding $Veto$. This will definitely happen if $v_1 + v_2 < 100$, because given that $t_1 + t_2 = 100$, one player or the other will foresee $v_i - t_i < 0$ and bid $Veto$. Even if $v_1 + v_2 \geq 100$, however, it may be that $v_i - t_i < 0$, because when the sum of values is close to 100, even a small difference between the tax and proportional payment will cause one player's tax to be greater than his benefit. Thus, we will have to check for that.

Let the probability the project is built be, for $m_1, m_2 > 0$,

$$p = 1 - \frac{\beta}{(m_1m_2)^\beta}$$

and player 1's tax be

$$t_1 = \frac{m_1}{m_1 + m_2}(100)$$

The first order conditions are:

$$\beta^2 m_2 (m_1 m_2)^{-\beta-1}\left(v1 - \frac{100 m_1}{m_1+m_2}\right) + \left(\frac{100 m_1}{(m_1+m_2)^2} - \frac{100}{m_1+m_2}\right)\left(1-\beta(m_1m_2)^{-\beta}\right) = 0$$

and

$$\beta^2 m_1 (m_1 m_2)^{-\beta-1}\left(v2 + \frac{100 m_1}{m_1+m_2} - 100\right) - \frac{100 m_1\left(1-\beta(m_1m_2)^{-\beta}\right)}{(m_1+m_2)^2} = 0$$

The second derivative of player 1's profit is

$$\begin{aligned}&(-\beta-1)\beta^2 {m_2}^2 (m_1m_2)^{-\beta-2}\left(v1 - \tfrac{100m_1}{m_1+m_2}\right)\\ &+2\beta^2 m_2\left(\tfrac{100m_1}{(m_1+m_2)^2} - \tfrac{100}{m_1+m_2}\right)(m_1m_2)^{-\beta-1} + \left(\tfrac{200}{(m_1+m_2)^2} - \tfrac{200m_1}{(m_1+m_2)^3}\right)\left(1-\beta(m_1m_2)^{-\beta}\right)\end{aligned}$$

Solving the two first-order conditions together results in the following equilibrium if $v_1 - v_2 < 100$ (where we assume without loss of generality that $v_1 \geq v_2$):[2] Define

$$z \equiv 1 - (v_1 - v_2)^2.$$

Then,

$$m_1^* = (100 + v_1 - v_2)\frac{\left(\frac{\beta z + 2\beta^2(v_1+v_2-100}{z}\right)^{\frac{1}{2\beta}}}{\sqrt{z}}$$

and

$$m_2^* = (100 - v_1 + v_2)\frac{\left(\frac{\beta z + 2\beta^2(v_1+v_2-100}{z}\right)^{\frac{1}{2\beta}}}{\sqrt{z}}$$

The probability of building the project when the values add up to more than the cost is

$$p^* \equiv p(m_1^*, m_2^*) = 1 - \frac{z}{z + 2\beta^2(v_1+v_2-100)}$$

and the tax share of player 1 is

$$t_1^* \equiv t_1(m_1^*, m_2^*) = (1/2)(100 + v_1 - v_2)$$

[2]We used Mathematica 12 to solve the system, though it required quite a few manual steps too since Mathematica was not able to solve it directly. See http://rasmusen.org.papers/00PublicGoods-Mathematica.nb or http://rasmusen.org.papers/00PublicGoods-Mathematica.pdf.

Note that t_1^* is increasing in v_1 and therefore t_2^* is increasing in v_2, as required for a fair tax system. We do not have $m_1^* = v_1$ or $m_2^* = v_2$, but I think that on observing both m_1 and m_2, the decider can back out the values of v_1 and v_2. We could therefore use a direct mechanism, a truthtelling one if we wanted to, but it is not convenient here, unlike in typical mechanism design problems. One reason it is not convenient is that the decider must observe *both* m_1 and m_2 to deduce v_1.

Note that the tax share does not depend on β. Are the taxes less than the values? Yes. For player 1, $t_1^* = 50 + v_1/2 - v_2/2 = v_1 - (v_1/2 + v_2/2 - 50) < v_1$ since $v_1 + v_2 \geq 100$. For player 2, $t_2^* = 100 - v_1^* = 50 - v_1/2 + v_2/2 = v_2 - (v_1/2 + v_2/2 - 50) \leq v_2$ since $v_1 + v_2 \geq 100$. Thus neither player will veto if $v_1 + v_2 \geq 100$ and we have avoided that type of inefficiency.

Thus,

$$Lim\ \beta \stackrel{p^*}{\rightarrow} 0 = 0$$

and

$$Lim\ \beta \stackrel{p^*}{\rightarrow} \infty = 1$$

Note that:

$$p^*(v_1 = v_2 = v) = 1 - \frac{1}{1 + 2\beta^2(2v - 100)} = \frac{2\beta^2(2v - 100)}{1 + 2\beta^2(2v - 100)}$$

If $v_1 - v_2 \geq 100$, then m_1^* approaches infinity, p^* approaches 1, and t_1^* approaches 100. Thus, if one player's value exceeds the other's by 100 or more then regardless of β, the project is built, but the tax is not "fair" because the higher-valuing player's tax ceases to rise with his value once he is paying the entire project cost.

Here aer some numerical solution results from solving the first order conditions together. Note that it seems $x_i = v_i$ when $v_1 + v_2 = 100$.

```
    "beta=2 ;   v1=80; v2=80  yields t1=50,  p =.99";
"beta=8 ;   v1=80; v2=30  yields t1=75,  p =.96";
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 "beta=8 ;   v1=80; v2=20  yields t1=80,  p =.875";
 "beta=8 ;   v1=98; v2=2   yields t1=98, p =.875";
 "beta=8 ;   v1=99; v2=1   yields t1=99, p =.875";
 "beta=8 ;   v1=120; v2=20  yields no solution";
 "beta=8 ;   v1=100; v2=2   yields t1=99, p =.99";
"beta=8 ;   v1=120; v2=2   yields  no solution";
"beta=8 ;   v1=199; v2=1    yields no solution;";
"beta=8 ;   v1=100; v2=0    yields no solution (of course);";
```

See what happens in the special caes taht $v_1 = v_2$ and $v_1 + v_2 = 100$.

There would be another equilibrium at (0,0) if we didn't rule that out. There is indeed another equilibrium, (Veto, Veto), but it is weak. Moreover, it can be ruled out if the mechanism is made more complex. Instead of submitting the messages simultaneously, let the mechanism specify that player 1 submit m_1 first, and the decider observes it but does not reveal it to player 2. If $m_1 = VETO$, the decider tells player 2 and player 2 is forbidden to choose $m_2 = VETO$ and instead must choose $m_2 \in (0, \overline{m}]$. This eliminates the (VETO, VETO) equilibrium.[3]

[3]Note that it would *not* work to simply cancel the project if the first message is VETO. The reason is that if the first player expects the second player to choose VETO, it is an equilibrium for him to choose VETO himself, even if by thereby doing he prevents the game from reaching the node where player 2 would actually get to make that decision.

Criteria for Mechanisms

We have started with what is novel in this paper: the breakdown mechanism for attaining close to efficiency, with budget-balancing, simplicity, and fairness. We will now discuss existing mechanisms. We believe that this part of the paper may be useful to economists who wish to understand mechanism design, even aside from its function of distinguishing our new mechanism. Mechanism design has earned a reputation as a field for nonspecialists to steer clear of, full of confusing results that lack intuition and realism— see, e.g. Rothkopf (2007). Also, while expositions of mechanism design are clear about whether their equilibria are Nash or dominant-strategy, they are often unclear in whether the results are weak or strong equilibria— that is, whether the players are indifferent about their equilibrium actions— and whether weak equilibria are intrinsically weak (as mixed-strategy equilibria are) or due to open-set problems (as with firms indifferent about continuing in operation with zero profits under perfect competition, or agents who are indifferent between working when they are paid their reservation wage). Thus, we think it will be a useful clarification to go through the standard mechanisms for our particular problem of the building and taxation of a public good for two agents who know each other's valuations. Let's think of the decider's thought process.

First, though, let's think about some properties of equilibria. [FINISH] An equilibrium is a pair of strategies, one for each player, with certain properties. A Nash equilibrium has each player's strategy maximizing his payoff given the other player's strategy.

Definition. A strategy profile is a pair (x_1, x_2) of strategies, one for each player.

Definition. The strategy profile (x_1^*, x_2^*) is a *strong Nash equilibrium* if $\pi_1(x_1^*, x_2^*) > \pi_1(x_1, x_2^*)$ for any alternative strategy x_1 (and $\pi_2(x_1^*, x_2^*) > \pi_2(x_1^*, x_2))$.

Definition. The strategy profile (x_1^*, x_2^*) is a *Nash equilibrium* or *weak Nash equilibrium* if $\pi_1(x_1^*, x_2^*) \geq \pi_1(x_1, x_2^*)$ for any alternative strategy x_1 (and $\pi_2(x_1^*, x_2^*)\ geq \pi_2(x_1^*, x_2)$).

DEFINITION. An undominated Nash equilibrium is a Nash equilibrium where no player is using a weakly dominated strategy (Jackson & Moulin [1992]). Thus, (x_1^*, x_2^*) is an undominated Nash equilibrium if for any alternative strategy x_1,
(i) $\pi_1(x_1^*, x_2^*) \geq \pi_1(x_1, x_2^*)$, and
(ii) $\pi_1(x_1^*, x_2) \leq \pi_1(x_1, x_2)$ for all x_2 implies that $\pi_1(x_1^*, x_2) = \pi_1(x_1, x_2)$ for all x_2.
(and similarly for player 2).

Note that this means a strategy profile is an undominated Nash equilibrium if x_1^* is equally as good as x_1 in response to all possible x_2, even if it is never strictly better.

Definition. DIFFERENT An undominated-strategy Nash equilibrium is a strategy profile in which no player plays strategy S that never has a higher payoff than strategy S' but in some strategy pairs has a lower payoff (Palfrey & Srivastava [1991]).

Jackson (1991) I think is relevant somehow. Maybe not. xxx

The Nash equilibrium is weak if at least one player could deviate to a different strategy and keep his payoff the same. It is strong if no player could do so. The weakeness of the equilibrium is an open-set problem if when a player is tempted to do that, the other player could alter his strategy slightly and end the temptation.

Definition. A Nash equilibrium (x_1^*, x_2^*) has an *open-set problem* if for any alternative strategy x_1 such that $\pi_1(x_1^*, x_2^*) = \pi_1(x_1, x_2^*)$ there is some number $\bar{\epsilon}$ such that for any number $\epsilon \in (0, \bar{\epsilon}]$, $\pi_1(x_1^*, x_2^* + \epsilon) > \pi_1(x_1, x_2^* + \epsilon)$ or $\pi_1(x_1^*, x_2^* - \epsilon) > \pi_1(x_1, x_2^* - \epsilon)$, or the analogous condition is true for player 2.

Definition. We will say an equilibrium is *open-set strong* if it would be strong except for an open-set problem.

[Needs a better definition; it might be that iff the weak Nash equilibrium is due to an open-set problem, it is locally unique. Think about mixed strategies.]

A dominant-strategy equilibrium has each player's strategy strictly maximizing his payoff given the other player's strategy.

Definition. The strategy profile (x_1^*, x_2^*) is a *dominant-strategy equilibrium* or *strong dominant-strategy equilibrium* if $\pi_1(x_1^*, x_2) > \pi_1(x_1, x_2)$ for any x_2 and any alternative strategy x_1 (and $\pi_2(x_1, x_2^*) > \pi_2(x_1, x_2)$)

Definition. The strategy profile (x_1^*, x_2^*) is a *weak dominant-strategy equilibrium* if $\pi_1(x_1^*, x_2) \geq \pi_1(x_1, x_2)$ for any x_2 and any alternative strategy x_1 (and $\pi_2(x_1, x_2^*) \geq \pi_2(x_1, x_2)$).

A strategy profile is a weak dominant-strategy equilibrium if at least one player could deviate to a different strategy and keep his payoff the same.

Definition. Palfrey (handbook) says that a mechanism *implements* a social choice function if its equilibrium always results in an outcome prescribed by that social choice function. This requires defining "equilibrium" and choosing a social choice function. The equilibrium might be defined as Nash equilibrium, dominant-strategy equilibrium, or strong Nash equilibrium, for example. The social choice function we will use here is that the outcome be pareto-efficient given available information and that the taxes be fair in the sense that a player with a higher value is never taxed less. Implementation does not require that the equilibrium be unique, so long as the outcome always satisfies the social choice function. In our setting, for example, efficiency just requires that the project be built when $v_1 + v_2 > 100$ and not built when $v_1 + v_2 < 100$, but it imposes no requirement when $v_1 + v_2 = 100$ and it imposes no requirement on the taxes. Also, fairness as defined here requires that t_1 not fall as v_1 rises, but this permits a large variety of tax arrangements.

Sometimes the only equilibrium that exists is a weak Nash equilibrium in dominated strategies, e. g. the Bertrand game, where choosing price to equal a constant marginal cost surely yields a payoff of zero but choosing a higher price would yield positive prices if the other players deviated from equilibrium. What would be nice and realistic to aim for in a mechanism, I think, is a unique strong Nash equilibrium, or a unique weak Nash equilibrium that is weak only because of an open-set problem.

That said, what kind of equilibrium does the Breakdown Mechanism have? A unique strong Nash equilibrium.

Another important kind of terminology in mechanism design is the distinction between ex ante, interim, and ex post individual rationality and efficiency. This will not be important to our discussion here, but we will explain them for completeness.

The "individual rationality" constraint is that the players must face an expected payoff greater than their outside option or reservation utility. As one of us has argued in Rasmusen (2004), this is better called the "participation constraint", since we assume that players are individually rational, using the words in their ordinary, separate, meanings, in every assumption we make about behavior in game theory. Ex ante, interim, and ex post refer to the point in time at which the constraint is applied. Let π_1 be player 1's payoff under the mechanism, v_1 be his type, v_2 be the other player's type, and x be some other variable or let it refer to the realization of a mixed strategy.

Definition. Player 1's *ex ante participation constraint* is satisfied if he is willing to participate before he knows his type: $E\pi_1(v_1, v_2, x) \geq 0$, normalizing the payoff from not building the project to zero.

Definition. Player 1's *interim participation constraint* is satisfied if he is willing to participate after he knows his type but before he knows the other player's: $E(\pi_1(v_2, x)|v_1) \geq 0$.

Definition. Player 1's *ex post participation constraint* is satisfied if the player is willing to participate after he knows everyone's type, but perhaps before he knows how other random variables will turn out or what will be the realization of mixed strategies: $E((\pi_1(x)|v_1, v_2) \geq 0$, normalizing the payoff from not building the project to zero.

We say that a mechanism is efficient if it is allocatively efficient in equilibrium given available information; that is, a social planner with the available information could not make a decision different from the mechanism's result without reducing some player's expected payoff. The "available information" can be "nothing" (ex ante efficiency), v_1 (interim efficiency) or v_1 and v_2 (ex post efficiency).

In this paper we are only concerned with ex ante efficiency and ex post participation.

Definition. An equilibrium is *efficient* if xxxx.

Definition. An equilibrium is *epsilon-efficient* if xxxx.

A mechanism viuay imemens a scia chice funcin2 if i can (exacy) imemen arbritrarilyclose approximations ot eh social coice function. Palfre y surve,

ABEUAND MATHSUahim 1992A– USE LOtteriesto get virtual implementabioitlity. Each player reports lots of value profiles and there is a lottery over whcih isuased.

related to Chung and Ely's (2003) study of the robustness undominated Nash implementation. Chung and Ely show that social choice rule is not Maskin monotonie but can be implemented in undominated Nash equilibrium9 under complete information, then there are information perturbations under which an undesir- able undominated Nash equilibrium appears. In contrast, we con- sider extensive-form mechanisms and show that only Maskin monotonie social choice rules can be implemented in the closure of the sequential equilibrium correspondence

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Other Mechanisms

Let's now consider other mechanisms that might address the problem of a decider trying to maximize social surplus by deciding whether to build a project while taxing two beneficiaries who know each others' values just enough to finance the cost.

1. The decider commits to build the project if $m_1 + m_2 \geq 100$ and not otherwise, with taxes if it is built of $t_1 = m_1$ and $t_2 = m_2$. This will have lots of strong Nash equilibria, mostly efficient, as Bagnoli & Lipman (1989) show. The problem is that they might not be "fair". One strong equilibrium if $v_1 = 150$ and $v_2 = 60$ is $(t_1 = 41, t_2 = 59)$, for example, with the higher-valuing player paying more for the project. Proportional taxation would have $(t_1 = 70, t_2 = 30)$ in that example. Also, using those same valuations, another strong Nash equilibrium is $(0, 0)$ and the project isn't built.

But the decider imposes a tax rule too, and can't that produce fairness? No, it can't. The problem is that the player will bid $m_1 + m_2$ combinations that equal 100, the only constraints being $m_1 \leq v_1$ and $m_2 \leq v_2$. Whatever fancy rule he might impose, it reduces to a tax for each that they can anticipate. How about a random tax rule? That, too, reduces to a known expected value of tax based on x_1 and x_2 and will work no better.

2. The decider uses a Groves mechanism.[4] He commits to building the project if $m_1 + m_2 \geq 100$ and not otherwise, with taxes if it is built of $t_1 = 100 - m_2$ and $t_2 = 100 - m_1$. Thus, if $v_1 = 40$ and $v_2 = 90$, the project is built and $t_1 = 10$ and $t_2 = 60$. This is a dominant-strategy mechanism. It works even if the two players do not know each others' values. The problem is that this is not budget-balancing; it may well happen (though not necessarily?) that the taxes do not add up to 100. And it's only a weakly dominant and weakly best-response strategy for each player to tell the truth, and it is not the unique Nash equilibrium. Another Nash equilibrium is $m_1 = 0, m_2 = 0$, when the players know each others' values. If they DON't know each other's values, the Groves Mechanism works better. Then, it is strictly dominant to bid your true value, to maximize your EXPECTED payoff.

3. The decider thinks about using a Clarke Tax, but realizes he can't. This is a variant on the Groves Mechanism. Clarke added an extra tax to make sure that the taxes exceed the cost, but the idea doesn't apply when there are only two players. Suppose there are three. Then an example of a Clarke tax is to build the project if $\sum t_j \geq 100$ and if it is built set the tax to $t_i = 100/3$ if $(\sum_{j \neq i} m_j > 100$ and $t_i = 100/3 + 100 - (\sum_{j \neq i} m_j)$ if $\sum_{j \neq i} m_j \leq 100$. Then player i's tax is 100/3 if his message makes no difference to the project being built, and more than 100/3 if he does make a difference, more by an amount

[4] This mechanism is often called the VCG Mechanism or the Vickrey-Clarke-Groves Mechanism. We do not like that terminology. The mathematical form of the idea is in Vickrey (1961), to be sure, but nobody applied it to anything but auctions and other private-goods situations. Clarke (1971) comes next chronologically, but what is distinctive about his mechanism is the tax he constructed to ensure that the taxes would at least equal and probably exceed the cost of the project. Groves (1976, 1979) figured out the "trick" independently at the same time as Clarke (though published later) and has the simplest version. Tideman (1977) tells us that Groves and Clarke discovered the idea independently as graduate students in work starting around 1967 and both presented it at conferences in September 1970. "Vickrey-Clarke-Groves Mechanism" is also unfortunate because dhort names are better than long ones (consider the abomination "Herfindahl-Hirschman Index") and it is more important that names be useful than that they commemmorate we scholars. If too long, they become acronyms, which are pernicious because meaningless in themselves.

depending on just the other players' messages. He is stuck paying 100/3 either way, and if the project is built he pays nothing extra if his reporting $m_i = 0$ would have made no difference, and an extra amount less than his report otherwise (since $100 - (\sum_{j \neq i} m_j \leq m_i$ when $\sum t_j \geq 100$). The result is that the bridge is built when it is efficient to do so, and although the budget is not balanced, at least the taxes exceed the expenditure. The taxes tend are never higher for players with lower values, and sometimes they are higher. For example, if $v_1 = 10$, $v_2 = 50$, and $v_3 = 80$, when they report the truth the result is that $t_1 = 100/3+0$, $t_2 = 100/3+(100-90)$, and $t_3 = 100/3+(100-60)$. A major flaw, however, is that a player can, like player 1 this example, end up paying more in taxes than he receives in value ($100/3 > 10$). Also, this is not a dominant-strategy mechanism; player 2 is not willing to send $m_2 = 50$ if he thinks players 1 and 2 are going to send $m_1 = 0$ and $m_3 = 80$ because then the bridge would be built and his tax would be $t_2 = 100/3 + 100 - 80 > 50$. It is just a weak Nash mechanism.

Note that this is different from the Groves mechanism having multiple equilibria. In this example, the Groves mechanisms would have as two of its equilibria (10, 50, 80) and (0,0,0). Under the Groves mechanism, $m_2 = 50$ is a weakly dominant strategy: it is as good as any other strategy regardless of what the other two players do. Under the Groves mechanism $m_2 = 0$ is not weakly dominant because if the other players play $m_1 = 10, m_3 = 89$, player 2 will want to deviate to $m_2 = 1$ so that the project is built and he will be taxed $t_1 = 1$. The message $m_2 = 0$ is, however, a weak Nash equilibrium strategy, because it is a weak best response to $m_1 = 0$, $m_3 = 0$. Under the Clarke mechanism, $m_1 = 50$ is not weakly dominant, because it is not a best response to $m_1 = 0, m_2 = 80$. It is, however, a weak Nash equilibrium strategy, because it's a weak best response to $m_1 = 10$, $m_3 = 80$.

I think this is a point of some confusion among economists, and Wikipedia is wrong on it. This is hard to see because commonly people talk about the Groves-Clarke Mechanism, conflating the two. Also, it may depend on whether the players know each others' types or not.

4. The decider uses a Jackson-Moulin (1992) two-stage mechanism. The two players submit reports w_1 and w_2 which are each supposed to equal $v_1 + v_2$. If both reports are less than 100, the project isn't built. If it is, the player reporting the highest w (breaking ties with a coin flip) is titled "the payer" and the players submit reports m_1 and m_2 of their own values. Suppose player 1 is the payer. After the second set of reports, if $m_1 + m_2 < w_1$, the project is not built, and player 1 pays to player 2 amount $w_1 - m_1 - \frac{w_1 - m_1}{w_1}(100)$ (which is player 2's equilibrium payoff if the project is built). If $m_1 + m_2 > w_1$ the project is built and $t_1 = \frac{m_1}{w_1}(100)$. If $m_1 + m_2 = w_1$, the equilibrium outcome, the payer chooses to pay player 2 or to build and pay the tax.

One equilibrium is for both players to report truthfully and for the payer to choose to build and pay the tax if $v_1 + v_2 \geq 100$. Player 1's payoff if he is honest and $v_1 + v_2 \geq 100$ is $v_1 - \frac{v_1}{v_1+v_2}100$. This is more than the 0 he would obtain if he underreported. Let's suppose he is the payer. If he deviated to $m_1 < v_1$, the project would not be built and he would have to pay player 2 amount $-(v_1 + v_2 - v_1 - \frac{v_1+v_2-v_1}{v_1+v_2}(100))$ and get a negative payoff, so he wouldn't do that. How about player 2? If he deviates to $m_2 < v_2$ in the second stage, he will get a payment from player 1 but the project won't be built, and the payment from player 1 was designed to be no greater than player 2's payoff from the project being built. So player 2 won't deviate either. Player 2 is indifferent as to what value of m_2 he chooses, because it doesn't matter to his payoff.

The equilibrium is not unique. Suppose $v_1 = v_2 = 60$. One equilibrium has $(m_1 = m_2 = 60)$. Another starts with $(w_1 = 0, w_2 = 0)$, and if one player deviates, the other sends $m = 30$. To see that this is an equilibrium, start from the end and suppose without loss of generality that player 1 deviated in the first stage and submitted $w_1 \geq 100$. Player 2 would be indifferent as to what he does in the second stage because m_2 does not affect player 2's payoff, so $m_2 = 30$ is one best response. Player 1, however, will now be faced with either paying a tax more than v_1, since if $m_2 = 30$ player 2 is not going to be taxed more than 30 and the cost is 100, or choosing an m_1 that kills the project and requires him to pay something to player 2. Either way player 1's payoff is negative, so he won't deviate to $w_1 \geq 100$. This equilibrium has player 2 playing a weakly dominated strategy, however, so it is not an undominated Nash equilibrium.

This mechanism has a bigger problem, though. It has the same multiple equilibrium outcomes as Bagnoli & Lipman's simple single-offer mechanism, and thus the same fair taxation problem. Our earlier example will convey the point. Under the single-offer mechanism, if $v_1 = 140$ and $v_2 = 60$ then a strong equilibrium is $(t_1 = 41, t_2 = 59)$, whereas proportional taxation would have $(t_1 = 70, t_2 = 30)$. The Jackson-Moulin two-stage mechanism has an equilibrium with $(t_1 = 70, t_2 = 30)$, but of course so does the single-offer mechanism. Does the Jackson-Moulin two-stage mechanism have an equilibrium with $(t_1 = 41, t_2 = 59)$? Yes, as we will see. Suppose $w_1 = w_2 = 200$ in the first step, and $(m_1 = 82, m_2 = 108)$ in the second. The equilibrium payoffs will be $140 - 41$ for player 1 and $60 - 59$ for player 2, both positive. Player 2's second-step choice of m_2 does not affect his payoff, so he will not deviate. Player 1 has the option of sending $m_1 < 82$, in which case the project is cancelled and he pays $w_1 - m_1 - \frac{w_1 - m_1}{w_1}(100) = 200 - m_1 - 70$ to player 2, which would leave player 1's payoff negative, so he won't deviate. The outcome is an unfair split, and it should be clear that as with the single-offer mechanism, any efficient allocation can be obtained as a Nash equilibrium.

The problem of multiple equilibria cannot be solved by having the players submit their entire strategies (their decision rules based on what happens so far in the game) simultaneously, in which case the question of subgame perfectness does not arise. Removing the perfectness constraint can only change the number of equilibria by increasing it.

Jackson and Moulin preferred mechanism is the two-stage mechanism, because when there are more than two players it does not require each player to know the v value of each other player, just the aggregate of the others. They also have a three-stage mechanism, though, which improves on the single-offer mechanism by having a single equilibrium, which can be designed to be fair.

REPLACE FRMO MOULIN MISTAKE SEMINAR SLIDES.

5. The decider uses the Jackson-Moulin (1992) three-stage mechanism (simplified from their presentation). This will have a unique equilibrium but it's slightly more complicated. The decider commits to the following scheme. First, the players submit reports w_1 and w_2 which are supposed to equal $v_1 + v_2$. If both reports are less than 100, the project isn't built. If it is, the player reporting the highest w (breaking ties with a coin flip) is titled "the payer" and the payer submits report m_1 of his value. Third, player 2 — who is *not* the payer— chooses between (a) building the bridge with $t_1 = \frac{w_1 - m_1}{w_1}(100)$ and (b) not building the bridge, but collecting amount $w_1 - m_1 - \frac{100(w_1-m_1)}{w_1}$ (which is player 2's equilibrium payoff) from player 1.

This works because (a) is better for player 2 if m_1 is too small and (b) is better if m_1 is too big. Player 1 strongly prefers (a), so he will tell the truth and set $m_1 = v_1$. Moreover, there is no inefficient cancellation equilibrium. If $v_1 = 80, v_2 = 70$, for example, and player 2 sends $w_2 = 0$, player 1 can send $w_1 = 150$ and $m_1 = 80$ and the equilibrium will play out efficiently— indeed, this is itself an equilibrium, though payoff-equivalent to the equilibrium described earlier.

The equilibrium is still weak, because in equilibrium player 2 does not care about the value of w_2 and is indifferent between (a) and (b), but that is just an open-set problem. And it is unique. The intuition for why this improves on the two-stage mechanism is that now only player 1 is reporting information in the second step, not player 2. Without the simultaneous reporting of the two-stage and single-offer mechanisms, we do not have the mutually dependent choices that give rise to multiple consistent sets of beliefs and multiple equilibria. Instead, player 1 reports m_1 and m_2 and after he finishes, player 2 chooses to build or take cash.

This is a good solution to the problem if complexity is not too much of a drawback. We never observe it, of course. Nonetheless, it is like "I divide the pie; you choose the piece", which usually is applied to situations where a good (or "bad") is being divided (see Crawford [1977] and Crawford & Heller [1979] for allocating an indivisible private good). Here, the aim is not an equal division of the tax, but the desired division is obtained cleverly by letting one player choose a tax division and the other choose between that division plan and direct cash.

For each of these, use the same example of v1 and v2 and show what happens.

Here, is the equilibrium unique? How about a 0,0 equilbirium?

6. The decider uses a leadership auction mechanism (Moulin [1981], inspired by Crawford [1979]). First, he auctions off the right to be the "leader", the players submitting bids of b_1 and b_2 simultaneously. If neither player bids a positive amount, the project is not built. The leader, the player who bids highest (ties being broken randomly), pays his bid to the other player. The leader then proposes a tax t for himself and $100-t$ for the other player. The other player then decides whether to accept or reject the tax scheme, resulting in the project being built or not built.

Consider what happens if $v_1 = 80$ and $v_2 = 60$. Working back from the end, if player 1 wins the auction he will choose a tax scheme barely acceptable to player 2, $t_1 = 40$, and player 2 as winner would similarly choose $t_1 = 80$. They would therefore bid $b_1 = b_2 = 20$, so that the winner would pay 20 in the loser but gain $v - t = 40$ and each would end up with an overall payoff of 20. This is a weak Nash equilibrium, but only due to the open-set problem of the loser being indifferent between accepting and rejecting the tax scheme. It is unique, I think. If the loser's strategy were to reject when indifferent, the winner would deviate to offer him a tax lower by small amount ϵ and the loser would no longer be indifferent and would strongly wish to accept.

The leadership auction mechanism is efficient and fair. It will not result in proportional taxation, but it will, like the bargaining mechanism we will consider next, result in the surplus being split and a player's tax rising in his value for given value of the other player.

7. The decider tells the two players to decide among themselves what to do, and he will follow their suggestion if they agree, and not build the project otherwise. What happens depends on how bargaining works. Using the Nash Bargaining Solution, the outcome is efficient and the taxes satisfy

$$Maximize(t_1, t_2)(v_1 - t_1)(v_2 - t_2) \text{ subject] to} t_1 \leq v_1, t_2 \leq v_2, t_1 \leq 100, t_2 \leq 100, t_1 + t_2 \geq 100$$

The result is to split the surplus from building the project, each player getting half, except that a player cannot be taxed less than 0 or more than 100. Rubinstein (1982) bargaining or my own Rasmusen (2019) breakdown bargaining reach much the same result. If $v_1 = 80$ and $v_2 = 40$, for example, the surplus of 20 is split so that $t_1 = 70$ and $t_2 = 30$ and $\pi_1 = \pi_2 = 10$. If $v_1 = 120$ and $v_2 = 10$, however, , the surplus of 30 is split so that $t_1 = 100$ and $t_2 = 0$, with $\pi_1 = 20 > \pi_2 = 10$. This works out very well, if bargaining is costless. It does not reach a proportional tax, but it reaches one that is fair in a different, equally sensible, meaning of "fair". If bargaining breaks down often, as in Rasmusen (2019) it is not efficient.

5. Concluding Remarks

Rothkopf (2007) gives thirteen reasons why mechanisms in the literature aren't used in the real world. Weyl (2019) talks about "price theory" approach to economic theory. Participants int eh literature worry about this—see xxx.

This paper is in the tradition. We have what we hope is a robust, simple, mechanism, suited for situations with a samll number of players who know each other well.

MORE NOTES

What if the agents do not know each others' values? Then Myerson-Satterthwaite comes into play.

$$\pi_1 = \int_0^\infty p(m_1, m_2(v_2))(v_1 - t_1(m_1, m_2(v_2)))f(v_2)dv_2$$

This will be very difficult to solve out.

How about the breakdown bargaining model where players know each others' values? That is interesting. It is different from just risk aversion, I think. Suppose player 1 values the pie at 1 and player 2 at 2. No– better: they must pay t_1 or t_2 to build the pie for the cost of 1, but its benefit is v_1 or v_2.

Myerson-Satterthwaite. This is not my bargaining basics situation, because it is incomplete information. What they showed was that if the supports of the values overlap, there is no efficient mechanism (and if their values are uncorrelated; if $v_1 = 3v_2$, for example, there's a mechanism that works). If we don't insist on budget balancing, there's a mechanism too.

Suppose v_1 and v_2 are both uniform over [0,200], and we say the price is $p = \frac{m_1+m_2}{2}$. The seller, player 1, will send $m_1 > v_1$ because it increases the price very often and kills a sale hardly ever, for a small price increase. Can adding a no-sale probability of $n(m_2 - m_1)$ help with that? Probably not.

Myerson & Satterthwaite show that the split-the-difference mechanism (which they say is due to Chaterjee-Samuelson) is second-best efficient, maximizing expected surplus even though it is not first-best efficient and sometimes trade does not take place. So I can improve a little bit on it? If I can, MS have a serious mistake. I think they are right, tho. They ahve put the breakdown into the cheapest possible place, where gains from trade are smallest. Probably I can do the same thing in my model. Make it incompelte info, adn say that there is no trade unless $v_1 + v_2 \geq 100 + K$ and find K, for a truthful mechanism.

The tax in Chatterjee-Samuelson is to have split-the-difference pricing. Under that mechanism, the players split the surplus, so it would probably be that here. That is, each gets a surplus of $\frac{v_1+v_2-100}{2}$. That means $t_1 = 50 + \frac{v_1-v_2}{2}$.

Under this mechanism, the project is built if and only if $m_1+m_2 \geq 100 + K$. Suppose player 2 tells the truth.

$$\pi_1 = \int_0^{100+K-m_1} (0) f(\mu_2) dv_2 + \int_{100+K-m_1}^{\infty} (v_1 - (50 + \frac{m_1 - v_2}{2})) f(v_2) dv_2$$

$$\pi_1 = \int_{100+K-m_1}^{\infty} (v_1 - 50 - \frac{m_1}{2} + \frac{v_2}{2}) f(v_2) dv_2$$

Now maximize over m_1. We get

$$\frac{d\pi_1}{dm_1} = (v_1 - 50 - \frac{m_1}{2} + \frac{100 + K - m_1}{2}) f(100+K-m_1) + \int_{100+K-m_1}^{\infty} (-\frac{1}{2} f(v_2) dv_2 = 0$$

If $f = 1/200$ because the values are uniform over [0,200] then

$$(v_1 + K/2 - m_1)(1/200) - |_{100+K-m_1}^{200} (\frac{1}{2}(v_2/200) = 0$$

$$v_1 + K/2 - m_1 - 100 + 50 + K/2 - m_1/2 = 0$$

$$m_1 = (2/3)(v_1 - 50 + K)$$

Well, that's not incentive compatible except at one value of v_1, if we pick K right.

Let's try not using hte REvelation principle, since incentive compatibility seems not to work.

What would be nice is to find the MS mechanism. It is not fully efficient either, but is realistic.

Appendix 1: Mathematica Code for the Proof

We'll have to clean this up for it to be understandable. It not only solves the first order conditions together, but checks the solution.

PROBLEM STATEMENT:
THE PROBLEM IS TO FIND P AND T1 FUNCTIONS WITH THE RIGHT PROPERTIES.
WE WANT THEM BOTH INCREASING IN M1,M2. THEY CANNOT DEPEND ON
V1 AND V2, EXCEPT INDIRECTLY, THROUGH THE PLAYERS' CHOICE OF M1,M2.
THE PROBABILITY P SHOULD BE BOUNDED BETWEEN 0 AND 1, AND THE TAX T1
BETWEEN 0 AND 100. ALSO, WE WANT A UNIQUE SOLUTION TO EXIST WHEN THE
PLAYERS MAXIMIZE THEIR PAYOFFS AND WE WANT THE EQUILIBRIUM PROBABILITY
TO BE ARBITRARILY CLOSE TO 1. AND OUR GOAL IS TO GET T1 AS CLOSE AS POSSIBLE
TO V1/(V1+V2). WE CAN LIMIT V1 AND V2 TO BE IN [0,100] IF NECESSARY.

Update: Here we don't want to SOLVE for p (we should make that clear) we just want to chose p qualitatively and then find m1^* and m2^*.

Eric: the PDE solutions below are exceptional cases where the m1^* is free and only m2^* is determined...these are special exceptional functions.

Note we cannot solve (in complete generality) for v1 and v2 in terms of t1 and t2 since t2 = 1 - t1, so only one parameter in general.

New version (we chose p[m1,m2] as a given function and simply solve for m1,m2).

Example 1. $\boldsymbol{p[\mathrm{m1\_},\mathrm{m2\_}] = 1 - b/(\mathrm{m1m2})\hat{}b}$

$\mathbf{t1[m1\_, m2\_] = m1/(m1 + m2)}$

$1 - b(\mathrm{m1m2})^{-b}$

$\frac{\mathrm{m1}}{\mathrm{m1+m2}}$

Here we are first treating v1 and v2 as constant in m1 and m2.

$\mathbf{Eqs = \{}D[p[\mathbf{m1, m2}], \mathbf{m1}](\mathbf{v1 - t1[m1, m2]}) - p[\mathbf{m1, m2}]D[\mathbf{t1[m1, m2], m1}],$

$D[p[\mathbf{m1, m2}], \mathbf{m2}](\mathbf{v2 - 1 + t1[m1, m2]}) - p[\mathbf{m1, m2}]D[\mathbf{1 - t1[m1, m2], m2}]\mathbf{\}//FullSimplify}$

$\left\{\frac{(\mathrm{m1m2})^{-b}\left(b\mathrm{m1m2}-\mathrm{m1m2}(\mathrm{m1m2})^{b}+b^{2}(\mathrm{m1+m2})(\mathrm{m1}(-1+\mathrm{v1})+\mathrm{m2v1})\right)}{\mathrm{m1}(\mathrm{m1+m2})^{2}}, \frac{(\mathrm{m1m2})^{-b}\left(b\mathrm{m1m2}-\mathrm{m1m2}(\mathrm{m1m2})^{b}+b^{2}(\mathrm{m1+m2})(\mathrm{m2}(-\right.}{\mathrm{m2}(\mathrm{m1+m2})^{2}}\right.$

$\mathbf{Solve[Eqs == 0, \{m1, m2\}]}$

Solve : This system cannot be solved with the methods available to Solve.

$\mathrm{Solve}\left[\left\{\frac{(\mathrm{m1m2})^{-b}\left(b\mathrm{m1m2}-\mathrm{m1m2}(\mathrm{m1m2})^{b}+b^{2}(\mathrm{m1+m2})(\mathrm{m1}(-1+\mathrm{v1})+\mathrm{m2v1})\right)}{\mathrm{m1}(\mathrm{m1+m2})^{2}}, \frac{(\mathrm{m1m2})^{-b}\left(b\mathrm{m1m2}-\mathrm{m1m2}(\mathrm{m1m2})^{b}+b^{2}(\mathrm{m1+}\right.}{\mathrm{m2}(\mathrm{m1+m2})^{2}}\right.\right.$

$\mathbf{Eqs2 = \{}\left(b\mathbf{m1m2 - (m1m2)}^{b+1} + b^2\mathbf{(m1 + m2)(m1(-1 + v1) + m2v1)}\right), \left(b\mathbf{m1m2 - (m1m2)}^{b+1} -\right.$

$\left\{b\mathrm{m1m2} - (\mathrm{m1m2})^{1+b} + b^2(\mathrm{m1 + m2})(\mathrm{m1}(-1 + \mathrm{v1}) + \mathrm{m2v1}), b\mathrm{m1m2} - (\mathrm{m1m2})^{1+b} + b^2(\mathrm{m1 + m}\right.$

$\mathbf{Solve[Eqs2 == 0, \{m1, m2\}]}$

Solve : This system cannot be solved with the methods available to Solve.

Solve $\left[\left\{b\text{m1m2} - (\text{m1m2})^{1+b} + b^2(\text{m1} + \text{m2})(\text{m1}(-1 + \text{v1}) + \text{m2v1}), b\text{m1m2} - (\text{m1m2})^{1+b} + b^2(\right.\right.$

x=m1 m2 and y=m1+m2

xytom = Solve[{x == m1m2, y == m1 + m2}, {m1, m2}]//FullSimplify

$\left\{\left\{\text{m1} \to \frac{1}{2}\left(y - \sqrt{-4x + y^2}\right), \text{m2} \to \frac{1}{2}\left(y + \sqrt{-4x + y^2}\right)\right\}, \left\{\text{m1} \to \frac{1}{2}\left(y + \sqrt{-4x + y^2}\right), \text{m2} \right.\right.$

Note these are equivalent solutions (just switched order)

Eqs3 = $\boldsymbol{\left\{bx - x\hat{}(b+1) + b\hat{}2y\left(-\frac{1}{2}\left(y + \sqrt{-4x + y^2}\right) + y\text{v1}\right), bx - x\hat{}(b+1) + b\hat{}2y\left(-\frac{1}{2}\left(y \right.\right.\right.}$

$\left\{bx - x^{1+b} + b^2y\left(\text{v1}y + \frac{1}{2}\left(-y - \sqrt{-4x + y^2}\right)\right), bx - x^{1+b} + b^2y\left(\text{v2}y + \frac{1}{2}\left(-y + \sqrt{-4x + y^2}\right)\right.\right.$

Eqs3p = $\boldsymbol{\left\{bx - x\hat{}(b+1) + b\hat{}2y\left(-\frac{1}{2}\left(y - \sqrt{-4x + y^2}\right) + y\text{v1}\right), bx - x\hat{}(b+1) + b\hat{}2y\left(-\frac{1}{2}\left(y \right.\right.\right.}$

$\left\{bx - x^{1+b} + b^2y\left(\text{v1}y + \frac{1}{2}\left(-y + \sqrt{-4x + y^2}\right)\right), bx - x^{1+b} + b^2y\left(\text{v2}y + \frac{1}{2}\left(-y - \sqrt{-4x + y^2}\right)\right.\right.$

Solve[Eqs3 == 0, {x, y}]

Solve : This system cannot be solved with the methods available to Solve.

Solve $\left[\left\{bx - x^{1+b} + b^2y\left(\text{v1}y + \frac{1}{2}\left(-y - \sqrt{-4x + y^2}\right)\right), bx - x^{1+b} + b^2y\left(\text{v2}y + \frac{1}{2}\left(-y + \sqrt{-4x\right.\right.\right.\right.$

Reduce[Eqs3 == 0&&b > 0, {x, y}]

$Aborted

Subtracting both equations and dividing by b^2 y we have...

repx = Solve $\boldsymbol{\left[\text{v1}y - \frac{1}{2}\left(y + \sqrt{-4x + y^2}\right) - \left(\text{v2}y + \frac{1}{2}\left(-y + \sqrt{-4x + y^2}\right)\right) == 0, x\right]}$ **[[1]]**

$$\left\{x \to \frac{1}{4}\left(y^2 - \text{v1}^2 y^2 + 2\text{v1v2}y^2 - \text{v2}^2 y^2\right)\right\}$$

$$\textbf{Solve}\left[\text{v1}y - \frac{1}{2}\left(y + \sqrt{-4x + y^2}\right) - \left(\text{v2}y + \frac{1}{2}\left(-y + \sqrt{-4x + y^2}\right)\right) == 0, y\right]$$

$$\textbf{repy} = \%[[2]]$$

$$\left\{\left\{y \to -\frac{2\sqrt{x}}{\sqrt{1-\text{v1}^2+2\text{v1v2}-\text{v2}^2}}\right\}, \left\{y \to \frac{2\sqrt{x}}{\sqrt{1-\text{v1}^2+2\text{v1v2}-\text{v2}^2}}\right\}\right\}$$

$$\left\{y \to \frac{2\sqrt{x}}{\sqrt{1-\text{v1}^2+2\text{v1v2}-\text{v2}^2}}\right\}$$

$$\textbf{test} = \{x \to .53, y \to .331, \text{v1} \to .76, \text{v2} \to .42, b \to 1.3\}$$

$$\textbf{test2} = \{x \to .53, y \to .331, \text{v1} \to 1.76, \text{v2} \to .42, b \to 1.3\}$$

$$\{x \to 0.53, y \to 0.331, \text{v1} \to 0.76, \text{v2} \to 0.42, b \to 1.3\}$$

$$\{x \to 0.53, y \to 0.331, \text{v1} \to 1.76, \text{v2} \to 0.42, b \to 1.3\}$$

$$\textbf{eqy0} = bx - x^{1+b} + b^2 y\left(\text{v1}y + \frac{1}{2}\left(-y - \sqrt{-4x + y^2}\right)\right) /.\text{repx}//\text{FullSimplify}$$

$$-\frac{1}{4}b\left(-1 + (\text{v1} - \text{v2})^2\right)y^2 - 4^{-1-b}\left(-\left(-1 + (\text{v1} - \text{v2})^2\right)y^2\right)^{1+b} - \frac{1}{2}b^2 y\left(y - 2\text{v1}y + \sqrt{(\text{v1} - \text{v2})^2 y^2}\right)$$

$$\textbf{eqx0} = bx - x^{1+b} + b^2 y\left(\text{v1}y + \frac{1}{2}\left(-y - \sqrt{-4x + y^2}\right)\right) /.\text{repy}//\text{FullSimplify}$$

$$bx - x^{1+b} + \frac{2b^2\left(x - 2\text{v1}x + \sqrt{-(-1+\text{v1}-\text{v2})(1+\text{v1}-\text{v2})}\sqrt{x}\sqrt{-\frac{(\text{v1}-\text{v2})^2 x}{-1+(\text{v1}-\text{v2})^2}}\right)}{-1+(\text{v1}-\text{v2})^2}$$

We reduce the above under the assumptions that v1>= v2 (Just so sqrt[(v1-v2)^2]=v1-v2 ow =v2-v1) and (v1-v2)<1 (just for single solutions in power function). Note that one square root needs to be <0 to avoid a 3

$$\mathbf{eqy1} = \tfrac{1}{4}b\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)y^2 - 4^{-1-b}\left(\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)y^2\right)^{1+b} - \tfrac{1}{2}b^2y(y-2\mathrm{v1}y+(\mathrm{v1}-\mathrm{v2})y)//\mathrm{F}$$

$$\mathbf{eqx1} = bx - x^{1+b} - \frac{2b^2x((1-2\mathrm{v1})+(\mathrm{v1}-\mathrm{v2}))}{1-(\mathrm{v1}-\mathrm{v2})^2}//\mathrm{FullSimplify}$$

$$-\tfrac{1}{4}b\left(-1+(\mathrm{v1}-\mathrm{v2})^2\right)y^2+\tfrac{1}{2}b^2(-1+\mathrm{v1}+\mathrm{v2})y^2-4^{-1-b}\left(-\left(-1+(\mathrm{v1}-\mathrm{v2})^2\right)y^2\right)^{1+b}$$

$$x\left(b-\frac{2b^2(-1+\mathrm{v1}+\mathrm{v2})}{-1+(\mathrm{v1}-\mathrm{v2})^2}-x^b\right)$$

Note the (1-3 v1+v2) should be (1- v1-v2) if v1>1+v2. Here is the check:

{eqx0, eqy0}/.#&/@{test, test2}

{eqx1, eqy1}/.#&/@{test, test2}

$$\{\{0.821415, 0.0479631\}, \{-8.23449+0.i, 0.0808255\ -0.000121906i\}\}$$

$$\{\{0.821415, 0.0479631\}, \{-2.20011, 0.0808255\ -0.000121906i\}\}$$

If (v1-v2)>1, then
eqx=
eqy=

$$-\tfrac{1}{4}b\left(-1+(\mathrm{v1}-\mathrm{v2})^2\right)y^2+\tfrac{1}{2}b^2(-1+\mathrm{v1}+\mathrm{v2})y^2-4^{-1-b}\left(-\left(-1+(\mathrm{v1}-\mathrm{v2})^2\right)y^2\right)^{1+b}$$

$$x\left(b-\frac{2b^2(-1+\mathrm{v1}+\mathrm{v2})}{-1+(\mathrm{v1}-\mathrm{v2})^2}-x^b\right)$$

$$\mathbf{eqy2} = \tfrac{1}{4}b\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)+\tfrac{1}{2}b^2(-1+\mathrm{v1}+\mathrm{v2})-4^{-1-b}\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)^{1+b}y\hat{}(2b)$$

$$\mathbf{eqx2} = b-\frac{2b^2(1-\mathrm{v1}-\mathrm{v2})}{1-(\mathrm{v1}-\mathrm{v2})^2}-x^b$$

$$\tfrac{1}{4}b\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)+\tfrac{1}{2}b^2(-1+\mathrm{v1}+\mathrm{v2})-4^{-1-b}\left(1-(\mathrm{v1}-\mathrm{v2})^2\right)^{1+b}y^{2b}$$

$b-\frac{2b^2(1-\text{v1}-\text{v2})}{1-(\text{v1}-\text{v2})^2}-x^b$

{eqx0, eqy0}/.#&/@{test, test2}

{eqx2x, eqy2y^2}/.#&/@{test, test2}

$\{\{0.821415, 0.0479631\}, \{-8.23449+0.i, 0.0808255\ -0.000121906i\}\}$

$\{\{0.821415, 0.0479631\}, \{-2.20011, 0.0808255\ -0.000121906i\}\}$

solx = Solve[eqx2 == 0, x][[1]]//FullSimplify

Appendix 2: The Error in the Jackson-Moulin (1992) Theorem for the Two-Stage Mechanism

The Theorem in Jackson & Moulin (1992) says, among other things, that under the two-stage mechanism

"At every undominated Nash equilibrium, ... the second stage bids reveal the agents' true benefits."

I think that's wrong, and there are undominated Nash equilibria in which agents lie in the second stage. Moreover, the decider cannot "back out" the true values from the messages, because many pairs of true values can generate the same pair of messages.

Take, for example, $v_1 = v_2 = 80$, with a cost of 100 for the project and proportional taxation, i.e. $t_1 = \frac{m_1}{m_1+m_2}100$. One equilibrium is for both players to report $m1 = m2 = 80$ so they'd each pay 50 of the 100 cost. But the equilibrium could be m1=70, m2=90, so player 2 would end up paying a bigger share. That's a superior outcome for player 1, so it's not in weakly dominated strategies for him. And it's a best response for player 2, because if he reports his true value, the project won't get built, but reporting 90 he still will get positive payoff even tho his tax is a bit higher. Thus, for player 2 to report 90 instead of his true value of 80 has a strictly higher payoff for him in that equilibrium than reporting 80. The message pair $(m_1 = 70, m_2 = 90)$ is similarly an equilibrium for $(v_1 = 90, v_2 = 90)$ and many other value pairs.

I think I've found the flaw in the proof of the Theorem right at the start, where it says (moving now to the 1992 paper's notation),

"Thus the truthful report $\beta_i = b_i$ is a dominating strategy (and the unique undominated strategy)."

β_i is the "losing" player's report of his value in the second stage, and b_i is his true value. A few lines above, the proof says

"Set $\alpha = v - \beta_{N/i}$",

which means to set alpha equal to the first-stage winning report of the sum of the two true values minus the winning player's equilibrium second-stage report of his own value (if there are just two players). That's fine. But what if the equilibrium has the winning player's second-stage report of his own value a false report? Then the losing player will also want to make a false report in the second stage.

The proof goes on to talk about the winning player's report in the second stage. It says,

"If $\beta_{N/i*} \leq v$, then send $\beta^* = v - \beta_{N/i*}$ and ...",

which means that if the equilibrium losing player's report is less than the combined value report, the winning player sends a message equal to the combined value report minus the losing player's message. If the losing player was truthful and sent $\beta_i = b_i$, the winning player will send $\beta_i^* = b_i^*$, but if he expects the losing player to be untruthful, so will he. So that allows for multiple equilibria.

Note that in these other equilibria I am proposing, no player ends up with a negative payoff, and in most of them his payoff is strictly positive. It's just that lots of sharing taxes are compatible with positive payoffs for both players.

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