Back to Bargaining Basics: A Breakdown Model for Splitting a Pie

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Abstract

Nash (1950) and Rubinstein (1982) give two different justifications for a 50-50 split of surplus to be the outcome of bargaining with two players. Nash's axioms extend to n players, but the search for a satisfactory n-player non-cooperative game theory model of bargaining has been fruitless. I offer a simple static model that reaches a 50-50 split (or $1/n$) as the unique equilibrium. Each player chooses a "toughness level" simultaneously, but greater toughness always generates a risk of breakdown. Introducing asymmetry, a player who is more risk averse gets a smaller share in equilibrium. "Bargaining strength" can also be parameterized to yield an asymmetric split. The model can be expanded to resemble Rubinstein (1982) by making breakdown mere delay, but with an exact 50-50 split if the player's discount rates are equal. The model only needs minimal assumptions on breakdown probability and pie division as functions of toughness. Its intuition is simple: whoever has a bigger share loses more from breakdown and hence has less incentive to be tough.

Notation

Players 1 and 2 are splitting a pie of size 1.

Each simultaneously chooses a toughness level x_i in $[0, \infty)$.

With probability $p(x_1, x_2)$, bargaining fails and each ends up with a payoff of zero.

Otherwise, player 1 receives $\pi(x_1, x_2)$ and Player 2 receives $1 - \pi(x_1, x_2)$.

Assume that player i prefers a lower value of x_i to a higher one if the payoffs are equal.

Example 1: The Basics.

Let $p(x_1, x_2) = \frac{x_1+x_2}{12}$ for $x_1 + x_2 \leq 12$, $p = 1$ otherwise.

Let $\pi(x_1, x_2) = \frac{x_1}{x_1+x_2}$.

Equilibrium: $x_1 = x_2 = 3$. Player 1's share is $\pi = 50\%$. The probability of breakdown is $p = .5$.

Each player's expected payoff is .25.

The assumption that players prefer lower levels of x if payoffs are equal rules out $x_1 = 8, x_2 = 10$ as an equilibrium.

The General Model

Effort costs $c(x_i)$ for player i, with $c \geq 0$, $c' \geq 0, c'' \geq 0$. Assume that player i prefers a lower value of x_i to a higher one if the payoffs are equal, even if $c = 0$.

Utility is quasilinear: $u_1(\pi) - c(x_1)$ and $u_2(\pi) - c(x_2)$ with $u_1' > 0, u_1'' \leq 0$ and $u_2' < 0, u_2'' \leq 0$.

Breakdown: $p_1 > 0$ $p_2 > 0$, $p_{11} \geq 0$, $p_{22} \geq 0$, and $p_{12} \geq 0$ for all values of x_1, x_2 such that $p < 1$, and $p_1 = p_2 = 0$ for greater values. $p(a, b) = p(b, a)$.

Player 1's share: $\pi \in [0, 1]$, $\pi_1 > 0$, $\pi_{11} \leq 0$, and $\pi_{12} \geq 0$. $\pi(a, b) = 1 - \pi(b, a)$.

The General Model's Equilibrium Has a 50-50 Split

Proposition 1. The general model has a unique Nash equilibrium, and that equilibrium is in pure strategies with a 50-50 split of the surplus: $x_1^* = x_2^*$ and $\pi(x_1^*, x_2^*) = .5$.

The intuition: Suppose we had an asymmetric equilibrium and the player with the bigger share doesn't want to risk being any tougher for fear of blowing up the pie.

The player with the smaller share has less at risk from being a little tougher and increasing the chance of exploding the pie– so he'll do it, to increase his share.

That's why the concavity assumptions can be very weak: the player with the bigger share will have the bigger marginal cost of being tougher even if the pie-sharing and breakdown functions are linear.

Getting Rid of Convexity Altogether

Even the weak convexity assumptions could be weakened (and the infinitesimal fixed cost assumption dropped) if we are willing to impose a maximum toughness level at which breakdown has probability less than 1.

Suppose $x_i \in [0, \overline{x}]$ with $p(\overline{x}, \overline{x}) < 1$, but we do not require $\frac{\partial p^2}{\partial x_i^2} \geq 0$ but that we still require $\frac{\partial p^2}{\partial x_i \partial x_j} \leq 0$. The marginal cost of toughness might fall in toughness now, so the equilibrium might either be interior and with equal toughness or an upper corner solution at $x_1 = x_2 = \overline{x}$, a 50-50 split in either case.

We still need $\frac{\partial p^2}{\partial x_i \partial x_j} \geq 0$ to prevent one player's marginal cost from rising in the other player's marginal cost, strategic substitutes, in which case the equilibrium might be asymmetric.

Example 2: A Vanishingly Small Probability of Breakdown.

Keep $\pi(x_1, x_2) = \frac{x_1}{x_1+x_2}$.

Let the breakdown probability be $p(x_1, x_2) = \frac{(x_1+x_2)^k}{12k^2}$ for k to be chosen.

$$x^* = .5(\frac{12k}{k+1})^{1/k} \tag{1}$$

The probability of breakdown is then $p = \frac{1}{k+1}$, which goes to zero as k increases.

In equilibrium with large k, $x_1 = x_2$ and p are close to 0, the shares are equal, and each player's expected payoff is close to .5.

N Players.

Assume $p(x_1, x_2, \ldots, x_N) = \frac{\sum_{i=1}^N x_i}{12}$

Player i's payoff function is

$$Payoff(i) = \left(1 - \frac{\sum_{i=1}^N x_i}{12}\right) \frac{x_i}{\sum_{i=1}^N x_i}$$

In equilibrium, $\pi = 1/N$. The same marginal risk intuition holds as for $N = 2$.

The equilibrium probability of breakdown is $p(x, \ldots, x) = \frac{(N-1)}{N}$. (The handout is wrong on that.)

Thus, for $N = 2, p = .5$, $N = 4, p = .75$, $N = 10, p = .9$.

The breakdown probabilities rise because each player's share gets small as N increases, so he is more willing to risk exploding the entire pie in return for the same increase in his share. The externality from being tougher is imposed on more and more other players.

Example 3: Unequal Bargaining Power.

A way to put this in is to use a specification in which one of the players can increase his toughness and get a bigger share but not increase the breakdown probability as much as the other player. He is somehow better at being tough without wrecking the deal. He is better at both parts of The Art of the Deal: increasing his own share, but increasing the expected pie size too.

Let the probability of breakdown be

$$p(x_1, x_2) = Min\{e^{(1-\theta)\beta x_1 + \theta\beta x_2} - 1, 1\}$$

where $\theta \in [0, 1]$ is player 1's bargaining power and $\beta > 0$ is a parameter for breakdown risk.

Let player 1's share of the pie be $\pi(x_1, x_2) = .5 + (x_1 - x_2)$.

In equilibrium, player 1's share is θ and player 2's is $1 - \theta$.

Risk Aversion

It is easy to put risk aversion into the model, though I have not found a way to make the parameters translate neatly into payoffs.

Note that this is genuine risk aversion, not just curvature of utility functions, because there is genuine risk, unlike in most bargaining models. In Rubinstein (1982), for example, there is always agreement in the first period.

Here, in equilibrium there is a real risk of breakdown. Some players will fear that more than others, and they will hold back more on being tough.

Proposition 2: *If player 1 is more risk averse than player 2, his share is smaller in equilibrium.*

CARA Risk Aversion Example: Toughness, (x_1/x_2) and Player 1's Share π As Risk Aversion (α_1, α_2) Changes (Rounded)

α_1 \ α_2	.01	.50	1.00	2.00	5.00
.01	3.00/3.00 **50**				
.50	2.82/2.99 **49**	2.81/2.81 **50**			
1.00	2.64/2.98 **47**	2.63/2.79 **49**	2.61/2.61 **50**		
2.00	2.33/2.95 **44**	2.31/2.75 **45**	2.28/2.56 **47**	2.21, 2.21 **50**	
5.00	1.64/2.79 **23**	1.60/2.57 **38**	1.55/2.35 **40**	1.43/1.95 **42**	1.10/1.10 **50**

Multiperiod Bargaining in the Style of Rubinstein (1982)

We can turn this into a model of bargaining delay by saying that if breakdown occurs, the players lose a little by the delay because of discounting but can try again.

Proposition 3. *In the multiperiod bargaining game, a player's toughness and equilibrium share falls in his discount rate.*

This will be a stationary model, like Rubinstein's, but with some probability of bargaining lasting 1 period, 2 periods, 3 periods, etc.

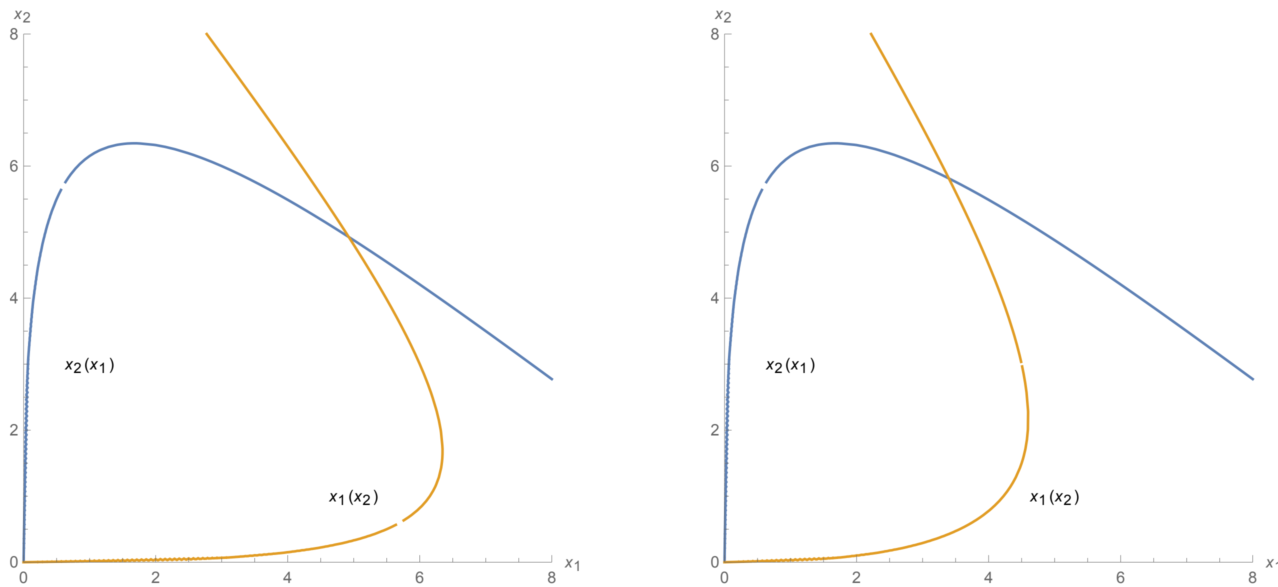
If the two players have the same discount rates, $x_1 = x_2$ in equilibrium. As r approaches zero, the probability of breakdown in any given period approaches 1, though the expected payoff tends towards .5. The expected payoff is lower with big r: it is .25 with $r = \infty$ (which is Example 1).

Toughness (x_1/x_2) and Player 1's Share (π) As Impatience (r_1, r_2) Increases (rounded)

		r_2					
		.001	.010	.050	.100	.500	2.000
	.001	5.5/5.5 **50**					
	.010	2.9/7.4 **28**	5.5/5.5 **50**				
r_1	.050	2.5/7.6 **25**	3.5/7.0 **33**	4.9/4.9 **50**			
	.100	1.4/9.6 **13**	2.9/7.4 **25**	4.2/5.4 **44**	4.6/4.6 **50**		
	.500	1.1/9.8 **10**	2.1/7.8 **21**	3.0/6.0 **33**	3.3/5.2 **39**	3.8/3.8 **50**	
	2.000	1.0/9.9 **9**	1.9/7.9 **19**	2.6/6.2 **30**	2.8/5.4 **34**	3.2/3.9 **45**	3.3/3.3 **50**

Reaction Curves for Toughness x_1 and x_2 in the Repeated Game

(a) $r_1 = r_2 = .05$ **(b)** $r_1 = .25$, $r_2 = .05$



In the relevant range, near where they cross, they are downward sloping. Not only does this make the equilibrium unique, it also tells us that the indirect effect of an increase in r_1 goes in the same direction as the direct effect. If r_1 rises, that reduces x_1, which increases x_2, which has the indirect effect of reducing x_2 further, and so the indirect effects continue ad infinitum.

Outside Options

Shaked and Sutton showed that if one of the players in Rubinstein (1982) has the outside option to leave and get .4, or gets that if bargaining breaks down, that has ZERO effect on his equilibrium share from the bargaining. The reason is that he can get close to .5 from bargaining, so his threat to take the outside option is not credible.

If the player had an outside option of .6, that WOULD matter, but it would just mean that his share would be .6 instead of about .5. It puts a bound on how little he can get, but he can't leverage that in the bargaining process to get, say, a share of .8, halfway between .6 and 1.

Example 6: Player 1 Has an Outside Option of z. As in Example 1, let the breakdown probability be $p(x_1, x_2) = \frac{x_1+x_2}{12}$ and player 1's share be $\pi(x_1, x_2) = \frac{x_1}{x_1+x_2}$. Player 1 has an outside option of z, a payoff he receives if bargaining breaks down. Player 2's reaction curve is

$$x_2 = \sqrt{12}\sqrt{x_1} - x_1, \tag{2}$$

Player 1's share is

$$\frac{1}{2 - z}$$

.

Proposition 4. *If player 1's outside option is* z*, his equilibrium bargaining share will be strictly greater than* $.5$ *and no greater than* $.5 + .5z$*, attaining the upper bound only if* p *and* π *are both linear.*

Why is it that the outside option does not operate the same way as a different threat point?.

If $z = .2$, then player 1's share would be .6 if the social surplus from reaching a bargain were split 50-50, but as an outside option, it yields him a smaller share: $\frac{1}{2-z} = 5/9$.

The outside option also helps player 1 if it is bigger than .5. If $z = .8$ it would be .83 (approximately). If the threat point were .8 instead, then player 1's share would be bigger: .9. Player 1's outside option improves his bargaining position, but not as much as if he started with .8 and bargaining occurred over the .2 difference between .8 and 1.

The reason is that the outside option is not a base level, but a replacement, and the toughness necessary for player 1 just to obtain the equivalent of his outside option can itself induce breakdown.