Splitting a Pie: Mixed Strategies in Bargaining under Complete Information

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Abstract

We characterize the mixed-strategy equilibria for the bargaining game in which two players simultaneously bid for a share of a pie and receive shares proportional to their bids, or zero if the bids sum to more than 100%. Of particular interest is the symmetric equilibrium in which each player's support is a single interval. This consists of a convex increasing density $f_1(p)$ on $[a, 1-a]$ and an atom of probability at a, and is unique for given $a \in (0, .5)$. The two outcomes with highest probability are breakdown and a 50-50 split. We use the same approach to characterize all symmetric and asymmetric equilibria (such as "hawk-dove") that mix over a finite set of bids, and for general sharing rules. We extend Malueg's 2010 existence proof with any "balanced" compact set $A \in (0, 1)$ as bid supports to uniqueness. We provide explicit formula for any equilibrium density in terms of finite discrete approximants, and an exact power series density for the case when the support is an interval.

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PART I: The Econ Part

The Situation

Rep. Pelosi and Presidet Biden each want 100% of the publicity for passing the stimulus bill. They meet to decide on what to say about each other. If they can't agree and end up being mean to each other, they both end up with a net 0% good publicity.

Thus, if they are both hard-nosed and go in asking for 80%, they both end up with payoffs of 0.

If Pelosi asks for 80% and Biden asks for 20%, they will agree, and that is what will happen.

What if Pelosi goes into the negotiation with a "soft" strategy of asking for 20%, and so does Biden?

What Happens If Both Players Adopt Soft Strategies?

What if Pelosi goes into the negotiation with a “soft” strategy of asking for 20%, and so does Biden?

I think if both go in equally soft, they will each end up with 50%. They won’t just agree to 20% each and give 60% to the Republicans.

I think a good prediction is that if Pelosi decides on a soft strategy of 20% and Biden decides on a moderate strategy of 40%, then Pelosi will come out with 1/3 and Biden with 2/3 of the credit.

The Model

Players 1 and 2 simultaneously choose bids p and q in the interval [0,1].

If their bids add up to more than 100%, then we say *breakdown* occurs, and both get zero.

Otherwise, Player 1 gets share $p/(p+q)$ as his payoff and Player 2 gets $q/(p+q)$.

Pure Strategy Equilibria

Splitting a Pie has a continuum of pure strategy Nash equilibria: every pair of bids with $p + q = 1$.

There are also many mixed-strategy equilibria, some with bidding over a finite set of points, some with bidding over a continuum.

Hawk-Dove Equilibria for Splitting a Pie

The simplest mixed strategy for Splitting a Pie has the two players each mix over the same two bids.

This is a hawk-dove equilibrium, mathematically the same as the well-known biological model of birds deciding whether to pursue aggressive or timid strategies.

One of the Hawk-Dove Equilibria: Each player chooses .30 with probability .6 and .70 with probability .4 The expected payoff is .42. With probability .16, they both choose .70 and the pie explodes.

Finding the Hawk-Dove Equilibria for Splitting a Pie

Each player chooses a with probability θ and $1 - a$ with probability $1 - \theta$ for $a \leq .5$. The two bids must add up to 1 because otherwise it would be a profitable deviation for one player to choose a bigger number for his lower bid, increasing his share without any greater likelihood of breakdown.

The mixing probabilities must make each action's expected payoff the same in equilibrium, so

$$\pi(a) = \theta(.5) + (1-\theta)a = \pi(1-a) = \theta(1-a) + (1-\theta)(0), \tag{1}$$

which solves to $\theta = 2a$ and $\pi_0 = 2a(1-a)$.

The players share the pie equally in equilibrium with probability a^2 and bargaining breaks down with probability $(1-a)^2$. Note that there are a continuum of equilibria since any value of a in (0,1) can support an equilbrium, and they can be pareto-ranked, with higher payoffs if a is closer to .5.

Observe that $\theta < 1/2$ for $a < 1/4$, so the players actually choose the higher of their two bids (hawk) with the highest probability, which is one difference from the standard hawk-dove game in biology.

Asymmetric Hawk-Dove Equilibria

We might have Player 1 choosing .1 and .4 with probabilities of about .28 and .72, for a payoff of about .11.

Player 2 choosing .6 and .9 with probabilities of about .93 and .07, for a payoff of about .84.

DEFINITION. The player's bid supports are *balanced* if and only if element s being in Player 1's support A implies that $1 - s$ is in Player 2's support B, and vice versa.

An example of a symmetric balanced support for an interval equilibrium is both players mixing over $[.2, .8]$.

An example of an asymmetric balanced support is Player 1 mixing over $[.2, .4]$ and Player 2 over $[.6, .8]$.

The Nash Demand Game: The "Take what you bid" Sharing Rule

In this game, the players bid p and q, and they receive p and q as payoffs if $p + q \leq 1$, but zero if $p + q > 1$ (Nash [1953]).

In the context of bargaining, if the players reach agreement but they have both bid low, they agree to abandon the remainder of the pie. This is not suitable for bargaining, but it provides a good model for other situations, e. g. two hunters independently choose how aggressively to pursue a deer that neither might catch if they are too timid or too aggressive.

The Nash Demand Game: Equilibrium

Example of a mixed-strategy equilibrium for the Nash Demand Game: each player bids a with probability K and then mixes using density f over the interval $[a, 1-a]$.

Player 2 can guarantee a payoff of a by bidding a, since $p + a \leq 1$ for any bid p Player 1 might play. Player 2 will have a payoff of $K(1-a)$ from bidding $(1-a)$, since $p + (1-a) > 1$ for any bid p Player 1 might play except for $p = a$, which has probability K. Since Player 2 is only willing to mix between bids if they have equal expected payoffs, this implies $\pi(a) = a = \pi(1-a) = K(1-a)$ and we can conclude that $K = \frac{a}{1-a}$.

The Nash Demand Game: Finding the Equilibrium Density

For bids between a and $1-a$, Player 2's expected payoff is

$$\pi(q) = Kq + \int_a^{1-q} qf(p)dp, \tag{2}$$

which we can rewrite using F as the cumulative distribution and our knowledge that $K = \frac{a}{1-a}$, and combine with the requirement that $\pi(q) = \pi(a) = a$ to yield

$$\pi(q) = \left(\frac{a}{1-a}\right) q + qF(1-q) = a. \tag{3}$$

Using the change-of-variables $p = 1-q$, this becomes $\left(\frac{a}{1-a}\right)(1-p) + F(p)(1-p) = a$, which solves to $F(p) = \frac{a}{1-p} - \frac{a}{1-a}$, which can be differentiated to yield the equilibrium mixing density, $f(p) = \frac{a}{(1-p)^2}$.

What's Easy To Solve about the Nash Demand Game

The Nash Demand Game is easy to solve because each player's payoff function depends on the other player's bid only if breakdown occurs. If the bids add up to less than one, a player's payoff is entirely independent of what the other player does. This is what allows us to move smoothly from equation equation (2)'s $\int_a^{1-q} qf(p)dp$ to equation (3)'s $qF(1-q)$. If Player 2's share of the pie depended on Player 1's bid via some function $v_2(q,p)$ instead of just being his own bid q, equation (2) would have the expression $\int_a^{1-q} v_2(q,p)f(p)dp$ and it would no longer be straightforward to extract $f(p)$ from the integral. The purpose of the present paper is to discover how to do this, with particular attention to the proportional-sharing case of $v_2 = \frac{p}{p+q}$.

Interval Equilibria

Each player bids over a single interval– e.g. both mix over [.2, .8], or Player 1 mixes over [.2, .4] and Player 2 over [.6, .8].

Proposition 1: Any equilibrium in which each player mixes over an interval

(a) consists of probability atom K_1 at a and density $f_1(p)$ on $[a, b]$ for Player 1

(b) and of probability atom K_2 at $1 - b$ and density $f_2(q)$ on $[1 - b, 1 - a]$ for Player 2.

(c) The densities are positive throughout, and if their derivatives exist, the densities start strictly positive with $f_1(a) = \left(\frac{a}{1-a}\right) K_1$ and $f_2(1-b) = \left(\frac{1-b}{b}\right) K_2$

(d) and increase convexly

(e) with positive derivatives of every order.

Four Examples

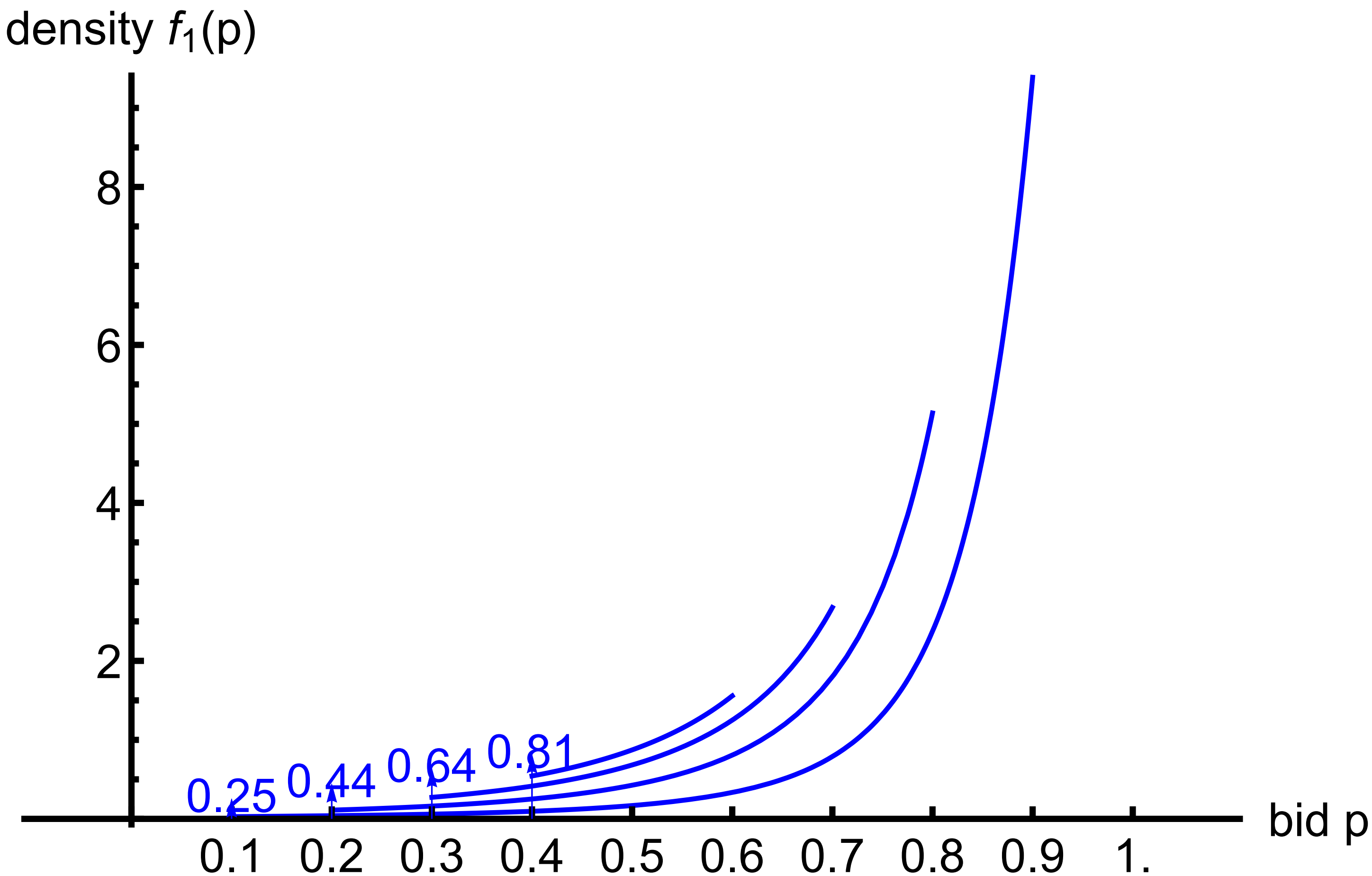

FIGURE 1. THE SYMMETRIC-EQUILIBRIUM MIXING DENSITY $f_1(p)$ AND ATOM K_1 FOR $a = .10, .20, .30, .40$

The Expected Payoff Function

The fundamental equation that characterizes the equilibrium then says that Player 1's f_1 and K_1 must be such that Player 2's expected payoff from any q in his support equals that from playing $q = 1 - a$:

$$\pi_2(q) = K_1\left(\frac{q}{q+a}\right) + \int_a^{1-q}\left(\frac{q}{q+p}\right) f_1(p)dp = K_1(1-a) \quad (\textit{the crucial equatio} \tag{4}$$

For every q in Player 2's support.

$$\frac{d\pi_2(q)}{dq} = \frac{aK_1}{(q+a)^2} + \int_a^{1-q}\left(\frac{p}{(p+q)^2}\right) f_1(p)dp - qf_1(1-q) = 0. \tag{5}$$

The Derivative Expression

For every q in Player 2's support.

$$\frac{d\pi_2(q)}{dq} = \frac{aK_1}{(q+a)^2} + \int_a^{1-q} \left(\frac{p}{(p+q)^2} \right) f_1(p)dp - q f_1(1-q) = 0.$$

The first two terms are the advantages of using a higher bid q. The first term represents the extra payoff from Player 2's share increasing when Player 1 chooses the atom bid of $1-b$, which never causes breakdown. The second term represents Player 2's extra payoff as the result of his share increasing when he raises q if Player 1 bids between $1-b$ and q so raising q does not cause breakdown. The third term is Player 1's disadvantage from bidding higher. It represents the increase in the probability of breakdown, and the resulting loss of the share q. The crucial level of p is $p = 1-q$, since that is at the edge of breakdown. The probability of Player 1 choosing that bid is $f_1(1-q)$, and since the two player's bids add up to 1 at that bid, Player 2's lost pie share is q if that happens.

More on the Derivative Expression

$$\frac{d\pi_2(q)}{dq} = \frac{aK_1}{(q+a)^2} + \int_a^{1-q} \left(\frac{p}{(p+q)^2} \right) f_1(p)dp - qf_1(1-q) = 0.$$

To see why the equilibrium density is increasing, think of how the marginal benefits and cost change for Player 2 as q rises. Imagine if the f_1 density were uniform instead of increasing. Both of the advantage terms would be larger for small q, because the effect on the share of increasing the bid is bigger from a lower base. Also, the second advantage term, the integral, is bigger for small q because it includes a larger probability range $[a, 1-q]$ over which the shares add up to less than one and breakdown is avoided. If we start with a large q, near the top of the support, then increasing$q$ yields a greater share, but obtained with small probability; most of the expected payoff from a high q is coming from the atom of probability with which Player 1 bids $1-b$. This gives one reason why $f_1(p)$ has to be rising— so that a small q's greater range of p's that don't cause breakdown is offset by a smaller density over that range.

Still More on the Derivative Expression

$$\frac{d\pi_2(q)}{dq} = \frac{aK_1}{(q+a)^2} + \int_a^{1-q} \left(\frac{p}{(p+q)^2} \right) f_1(p)dp - qf_1(1-q) = 0.$$

The second reason why f_1 must increase is the third-term marginal cost of increasing q and increasing the probability of breakdown, a marginal cost of $qf_1(1-q)$. The density takes the argument $1-q$ because $p = 1-q$ is the threshold for breakdown, and hence $f_1(1-q)$ is the rate of increase of the probability of breakdown as q increases. If f_1 were uniform, the marginal cost would increase with q, because a constant rate of increase in the probability of breakdown causes Player 2 more harm when his share is bigger. Thus, f_1 must be rising, so it is smaller for the critical $1-q$ threshold when q is large.

Asymmetric Equilibria

There also exist asymmetric interval equilibria, as shown in Figure 2 (again, using our explicit formula from later in the paper). Player 1 bids an atom of .94 at $a = .2$ and mixes over $[.2, .4]$,while Player 2 bids an atom of .33 at $a = .6$ and mixes over $[.6, .8]$.

The supports are balanced because .2+.8 and .4+.6 equal 1. Player 2 always bids more and will have a higher expected payoff, and the modal outcome will be a share of .25 for Player 1 and .75 for Player 2, as a result of bids of $p = .2$ and $q = .6$.

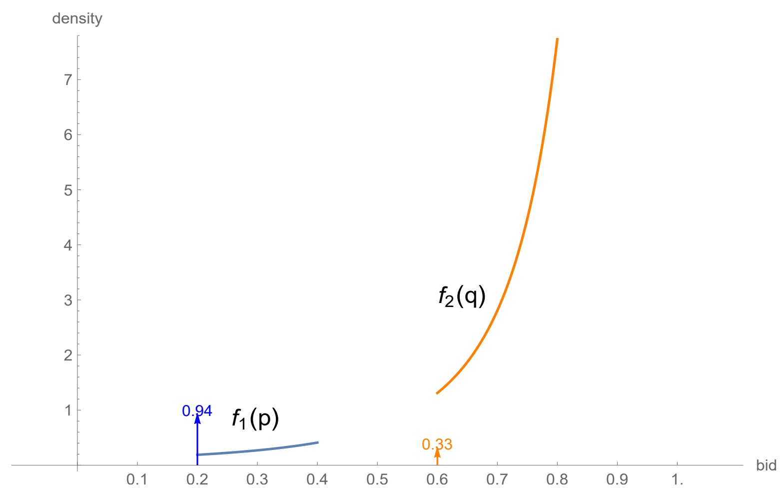

FIGURE 2. THE ASYMMETRIC-EQUILIBRIUM MIXING DENSITIES $f_1(p)$ OVER [.2,.4] AND $f_2(q)$ OVER [.6,.8]

PART II: The Math Part

Proposition 1. *Splitting a Pie with the proportional sharing rule has unique interval equilibrium mixing strategies among Borel measures for each pair of supports. Players 1 and 2 use increasing, convex (with all derivatives positive), real-analytic densities* $f_1(p)$ *on* $[a,b]$ *and* $f_2(q)$ *on* $[1-b, 1-a]$ *with* $0 < a < b < 1$ *and atoms of probability* K_1 *at* a *and* K_2 *at* $1-b$ *such that:*

$$f_1(p) = aK_1 \sum_{i\geq 0} \frac{m^{(i)}(a)}{i!}(p-a)^i, \quad K_1 = \frac{1}{1 + a\sum_{i\geq 0} \frac{m^{(i)}(a)}{(i+1)!}(b-a)^{i+1}}, \tag{6}$$

and

$$f_2(q) = (1-b)K_2 \sum_{i\geq 0} \frac{m^{(i)}(1-b)}{i!}(q-(1-b))^i, \quad K_2 = \frac{1}{1 + (1-b)\sum_{i\geq 0} \frac{m^{(i)}(1-b)}{(i+1)!}(} \tag{7}$$

with $m^{(i)}(\cdot)$ *for both densities recursively expressed as:*

$$m^{(i)}(x) = \frac{(i+1)!}{(1-x)} + \sum_{j=0}^{i-1} \frac{((j+1)(i-1-j)! + (i-j)!x)}{1-x} m^{(j)}(x). \tag{8}$$

If we expand out the first few terms f_1 evaluates to:

$$f_1(p) = aK_1\left(\frac{1}{(1-a)} + \left(\frac{3-a}{(1-a)^2}\right)(p-a) + \left(\frac{13-10a+3a^2}{2(1-a)^3}\right)(p-a)^2 + \left(\frac{71-89a+55a^2-13a^3}{6(1-a)^4}\right)(p-a)^3 + \sum_{i\geq 4}\left(\frac{m^{(i)}(a)}{i!}\right)(p-a)^i\right). \tag{9}$$

Existence and Uniqueness Generally: Compact Set Supports and General Sharing Functions

We will make the following assumptions on our sharing rules.

Assumptions ($\star$). *For players $i = 1$ and 2 bidding p and q the sharing rules v_i satisfy the following properties:*

(1) $v_i : [0,1] \times [0,1] \to [0,\infty)$ *are Borel measurable and continuous on* $(0,1] \times (0,1]$.

(2) $v_1(p,0) > v_1(0,0)$ *and* $v_2(0,q) > v_2(0,0)$ *for all* $p, q \in (0,1]$.

(3) $v_1(p,q)$ *is nondecreasing in* p *and nonincreasing in* q*, and strictly increasing in* p *if* $q > 0$.

(4) $v_2(p,q)$ *is nonincreasing in* p *and nondecreasing in* q*, and strictly increasing in* q *if* $p > 0$.

Proposition (Malueg)**.** *Let A and B be balanced nonempty finite subsets of the open interval (0, 1): if and only if element s is in A then $1 - s$ is in B. There exists a unique equilibrium in which Player 1 mixes over A and Player 2 mixes over B.*

Proposition 2. *Given the general assumptions ($\star$) on the players' payoffs, for any compact set $A \subset (0,1)$, there exists a unique equilibrium among positive measures with Player 1's support set exactly A. Player 2's support will be exactly $1 - A$.*

Proposition 3 is more radical than it might seem. The term "any compact set $A \in (0,1)$" covers more than just finite sets of points and intervals. It also includes, for instance, the Cantor set, which has measure zero but an uncountable number of points.

Proposition 3. *Given the general assumptions* $(\star)$ *on the players' payoffs, assume further that a player's payoff is 0 from the action 0 (the same payoff as for breakdown), so* $v_1(0,q) = 0$ *for all* $q > 0$ *and* $v_2(p,0) = 0$ *for all* $p > 0$*. All equilibrium measures with supports A and B for players 1 and 2 that include an endpoint* $\{0\}$ *or* $\{1\}$ *are classified as follows.*

(1) *If* $0 \in A$ *then Player 1 has an atom at 0 and Player 2 always bids 1:* $\mu_1(0) > 0$ *and* $\mu_2 = \delta_1$*. Player 1's expected payoff is* $\pi_1 = 0$ *and Player 2's expected payoff can be any value* $\pi_2 \in (0, v_2(0,1)]$*.*

(2) *If* $1 \in A$ *and* $1 \notin B$ *then* $\mu_1 = \delta_1$ *and* $\mu_2(0) > 0$*. Player 2's expected payoff is* $\pi_2 = 0$ *and provided* v_1 *is continuous, Player 1's expected payoff can be any value* $\pi_1 \in (0, v_1(1,0)]$*.*

(3) *If* $1 \in A$ *and* $1 \in B$ *then either* $\mu_1 = \mu_2 = \delta_1$*, or* $\mu_1 = \delta_1$ *and* $\mu_2(0) > 0$*, or* $\mu_2 = \delta_1$ *and* $\mu_1(0) > 0$*. For any pair of values* $(\pi_1, \pi_2) \in \{0\} \times [0, v_2(0,1)) \bigcup [0, v_1(1,0)) \times \{0\}$*, there exists an equilibrium with expected payoffs* (π_1, π_2)*.*

All remaining cases (0 *or* 1 *in* B*) are found by switching* A *and* B *and the roles of Player 1 and Player 2. Moreover, any choice of supports* A, B *satisfying the above conditions generate some equilibrium.*

Proposition 4. *For any finite support* $A = \{p_1, p_2, \ldots, p_n\}$ *with* $0 < p_1 < p_2 < \cdots < p_n < 1$*, provided* $v(p_j, 1-p_j) \neq 0$ *for* $j = 1, \ldots, n$*, the unique equilibrium is the probability measure* $\mu = \sum_{j=1}^{n} w_j \delta_{p_j}$*, where the weights* $w_j > 0$ *are given explicitly by:* $w_j = \pi_1 \omega_j$*, where*

$$\omega_j = \frac{1}{v(p_j, 1-p_j)} + \sum_{r=1}^{j-1} (-1)^r \sum_{1 \le i_1 < i_2 < \cdots < i_r < j} \frac{v(p_{i_1}, 1-p_{i_2}) \cdots v(p_{i_{r-1}}, 1-}{v(p_{i_1}, 1-p_{i_1}) v(p_{i_2}, 1-p_{i_2}) \cdots v(}$$

and the payoff is $\pi_1 = \dfrac{1}{\sum_j \omega_j}$.

Since $v(p_j, 1-p_j) = 1-p_j > 0$ provided $p_j \neq 1$, for the proportional sharing rule $v(p,q) = \frac{q}{q+p}$, the weights $w_i = \pi_1 \omega_j$ simplify to

$$\omega_j = \frac{1}{1-p_j} + \sum_{r=1}^{j-1} (-1)^r \sum_{1 \le i_1 < i_2 < \cdots < i_r < j} \frac{1}{(1-p_{i_1})(1+p_{i_1}-p_{i_2})(1+p_{i_2}-p_{i_3}) \cdots (}$$

and the payoff is $\pi_1 = \left(\sum_i \omega_i\right)^{-1}$. The first three ω_i are:

$$\omega_1 = \frac{1}{1 - p_1},$$

$$\omega_2 = \frac{1}{1 - p_2} - \frac{1}{(1 - p_1)(1 - p_2 + p_1)},$$

$$\omega_3 = \frac{1}{1 - p_3} - \frac{1}{(1 - p_1)(1 - p_3 + p_1)} - \frac{1}{(1 - p_2)(1 - p_3 + p_2)} + \frac{1}{(1 - p_1)(1 - p_2 +$$

Let us now think what happens as the number of discrete bids in the support becomes large. . It will turn out that as we increase the number of points in a discrete support over that interval, the discrete-mixing equilibrium will indeed approach the continuum-mixing equilibrium, but if the discrete support is over just a few points, the shape can be very different.

Suppose there are three bids used for mixing. The lowest must be chosen so it and the highest add up to one, to satisfy the balancing condition. The middle bid must equal .5 to satisfy the balancing condition, since it will have no matching bid with which to sum to one.

For $a = .3$ the probabilities are approximately 0.61 at 0.3, 0.10 at 0.5, 0.29 at 0.7. This is the same pattern we have found for our continuous-support mixed-strategy equilibrium—a high atom at the minimum, then low density, but rising monotonically to the maximum bid.

This turns out not to be always true for multi-bid mixing equilibria, as Figure 3 shows. Monotonicity of the ω_i for $i > 1$ fails for the simple example of support $\left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right\}$. Here, the bids in the support are unevenly spaced. In a loose sense the third bid, $\frac{2}{3}$, has more space to cover than the fourth bid, and so takes a bigger probability.

But perhaps if the number of the points in the support increased enough, the discrete equilibria would converge to the continuous equilibrium. This conjecture turns out to be true, once we formalize "increased enough" and "converge" to make it precise. In our last proposition we will show that as the finite support becomes more dense, even if it is unevenly spaced, the equilibria will converge to the unique continuous-interval equilibrium we found in Proposition 2 and thus will approach an atom followed by a convexly increasing density.

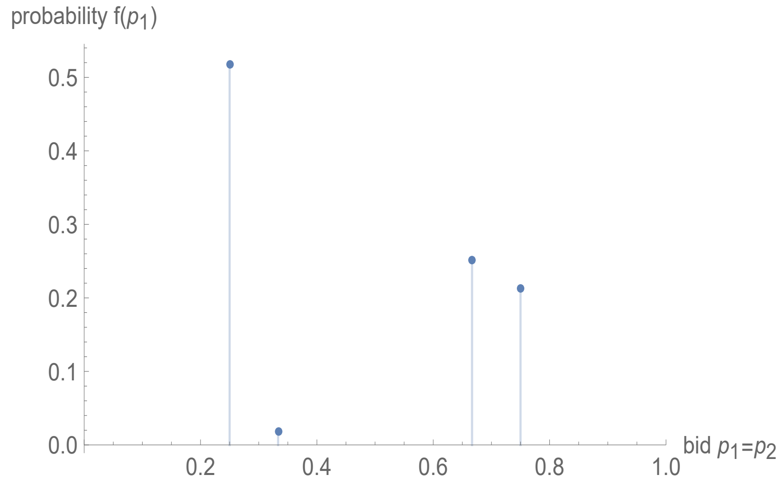

FIGURE 3. A NONMONOTONIC SYMMETRIC EQUILIBRIUM $f_1(p)$ ARISING FROM AN UNEVENLY SPACED SUPPORT $\{\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\}$

Proposition 5. *Given the general assumptions $(\star)$ on the players' payoffs, for any compact subset $K \subset (0,1)$ let μ_K be the unique equilibrium probability measure guaranteed by Proposition 2. If K_i is any sequence of compact sets converging to K in the Hausdorff metric, then μ_{K_i} weak-* converges to μ_K.*

In particular, using the proportional sharing rule, as the number of bids in a discrete support becomes dense in an interval $[a,b] \subset (0,1)$, the mixing probabilities for the equilibrium with the discrete support converge to the mixing atom and density of Proposition 1.

Thus, although the mixing distribution for an equilibrium mixing over a discrete set of bids may appear qualitatively different from that mixing over a continuous interval of bids, as the number of discrete bids increases and fills up the interval, the discrete equilibrium will come to look more and more like the continuous equilibrium. Similarly, we could even take a sequence of Cantor sets Hausdorff converging to $[a, b]$ and their equilibria would also converge.