Extra Proofs for: "The Learning Curve in a Competitive Industry"

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One of the implications of assumption (A5) is that the minimum one-period average cost with no learning, p_m, is attained at some finite positive output level. This is part of a more general lemma:

LEMMA I. Under assumptions (A1) - (A5), the functions $[C(q,0)/q]$, $[f(q,0)/q]$ and $[f(0,q)/q]$ attain their minima at finite positive output levels.

PROOF. First note that since $C(0,0) > 0$, the average costs in the statement of the lemma diverge to $+\infty$ as $q \to 0$.

Let $m_1 = inf\{[f(q,0)/q] : q \geq 0]\}$. Suppose the infimum is not attained at any finite q. Then, there exists $\{q_n\} \to +\infty$ such that $f(q_n,0)/q_n \to m_1$ and, further, for each n,

$$[f(q,0)/q] \geq [f(q_n,0)/q_n] \;\; for \;\; q \in [0, q_n] \tag{1}$$

There exists N such that $q_n > K$ for all $n \geq N$. Then, using (A5), for each $n \geq N$, there exists $z < q_n, y < q_n$ such that $z + y = q_n$ and: $f(q_n,0) > f(z,0) + f(y,0)$, that is,

$$\begin{aligned} & f(q_n,0)/q_n > [z/(z+y)][f(z,0)/z] + [y/(z+y)][f(y,0)/y] \\ & \geq [z/(z+y)][f(q_n,0)/q_n] + [y/(z+y)][f(q_n,0)/q_n] = f(q_n,0)/q_n, \end{aligned}$$

(using (1)), a contradiction.

A similar method can be used to show that $f(0,q)/q$ attains its minimum at a finite positive output. Lastly, note that (A5) implies that for any $q > K$, there exists $z, y \geq 0, z + y \leq q$,

$$C(0,0) + \delta C(q,0) > 2C(0,0) + \delta C(z,0) + \delta C(y,0)$$

which implies

$$C(q,0) > C(z,0) + C(y,0)$$

Again, the same set of steps can be replicated to show that $[C(q,0)/q]$ attains its minimum at a finite positive level. //

PROPOSITION 2. Under assumptions (A1)-(A6), an equilibrium exists. It is unique in prices, and it is socially optimal.

PROOF. Section III of the text defines the social planner's problem. Based on that definition, we can define the social cost minimization problem for any $Q_1 \geq 0, Q_2 \geq 0$ to be produced by S-type firms as:

$$(SCM1) \qquad Minimize \int_0^{n_S} [C(q_1(i), 0) + \delta C(q_2(i), q_1(i))]di$$

w.r.t. $n_S \geq 0, q_t : [0, n_S] \to \mathbf{R}_+, t = 1, 2, q_t(.)$ integrable w.r.t. Lebesgue measure, subject to the restrictions:

$$\int_0^{n_S} q_t(i)di \geq Q_t, t = 1, 2.$$

Let $\psi(Q_1, Q_2)$ be the value of the minimization problem.

Similarly, if $Q_E \geq 0$ is the amount to be produced by E-type firms in period 1, then the social cost minimization problem is given by:

$$(SCM2) \qquad Minimize \int_0^{n_E} C(q_E(i), 0)di$$

w.r.t. $n_E \geq 0, q_E : [0, n_E] \to \mathbf{R}_+, q_E(.)$ integrable w.r.t. Lebesgue measure, subject to the restriction:

$$\int_0^{n_E} q_E(i)di \geq Q_E.$$

Let $\psi_E(Q_E)$ be the value of the minimization problem.

Lastly, if $Q_L \geq 0$ is the amount to be produced by L-type firms in period 2, then the social cost minimization problem is given by:

$$(SCM3) \qquad Minimize \int_0^{n_L} \delta C(q_L(i), 0)di$$

w.r.t. $n_L \geq 0, q_L : [0, n_L] \to \mathbf{R}_+, q_L(.)$ integrable w.r.t. Lebesgue measure, subject to the restriction:

$$\int_0^{n_L} q_L(i)di \geq Q_L(*)$$

Let $\psi_L(Q_L)$ be the value of this minimization problem.

First consider (SCM1). For $Q_1 = Q_2 = 0$, the solution is obviously $n_S = 0$. Suppose $Q_2 = 0$ and $Q_1 > 0$. Let

$$m_1 = min\{[(C(q, 0) + \delta C(0, q))/q] : q \geq 0\} \qquad (2)$$

From Lemma I we have the existence of finite $q > 0$ which solves this minimization problem. Let $q(m_1) > 0$ be any such solution. Consider the feasible set in (SCM1). One may without loss of generality confine attention to the subset of the feasible set where $q_2(i) = 0$ a.e. and

$$\int_0^{n_S} q_1(i)di = Q_1.$$

Let $(n_S, q_1(i), q_2(i))$ be any such feasible solution. Then

$$\begin{aligned}\textstyle\int_0^{n_S} [C(q_1(i),0) + \delta C(0, q_1(i))]di & \\ &= \textstyle\int_0^{n_S} [(C(q_1(i),0) + \delta C(0, q_1(i)))/q_1(i)]q_1(i)di \\ &\geq \textstyle\int_0^{n_S} m_1 q_1(i)di \\ &= m_1 Q_1 \\ &= [C(q(m_1),0) + \delta C(0, q(m_1))]\hat{n}\end{aligned}$$

where $\hat{n} = Q_1/q(m_1)$.

Thus, there exists a solution to SCM1 for $Q_1 > 0, Q_2 = 0$ and $\psi(Q_1, 0) = m_1 Q_1$.

Similarly, let m_2 be defined by

$$m_2 = min\{[(C(0,0) + \delta C(q,0))/q] : q \geq 0\}. \tag{3}$$

Using Lemma I, there exists a finite positive solution to the minimization problem in (3). Consider (SCM1) for the case where $Q_1 = 0, Q_2 > 0$. One can show by similar arguments as above, that there exists a solution to (SCM1) for this case and that $\psi(0, Q_2) = m_2 Q_2$.

In fact, the same set of arguments will show that there exists solution to (SCM2) and (SCM3) and that $\psi_E(Q_E) = p_m Q_E$ and $\psi_L(Q_L) = \delta p_m Q_L$.

Lastly, consider (SCM1) for the case where $Q_1 > 0, Q_2 > 0$. First note that $(n_S = 1, q_1(i) = Q_1, q_2(i) = Q_2)$ is a feasible solution. Let N be defined by

$$N = [C(Q_1, 0) + \delta C(Q_2, Q_1)]/[C(0,0)]$$

Suppose there is a feasible solution $(n_S, q_1(i), q_2(i))$ where $n_S > N$. Then

$$\int_0^{n_S} [C(q_1(i),0)+\delta C(q_2(i), q_1(i))]di \geq n_S C(0,0) > NC(0,0) = [C(Q_1,0)+\delta C(Q_2,Q_1)].$$

We may, therefore, without loss of generality confine attention to feasible points where $n_S \leq N$. Given assumption (A5), we may confine attention to feasible solutions where $q_t(i) \leq K, i = 1, 2$. Lastly, it would be wasteful to introduce an S-type firm which produces zero output in both periods. So, w.l.o.g. one can confine attention to feasible points where $q_t(i) > 0$ for some t, for all $i \in [0, n_S]$.

Note that it is possible to extend any function $q_t(i)$ on $[0, n_S]$ to an integrable function on $[0, N]$ by setting $q_t(i) = 0$ for $i \in (n_S, N]$.

Let $I(q_1, q_2) = 1$, if $q_t > 0$ for some $t, q_t \leq K$ for $t = 1, 2$ and $I = 0$ otherwise. One can rewrite (SCM1) for $Q_1, Q_2 >> 0$ as

$$(SCM1') \quad Minimize \int_0^N G[q_1(i), C(q_2(i)]di$$

w.r.t. $q_t : [0, N]x[0, K], q_t(.)$ integrable, subject to

$$\int_0^N q_t(i)di \geq Q_i,$$

where $G : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+$ is given by

$$G(q_1, q_2) = C(q_1, 0) + \delta C(q_2, q_1)I(q_1, q_2).$$

Check that G is a bounded function on $\mathbf{R}_+^2$ (bounded above by $(1+\delta)C(K, 0)$). A direct appeal to Theorem 6.1 in Aumann and Perles (1965) shows that there exists a solution to (SCM1'). This, in turn, implies that there exists a solution to (SCM1) for all $Q_1, Q_2 \geq 0$.[1]

Let λ_1 and λ_2 be the Lagrangean multipliers associated with the constraints (*) in (SCM1). Then at any optimal solution $(n_S, q_1(i), q_2(i))$ the necessary conditions

$$q_1(i) > 0 \text{ implies } C_q(q_1(i), 0) + \delta C_w(q_2(i), q_1(i)) = f_1(q_1(i), q_2(i)) = \lambda_1(Q_1, Q_2), \tag{4}$$

and

$$q_2(i) > 0 \text{ implies } C_q(q_2(i), q_1(i)) = f_2(q_1(i), q_2(i)) = \lambda_2(Q_1, Q_2). \tag{5}$$

[1] R.J. Aumann and M. Perles (1965) "A Variational Problem Arising in Economics." *Journal of Mathematical Analysis and Applications*, 11: 488 - 503.

We claim that $\psi(Q_1, Q_2)$ is a convex function on $\mathbf{R}^2_+$. Let us define the "cost-possibility set" of any firm $i \in \mathbf{R}_+$ by

$$f(i) = \{(q_1, q_2, -y)q_t \geq 0, y \geq C(q_1, 0) + \delta C(q_2, q_1)\} \cup \{(0, 0, 0)\}$$

(no-entry is equivalent to (0,0,0))

$f(i)$ is identical for all i. Let F be the "cost-possibility set" of the social planner, i.e. the set of all output-cost combinations that are feasible for the social planner by using any number of S-type firms and any distribution of output across such firms. Thus, F is the integral of the set-valued correspondence $f(i) : \mathbf{R}_+ \to \mathbf{R}_+$ with respect to Lebesgue measure. A direct appeal to the Lyapunov-Richter theorem [2] shows that F is a convex set. By definition of ψ in (SCM1), F is also the set of all $\{(Q_1, Q_2, -Y) : Y \geq \psi(Q_1, Q_2), Q_t \geq 0, t = 1, 2\}$. It is easily checked that convexity of F implies that ψ is a convex function on $\mathbf{R}^2_+$.

Next, we claim that ψ is continuous on $\mathbf{R}^2_+$. Continuity on $\mathbf{R}^2_{++}$ follows from its convexity. Continuity on the border can be verified directly. For example, choose any sequence $\{Q_1^m, Q_2^m\} \to (Q_1, Q_2)$ such that $Q_1 = 0, Q_2 > 0$. Let $q(m_2)$ be a solution to the minimization problem in (2). Then

$$m_2 Q_2^m = \psi(0, Q_2^m) \leq \psi(Q_1^m, Q_2^m) \leq \{C([(Q_1^m q(m_2))/Q_2^m], 0) + \delta C(q(m_2),$$

and $[(Q_1^m q(m_2))/Q_2^m])\}\{Q_2^m/q(m_2)\} \to m_2 Q_2 = \psi(0, Q_2)$ as $m \to +\infty$.

Since the left hand side of the inequality equals $m_2 Q_2^m \to m_2 Q_2 = \psi(0, Q_2)$, we have that $\psi(Q_1^m, Q_2^m) \to \psi(0, Q_2)$. Similar arguments can be used when the limits (Q_1, Q_2) of the sequence $\{Q_1^m, Q_2^m\}$ are such that $Q_2 = 0, Q_1 > 0$ and also when both $Q_t = 0$.

Next we want to establish that the partial derivatives of $\psi(Q_1, Q_2)$ exist on $\mathbf{R}^2_{++}$. For any fixed $Q_2 = z > 0$, let $g(Q_1) = \psi(Q_1, z)$. We will show that g is differentiable on $\mathbf{R}_{++}$.

Firstly, note that g is a convex function and so its right and left hand derivatives exist. Furthermore,

$$g'_+ \geq g'_-. \tag{6}$$

Let $(n_S, q_1(i), q_2(i))$ be the optimal solution of (SCM1) at (Q_1, z). For $\varepsilon > 0$ small enough, consider the vector $(Q_1 + \varepsilon, z)$. Let $\hat{q_1}(i) = q_1(i) + \varepsilon[q_1(i)/Q_1]$. Note that

[2] See L.1.3 in A. Mas-Colell (1985) *The Theory of General Economic Equilibrium: A Differentiable Approach.* Cambridge University Press, Cambridge, England.

$\hat{q}_1(i) = 0$ if $q_1(i) = 0$. It is easy to check that $(n_S, \hat{q}_1(i), q_2(i))$ is a feasible way to produce $(Q_1 + \varepsilon, z)$. Thus,

$$g(Q_1+\varepsilon) = \psi(Q_1+\varepsilon, z) \le \int_0^{n_S} [f(\hat{q}_1(i), q_2(i))]di$$
$$= \int_0^{n_S} [f([q_1(i) + \varepsilon(q_1(i)/Q_1)], q_2(i))di$$

so that

$$g'_+(Q_1) = \lim_{\varepsilon\to 0} \{[g(Q_1+\varepsilon) - g(Q_1)]/\varepsilon\}$$
$$\le \lim_{\varepsilon\to 0} (1/\varepsilon) \int_0^{n_S} \{f([q_1(i) + \varepsilon(q_1(i)/Q_1)], q_2(i))$$
$$-f(q_1(i), q_2(i))\}di.$$

(Use the dominated convergence theorem and note that in the last expression the term within curly brackets equals zero if $q_1(i) = 0$.)

$$g'_+(Q_1) = \lim_{\varepsilon\to 0} (1/\varepsilon) \int_0^{n_S} \{f([q_1(i) + \varepsilon(q_1(i)/Q_1)], q_2(i)) - f(q_1(i), q_2(i))\} I(i)di$$

where $I(i) = 1$ if $q_1(i) > 0$ and $I(i) = 0$ otherwise.

Since $f(q_1, q_2)$ is differentiable and further using (4), we have from the above inequality that:

$$g'_+(Q_1) \le \int_0^{n_S} f_1(q_1(i), q_2(i))[q_1(i)/Q_1]I(i)di$$
$$= \int_0^{n_S} \{[\lambda_1(Q_1, z)]/Q_1\} q_1(i)di \quad (using (4)) \tag{7}$$
$$= \lambda_1(Q_1, z).$$

By a similar argument one can show that:

$$g'_-(Q_1) = \lim_{\varepsilon\to 0} [g(Q_1) - g(Q_1 - \varepsilon)](1/\varepsilon) \ge \lambda_1(Q_1, z). \tag{8}$$

Thus (7) and (8) imply

$$g'_+(Q_1) \le \lambda_1 \le g'_-(Q_1),$$

which combined with (6) yields

$$g'_+(Q_1) = g'_-(Q_1) = \lambda_1,$$

that is, g is differentiable at Q_1 and

$$g'(Q_1) = \psi_1(Q_1, Q_2) = \lambda_1(Q_1, Q_2). \tag{9}$$

Note that at $Q_2 = 0, \psi(Q_1, Q_2) = m_1 Q_1$ so that $\psi_1(Q_1, 0) = m_1$. Similar reasoning shows that given $Q_1 \geq 0$, the partial derivative of ψ w.r.t. Q_2 exists on $\mathbf{R}_{++}$ and is given by

$$\psi_2(Q_1, Q_2) = \lambda_2(Q_1, Q_2)$$

and, in particular, $\psi_2(0, Q_2) = m_2$.

As the partial derivatives of ψ exist at each point in $\mathbf{R}^2_{++}$ and ψ is convex, it follows that ψ is continuously differentiable on $\mathbf{R}^2_{++}$ (Section 42, Theorem D and Section 44, Theorem E in Roberts and Varberg [1973]).[3]

We summarize the above discussion in the Lemma II:

Lemma II: (a) There exists a solution to the (SCM1), (SCM2) and (SCM3).

(b) For any $Q_E \geq 0, \psi_E(Q_E) = p_m Q_E$ and in the solution to (SCM2), $q_E(i) \in \{q : [C(q, 0)/q] = p_m\}$

(c) For any $Q_L \geq 0, \psi_L(Q_L) = \delta p_m Q_L$ and in the solution to (SCM3), $q_L(i) \in \{q : [C(q, 0)/q] = p_m\}$.

(d) For $Q_1 = 0, \psi(Q_1, 0) = m_1 Q_1$ where $m_1 > 0$ is defined by (2); For $Q_2 \geq 0, \psi(0, Q_2) = m_2 Q_2$, where m_2 is defined by (3).

(e) ψ is continuous and convex on $\mathbf{R}^2_{+}$;

(f) For any $(Q_1, Q_2) \geq 0$, there exists Lagrangean multipliers $\lambda_1(Q_1, Q_2), \lambda_2(Q_1, Q_2)$, such that the solution $(n_S, q_1(i), q_2(i))$ to (SCM1) is characterized by (4) and (5).

(g) ψ is continuously differentiable on $\mathbf{R}^2_{++}$ and its partial derivatives are given by $\psi_t(Q_1, Q_2) = \lambda_t(Q_1, Q_2), t = 1, 2$; further, $\psi_1(Q_1, 0) = m_1$ and $\psi_2(0, Q_2) = m_2$.

We now rewrite (SPP*) as the following problem (hereafter referred to as SPP)

$$\underset{Q_1, Q_2, Q_E, Q_L \geq 0}{Maximize} \; W(Q_1, Q_2, Q_E, Q_L)$$

where

$$W(Q_1, Q_2, Q_E, Q_L) = \{S(Q_1 + Q_E) + \delta S(Q_2 + Q_L) - \psi(Q_1, Q_2) - \psi_E(Q_E) - \psi_L(Q_L)\},$$

and $S(y) = \int_0^y P(q)dq$.

[3]Roberts, A. and D.F. Varberg (1973) *Convex Functions*, Academic Press, New York.

Note that S is a strictly concave function, continuous on $\mathbf{R}_+$, continuously differentiable on $\mathbf{R}_{++}$ and $S'(y) = P(y)$.

Thus, using Lemma II, W is continuous and strictly concave on $\mathbf{R}^4_+$. Further, at any point $(Q_1 > 0, Q_2 > 0, Q_E \geq 0, Q_L \geq 0)$, the partial derivatives of W w.r.t all arguments exist. In particular,

$$\partial W/\partial Q_1 = P(Q_1 + Q_E) - \psi_1(Q_1, Q_2) = P(Q_1 + Q_E) - \lambda_1(Q_1, Q_2) \quad (10)$$

$$\partial W/\partial Q_2 = P(Q_2 + Q_L) - \psi_2(Q_1, Q_2) = P(Q_2 + Q_L) - \lambda_2(Q_1, Q_2) \quad (11)$$

$$\partial W/\partial Q_E = P(Q_1 + Q_E) - \psi'_E(Q_E) = P(Q_1 + Q_E) - p_m \quad (12)$$

$$\partial W/\partial Q_L = \delta P(Q_2 + Q_L) - \psi'_L(Q_L) = \delta[P(Q_2 + Q_L) - p_m]. \quad (13)$$

Since $P(y) \to 0$ as $y \to +\infty$ and $p_m > 0$, there exists $Q_0 > 0$, such that $\partial W/\partial Q_E < 0$ for any (Q_1, Q_2, Q_E, Q_L) where $Q_E > Q_0$ and $\partial W/\partial Q_L < 0$ for $Q_L > Q_0$. Note that $\lambda_2(Q_1, Q_2) = C_q(q_2(i), q_1(i)) \geq min\{C_q(q, x) : \; 0 \leq q \leq K, 0 \leq w \leq K\}$. Using (A1) we can check that the minimum in the previous expression is actually attained at some $(q'', w'') \in [0, K] \times [0, K]$. Thus, for all (Q_1, Q_2) such that $Q_2 > 0$, we have $\psi_2(Q_1, Q_2) \geq C_q(q'', h'') = h$ where, using (A2), we have $h > 0$. So, there exists $Q^\$ > 0$ such that $\partial W/\partial Q_2 < 0$ for $Q_2 \geq Q^\$$. Let $\hat{Q} = max(Q^\$, Q_0)$. One can without loss of generality, rewrite SPP as

$$\underset{Q_1 \geq 0, Q_2, Q_E, Q_L \in [0, \hat{Q}]}{Maximize} \quad W(Q_1, Q_2, Q_E, Q_L)$$

We claim there exists a solution to this maximization problem. Let $W*$ be the supremum of the maximand (which can be $+\infty$). Then, there exists a sequence $\{Q_1^m, Q_2^m, Q_E^m, Q_L^m\}, m = 1, 2, ...,$ where $W(Q_1^m, Q_2^m, Q_E^m, Q_L^m) \to W^*$. Suppose $\{Q_1^m\}$ is bounded above. Then the sequence $\{Q_1^m, Q_2^m, Q_E^m, Q_L^m\}$ is bounded above and has a convergent subsequence, whose limit is the optimal solution, using continuity of W.

Now suppose that $\{Q_1^m\}$ is not bounded above; abusing notation somewhat, suppose that $\{Q_1^m\} \to +\infty$. Choose M such that $Q_1^M > 0$. Then for $m \geq M$,

$$\psi_1(Q_1^m, Q_2^m) \geq \psi_1(Q_1^M, Q_2^m) \geq inf\{\psi_1(Q_1^M, x) : \; 0 \leq x \leq K\} = \Delta(say).$$

Let $f(x) = \psi_1(Q_1^M, x)$. There exists a sequence $\{x_k\} \in [0, K]$, such that $f(x_k) \to \Delta$. We claim that $\Delta > 0$. To see this, consider a convergent subsequence of $\{x_k\}$, e.g. $\{x'_k\} \to x*$. If $x* > 0$, then from continuous differentiability of ψ

(implying continuity of partial derivatives) on $\mathbf{R}^2_{++}$, we have that $f(x_n) \to f(x*) = \psi_1(Q_1^M, x*) = \lambda_1(Q_1^M, x*) > 0$. Suppose $x* = 0$. Note that $\psi_1(Q_1^M, 0) = m_1$. For $\varepsilon > 0, \psi_1(Q_1^M, \varepsilon) \geq [\psi(Q_1^M, \varepsilon) - \psi(0, \varepsilon)]/Q_1^M$ (using convexity of ψ) so that taking the limit as $\varepsilon \to 0$ (using the continuity of ψ on $\mathbf{R}^2_+$ and the fact that $\psi(0,0) = 0$) yields:

$$\lim_{\varepsilon \to 0} \psi_1(Q_1^M, \varepsilon) \geq \psi(Q_1^M, 0)/Q_1^M = m_1$$

so that $\lim_{k' \to +\infty} f(x'_k) = \Delta > 0$.

Since $Q_1^m \to +\infty$, there exists $M' > 0$ such that for $m \geq M', P(Q_1^m) < \Delta$. Thus, for $m \geq max(M, M'), P(Q_E^m + Q_1^m) < \psi_1(Q_1^m, Q_2^m)$ i.e. $\partial W/\partial Q_1 < 0$ when evaluated at m large enough. This is a contradiction. Thus, $\{Q_1^m\}$ must be bounded above. The proof of existence is complete.

Note that since W is strictly concave on $\mathbf{R}^4_+$, the solution to (SPP) is unique.

It is easy to check that assumption (A6) implies $P(0) > \psi'_E(0) = p_m$ and $\delta P(0) > \psi'_L(0) = \delta p_m$ which, in turn, is sufficient to assert that if (Q_1, Q_2, Q_E, Q_L) solves (SPP) then $Q_1 + Q_E > 0$ and $Q_2 + Q_L > 0$. Further, it is impossible that in the optimal solution $Q_t = 0$ for either $t = 1$ or $t = 2$ or both. Suppose $Q_1 = 0$ and $Q_2 > 0$. Now, consider the original form of the social planner's problem (SPP*). Since $Q_1 + Q_E > 0$, it must be true that $Q_E > 0$. Then the social planner can easily reduce total cost by letting the existing n_S firms of type S (who currently produce zero in period 1) produce a total amount $Q_1 = Q_E$ (setting $n_E = 0$). The cost in period 1 is unchanged and that in period 2 is reduced (using assumption (A2)), a contradiction. Similarly, $Q_1 > 0, Q_2 = 0$ is ruled out as this implies $Q_L > 0$ and total cost can be reduced by letting S-type firms (who currently produce nothing in period 2) produce a total amount $Q_2 = Q_L$ (setting $n_L = 0$). Lastly, if both Q_t are equal to zero, then $Q_E > 0, Q_L > 0$. Suppose we let ε be the measure of E-type firms that produce in period 2— i.e. convert them to S-type firms, reduce the number of L-type firms by ε and transfer their output to these ε S-type firms. Then it is easy to see that total costs are reduced, a contradiction. To summarize:

Lemma III: There exists a solution to SPP and, hence, (SPP*). The solution to the social planner's problem is unique in (Q_1, Q_2, Q_E, Q_L). If $(Q_1^*, Q_2^*, Q_E^*, Q_L^*)$ is an optimal solution in total output produced by different "types" of firms then $Q_1^* > 0$, $Q_2^* > 0$.

Let us now write down the first order necessary conditions for $(Q_1^*, Q_2^*, Q_E^*, Q_L^*)$

to be an optimal solution to (SPP)

$$P(Q_1^* + Q_E^*) = \psi_1(Q_1^*, Q_2^*)[= \lambda_1(Q_1^*, Q_2^*)] \le \psi_E'(Q_E^*)[= p_m] \tag{14}$$

$$P(Q_1^* + Q_E^*) = p_m \;\; if \;\; Q_E^* > 0 \tag{15}$$

$$\delta P(Q_2^* + Q_L^*) = \psi_2(Q_1^*, Q_2^*)[= \lambda_2(Q_1^*, Q_2^*)] \le \psi_L'(Q_L^*)[= \delta p_m] \tag{16}$$

$$P(Q_2^* + Q_L^*) = p_m \;\; if \;\; Q_L^* > 0 \tag{17}$$

Define $p_1^* = P(Q_1^*+Q_E^*), p_2^* = P(Q_2^*+Q_L^*)$. Let $(n_S^*, n_E^*, n_L^*, q_1^*(i), q_2^*(i), q_E^*(i), q_L^*(i))$ be the solutions to (SCM1), (SCM2), and (SCM3) associated with $(Q_1^*, Q_2^*), Q_E^*$ and Q_L^* respectively.

We want to show that $[p_1^*, p_2^*, n_S^*, n_E^*, n_L^*, (q_1^*(i), q_2^*(i), 0 \le i \le n_S^*), (q_E^*(i), 0 \le i \le n_E^*, (q_L^*(i), 0 \le i \le n_L^*)]$ constitutes an equilibrium. Recall the conditions (i)-(xi) that define an equilibrium. By definition of the prices, conditions (i) and (ii) in the definition of equilibrium are satisfied. From (14) and (15) we have that $p_1^* \le p_m, p_2^* \le p_m$ which implies conditions (vii) and (viii) of the definition of equilibrium are satisfied. If $n_E^* > 0$, then it must be the case that $Q_E^* > 0$ so that (15) implies $p_1^* = p_m$ and from Lemma II(b) we have that $q_E^*(i)$ maximizes one period profit at price p_m and the maximum profit is equal to 0. Thus, conditions (iv) and (x) of the definition of equilibrium are satisfied. Similarly (17) and Lemma II(c) imply that conditions (v) and (xi) are met. Since $n_S^*, Q_1^*, Q_2^* >> 0$, it just remains to show that conditions (iii) and (ix) are satisfied (condition (vi) then holds automatically). In other words, we need to show that $(q_1^*(i), q_2^*(i))$ maximizes two period discounted sum of profits at prices (p_1^*, p_2^*) and that this maximum is equal to 0.

Consider (SCM1) at $(Q_1^*, Q_2^*) >> 0$. From (14) and (16) we have that $\lambda_1(Q_1^*, Q_2^*) = p_1^*, \lambda_2(Q_1^*, Q_2^*) = p_2^*$. Then $(\lambda_1 = p_1^*, \lambda_2 = p_2^*, q_1^*(i), q_2^*(i), n_S^*)$ minimizes the Lagrangean function:

$$L = \int_0^{n_S} f(q_1(i), q_2(i))di + \lambda_1(Q_1 - \int_0^{n_S} q_1(i)di) + \lambda_2(Q_2 - \int_0^{n_L} q_2(i)di)$$

with respect to $n_S \ge 0, \lambda_j \ge 0, q_1 : [0, n_S] \to \mathbf{R}_+, q_2 : [0, n_S] \to \mathbf{R}_+, q_t(.)$ integrable.

Then it must, in particular, be true that given $(\lambda_1 = p_1^*, \lambda_2 = p_2^*)$, the vector $(n_S^*, q_1^*(i), q_2^*(i))$ maximizes:

$$\int_0^{n_S} [p_1^* q_1(i) + p_2^* q_2(i) - f(q_1(i), q_2(i))]di$$

with respect to $n_S \geq 0, q_1 : [0, n_S] \to \mathbf{R}_+, q_2 : [0, n_S] \to \mathbf{R}_+, q_t(.)$ integrable. But this implies that (almost everywhere)

(a) $(q_1^*(i), q_2^*(i))$ maximizes $[p_1^* q_1 + p_2^* q_2 - f(q_1, q_2)]$ with respect to $(q_1, q_2) \geq 0$, and

(b) $[p_1^* q_1(i) + p_2^* q_2(i) - f(q_1(i), q_2(i))] = 0$.

Proof of (a) is obvious (for otherwise we could increase the maximand by choosing a different value for $(q_1(i), q_2(i))$ on a positive measure of firms. To see (b), suppose not. There are two possibilities:

(1) $[p_1^* q_1(i) + p_2^* q_2(i) - f(q_1(i), q_2(i))] < 0$ in which case the maximand is increased by simply eliminating a positive measure of such firms (reducing n_S below n_S^*), a contradiction;

(2) $[p_1^* q_1(i) + p_2^* q_2(i) - f(q_1(i), q_2(i))] > 0$ in which case the maximand can be increased to $+\infty$ by setting $n_S = +\infty$ and letting all $j \geq n_S^*$ produce the same output vector $(q_1(i), q_2(i))$, a contradiction as $n_S^* < \infty$.

This proves (b). (a) and (b) imply that conditions (iii) and (ix) in the definition of equilibrium hold. We have therefore proved that:

<u>Lemma IV:</u> Every solution to the social planner's problem is implementable as a competitive equilibrium. In particular, let $(n_S^*, n_E^*, n_L^*, q_1^*(i), q_2^*(i), q_E^*(i), q_L^*(i))$ be a solution to (SPP*) with associated total output $(Q_1^*, Q_2^*, Q_E^*, Q_L^*)$. Then, if $p_1^* = P(Q_1^* + Q_E^*), p_2^* = P(Q_2^* + Q_L^*)$, then $[p_1^*, p_2^*, n_S^*, n_E^*, n_L^*, (q_1^*(i), q_2^*(i), 0 \leq i \leq n_S^*), (q_E^*(i), 0 \leq i \leq n_E^*, (q_L^*(i), 0 \leq i \leq n_L^*)]$ is a competitive equilibrium.

Lemmas III and IV imply:

<u>Lemma V:</u> There exists an equilibrium.

Next, we show that every equilibrium is socially optimal. Let $[p_1^\$, p_2^\$, n_S^\$, n_E^\$, n_L^\$, (\hat{q_1}^\$(i), q_2^\$(i), 0 \leq i \leq n_S), (q_E^\$(i), 0 \leq i \leq n_E^\$), (q_L^\$(i), 0 \leq i \leq n_L^\$)]$ be a equilibrium. Let $(Q_1^\$, Q_2^\$)$ be total output produced by S-type firms in this equilibrium. From Lemma III, we have that $Q_1^\$ > 0, Q_2^\$ > 0$. Let $Q_E^\$$ and $Q_L^\$$ be the total quantity produced by E and L type firms in their period of stay.

Our first claim is that $(n_S^\$, \hat{q_1}^\$(i), q_2^\$(i))$ is a socially cost minimizing way of producing $(Q_1^\$, Q_2^\$)$ i.e. it solves (SCM1) given $(Q_1^\$, Q_2^\$)$. Suppose not. Then there exists $(\hat{n}, \hat{q_1}(i), \hat{q_2}(i))$ which solves (SCM1) given $(Q_1^\$, Q_2^\$)$ and

$$\psi(Q_1^\$, Q_2^\$) < \int_0^{n_S^\$} f(\hat{q_1}\$(i), q_2^\$(i))di. \tag{18}$$

The sum of total profits of all S type firms in equilibrium is zero. Therefore,

$$0 = \int_0^{n_S^\$}[p_1^\$\hat{q_1}^\$(i) + \delta p_2^\$ q_2^\$(i) - f(\hat{q_1}^\$(i), q_2^\$(i))]di < p_1^\$ Q_1^\$ + \delta p_2^\$ Q_2^\$ - \psi(Q_1^\$, Q_2^\$) \;\; (using \;\; (18))$$
$$= \int_0^{\hat{n}} [p_1^\$\hat{q_1}(i) + \delta p_2^\$\hat{q_2}(i) - f(\hat{q_1}(i), \hat{q_2}(i))]di$$

which implies, in turn, that there exists some i for which $[p_1^\$\hat{q_1}(i) + \delta p_2^\$\hat{q_2}(i) - f(\hat{q_1}(i), \hat{q_2}(i))] > 0$. But by definition of equilibrium, the maximum possible discounted sum of profit at prices $(p_1^\$, p_2^\$)$ is 0. We have a contradiction. Hence,

$$(Q_1^\$, Q_2^\$) = \int_0^{n_S^\$} f(\hat{q_1}^\$(i), q_2^\$(i))di \tag{19}$$

and $(n_S^\$, \hat{q_1}^\$(i), q_2^\$(i))$ does solve (SCM1) given $(Q_1^\$, Q_2^\$)$.

Next, suppose $n_E^\$ > 0$. Then from Proposition 1, $p_1^\$ = p_m$ and $q_E^\$(i) \in \{q \geq 0 : [C(q,0)/q] = p_m\}$ which means that total cost of production of $Q_E^\$$ is equal to $p_m Q_E^\$$ which is equal to $\psi_E(Q_E^\$)$, i.e. $(n_E^\$, q_E^\$(i))$ solves (SCM2) given $Q_E^\$$. Similarly, one can show that if $n_L^\$ > 0$, then $(n_L^\$, q_L^\$(i))$ solves (SCM3) given $Q_L^\$$.

Therefore, in equilibrium, total output $(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)$ produced by different types of firms are produced in the socially cost minimizing way. Let the total social welfare in equilibrium is equal to $W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)$, where the function W is as defined before introducing problem (SPP). As noted earlier, $W(.)$ is strictly concave on $\mathbf{R}_+^4$. The partial derivatives of W exist at all $(Q_1, Q_2, Q_E, Q_L) \geq 0$, where $Q_1 > 0, Q_2 > 0$.

Suppose equilibrium is not socially optimal. Let $(Q_1^*, Q_2^*, Q_E^*, Q_L^*)$ maximize social welfare. Then,

$$W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$) - W(Q_1^*, Q_2^*, Q_E^*, Q_L^*) < 0. \tag{20}$$

From Lemma III, $Q_1^* > 0, Q_2^* > 0$. As noted above, $Q_1^\$ > 0$, $Q_2^\$ > 0$. So strict concavity of W implies

$$\begin{aligned} &W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$) - W(Q_1^*, Q_2^*, Q_E^*, Q_L^*) \\ &\geq [\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_1][Q_1^\$ - Q_1^*] \\ &+[\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_2][Q_2^\$ - Q_2^*] \\ &+[\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_E][Q_E^\$ - Q_E^*] \\ &+[\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_L][Q_L^\$ - Q_L^*]. \end{aligned} \tag{21}$$

Note that:

$$\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_1 = P(Q_1^\$ + Q_E^\$) - \psi_1(Q_1^\$, Q_2^\$) = p_1^\$ - \lambda_1(Q_1^\$, Q_2^\$) \quad (22)$$

$$= p_1^\$ - f_1(\hat{q_1}^\$(i), q_2^\$(i)) \ \ (from \ \ (4)). \quad (23)$$

As $Q_1^\$ > 0, Q_2^\$ > 0$, there exists positive measure of i such that $\hat{q_1}^\$(i) > 0, q_2^\$(i) > 0$, so that the first order conditions of profit maximization (condition (iii) in definition of equilibrium) implies that right hand side of (22) is equal to zero, i.e.

$$\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_1 = 0. \quad (24)$$

Similarly, one can show that

$$\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_2 = 0. \quad (25)$$

Next note that, from Proposition 1, $p_1^\$ \leq p_m$ so that $\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_E = p_1 - \psi_E(Q_E^\$) = p_1 - p_m \leq 0$ and it is equal to zero if $Q_E^\$ > 0$ (since $p_1^\$ = p_m$). Thus,

$$[\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_E][Q_E^\$ - Q_E^*] \geq 0. \quad (26)$$

Similarly, one can show that

$$[\partial W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$)/\partial Q_L][Q_L^\$ - Q_L^*] \geq 0. \quad (27)$$

From (21) , (24) - (27) we have $W(Q_1^\$, Q_2^\$, Q_E^\$, Q_L^\$) - W(Q_1^*, Q_2^*, Q_E^*, Q_L^*) \geq 0$ contradicting (20) . The proof is complete. We have thus shown:

Lemma VI: Every competitive equilibrium is socially optimal.

Combining Lemmas (III) - (VI) yields Proposition 2.

PROPOSITION 3. Under assumptions (A1)-(A7), an equilibrium exists. It is unique in prices, output, and number of firms, and it is socially optimal.

Proof. It is sufficient to show that under (A7) there exists a unique solution to (SCM1), (SCM2) and (SCM3). Proposition 2 then implies the result. To see uniqueness in (SCM1) let $(n_S, q_1(i), q_2(i)), (n', q_1'(i), q_2'(i))$ be any two optimal production plans producing $(Q_1, Q_2) >> 0$. Let $N = max(n_S, n')$. Suppose $n_S < n'$. Then extend $q_t(i)$ to the entire interval $[0, n']$ by setting $q_t(i) = 0$ for $i > n_S$. Then let $\hat{q}_t(i) = (1/2)[q_t(i) + q_t'(i)], t = 1, 2$ be defined on $[0, n']$. It is easy to check that this is a feasible production plan for (Q_1, Q_2). Further,

$$\int_0^{n'} [f(\hat{q}_1(i), \hat{q}_2(i))]di < \int_0^{n'} (1/2)[f(q_1(i), q_2(i)) + f(q_1'(i), q_2'(i))]di = \psi(Q_1, Q_2),$$

a contradiction. Uniqueness in (SCM2) and (SCM3) are similarly established. //

PROPOSITION 4. Under assumptions (A1)-(A7), the following is true in equilibrium:

(a) Each of the staying firms behaves identically, and there exists a positive measure of staying firms. There exist q_1^* and q_2^* such that $q_1^*(i) = q_1^*$ and $q_2^*(i) = q_2^*$ for all active staying firms i.

(b) If exiting firms exist, they produce at the initial minimum efficient scale, which is less than the q_1 produced by the staying firms. If $n_E > 0$, then $q_E(i) = q_m$ for all $i \in [0, n_E]$, where q_m is the unique solution to minimization of $[C(q,0)/q]$ with respect to $q \geq 0$, and $q_E < q_1^*$.

(c) There exist no late-entering firms: $n_L = 0$.

Proof. The first part of (a) and (b) follow immediately from strict concavity of the profit function for each type of firm. (Note that since the total amount (Q_1, Q_2) produced by all S-type firms is always strictly positive, $(q_1^*, q_2^*) >> 0$.) The second part of (b) results from Proposition 1, because the negative first-period profits of the staying firms result from their high production for the sake of learning. It remains to show that $n_L = 0$ for part (c).

Suppose that $n_L > 0$. Then from Lemma II, $n_E = 0$ and $p_2 = p_m$. Under (A7), there exists a unique q_m which minimizes $[C(q,0)/q]$ over $q \geq 0$. So, $q_L(i) = q_m$ and

$$p_2 = p_m = C(q_m, 0)/q_m = C_q(q_m, 0). \tag{28}$$

Furthermore,

$$D(p_2) = D(p_m) = n_S q_2^* + n_L q_m > n_S q_2^*. \tag{29}$$

From first order condition of profit maximization for firms which produce in both periods we have that $C_q(q_2^*, q_1^*) = p_2 = p_m$ and, therefore (using (A7), (28) and $q_1^* > 0$)

$$q_2^* \geq q_m. \tag{30}$$

Next we claim that the following inequality is true:

$$C_w(q_2^*, q_1^*)q_1^* + C_q(q_2^*, q_1^*)q_2^* - C(q_2^*, q_1^*) \geq 0. \tag{31}$$

By convexity of C on $\mathbf{R}_+^2$,

$$C(q_m, 0) - C(q_2^*, q_1^*) \geq C_q(q_2^*, q_1^*)(q_m - q_2^*) + C_w(q_2^*, q_1^*)(0 - q_1^*)$$

which implies that $C_w(q_2^*, q_1^*)q_1^* + C_q(q_2^*, q_1^*)q_2^* - C(q_2^*, q_1^*) \geq C_q(q_2^*, q_1^*)q_m - C(q_m, 0) = p_2 q_m - C(q_m, 0) = [p_2 - (C(q_m, 0)/q_m)] = 0$ (using (28)).

From the first order conditions of profit maximization for S-type firms and the fact that in equilibrium, the discounted sum of profits is zero, we have:

$$C_q(q_1^*, 0)q_1^* + \delta C_w(q_2^*, q_1^*)q_1^* + \delta C_q(q_2^*, q_1^*)q_2^* - C(q_1^*, 0) - \delta C(q_2^*, q_1^*) = 0.$$

Using (31) in the above equation we have:

$$C_q(q_1^*, 0)q_1^* - C(q_1^*, 0) \leq 0$$

which implies that

$$q_1^* \leq q_m \tag{32}$$

so that, from (30) , we have $q_1^* \leq q_2^*$. Thus,

$$n_S q_1^* \leq n_S q_2^*. \tag{33}$$

From (29) and (33) we have

$$D(p_2) > n_S q_2^* \geq n_S q_1^* = D(p_1),$$

and so, $p_1 > p_2 = p_m$, which violates condition (vii) of the definition of equilibrium. //