When Does Extra Risk Strictly Increase an Option's Value?

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Abstract

It is well known that risk increases the value of options. This paper makes that precise in a new way. The conventional theorem says that the value of an option does not fall if the underlying asset becomes riskier in the conventional sense of the mean-preserving spread. This paper uses two new definitions of "riskier" to show that the value of an option strictly increases (a) if the underlying asset becomes "pointwise riskier," and (b) only if the underlying asset becomes "extremum riskier."

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I. Introduction

A call option is the right to buy the asset at a strike price, P. It has been well known at least since Merton (1973) that the value of a call option increases with the riskiness of the underlying asset. If extra risk increases the probability that the market price exceeds P, then the value of the option increases. A standard finance text says

> "The holder of a call option will prefer more variance in the price of the stock to less. The greater the variance, the greater the probability that the stock price will exceed the exercise price, and this is of value to the call holder." (Copeland & Weston, 3rd edition, p. 243)

But this is not quite correct, despite being the sort of thing that even experts say in conversation and in textbooks. As I am sure Copeland and Weston knew as they wrote this passage, it quite possible for the risk and variance of the underlying asset to increase while the value of the option remains does not increase. The value will not fall, but it might remain unchanged. Suppose the strike price is $50 and the current price of the asset is $40. If the probability of the price being between $10 and $15 or $45 and $49 increases, while the probability it is between $38 and $42 falls, the asset has become riskier, but there is no effect on the option value, because the probabilities of asset values above the strike price of $50 have not changed. This, too, is well-known, but it leaves open the question of what kind of risk does actually increase the value of options. It is false, strictly speaking, to say that additional risk increases the value. On the other hand it is true but uninteresting to say that additional risk does not reduce the value. A great many variables do not reduce the value of an option, usually because they never affect the value either way. For introductory textbooks no great harm is done in stating a risk-value proposition loosely, but it is worth thinking about how we can come up with a proposition for this basic intuition that is both interesting and true. One way out is to surrender generality in the kinds of asset distributions that we describe. Bliss (2001), noting the problem of coming up with a rigorous proposition, points out that a sufficient condition for option value to increase with risk is that the underlying asset value has a two-parameter distribution such as the normal or lognormal. The relationships between option value and risk, however, clearly holds for much more general distributions.

The options literature has travelled down the route of studying particular stochastic processes for asset returns— diffusion or jump processes— rather than looking at general distributions for end-states as Merton (1973) did. This began with the log-normal diffusion processes of Black & Scholes (1973) and continued with such generalizations as Cox & Ross (1976), Merton (1976), and Heston (1993). More recent entries in the literature that look at option properties as well as pricing include Bergman, Grundy & Wiener (1996) and Kijima (2002).

Other papers look at other considerations absent in the simplest model of one underlying asset, risk-neutral investors, and zero transaction costs. Jagannathan (1984), for example, looks at values when investors are not risk neutral and value wealth more in particular states of the world. In such a situation, a riskier asset might not have a higher option value because the option might yield its highest returns in a state of the world when investors are wealthier anyway and hence value the return less. While the "extreme value theory" of, e.g., Chavez-Demoulin & Embrechts (2004) has turned to looking at the effects of unusual events on financial valuation, it is oriented towards estimation of the value of particular assets.

In this article I return to the original problem of how risk affects option value, for very general distributions of the underlying asset but without looking at how values evolve slowly over time. First, we will see that if the underlying asset becomes riskier, then we can at least say that for *some* strike prices a call or put option will become more valuable— a very simple result, but worth noting. Second, I will show that only if the underlying asset becomes riskier in the special way I call "extremum riskier" will every option rise in value regardless of the strike price— a necessary condition for a rise in value. Third, I will show that if the underlying asset becomes riskier in the special way I call "pointwise riskier" then every option will rise in value regardless of the strike price— a sufficient condition for a rise in value.

The article's main contribution is to tidy up of one of the fundamental ideas in finance theory. This will be useful for those analysts who do not wish to assume normality of asset returns, particularly in real option theory, where option value enters only as part of a larger model of business decisionmaking (see, e.g. Dixit & Pindyck [1994] or section 25.6 of Gollier [2001]). The definitions here may also be useful in other areas of economics. Arrow & Fischer (1974) applied the idea of changing option value to cost-benefit analysis in environmental projects with irreversibility. Search theory is another application; see Weitzman (1979) for a classic model in which the value of searches increases with uncertainty, or Varian (1999) for a more recent article. In such models it may be useful to identify assumptions on changes in distributions so that propositions can be found that say when a change in uncertainty strictly increases the payoff from the option-creating action rather than just not reducing the payoff.

II. The Model

Let there be an asset which has terminal value x_i with probability $f(x_i)$, where the values of x_i with positive probability are $x_1 < x_2 < ... < x_m$. A call option entitles its owner to buy the asset at price p at the terminal time if he wishes. Denote by $V_{call}(f, p)$ the current value to a risk-neutral owner of a call option on that asset with strike price p such that $x_1 < p < x_m$. This rules out strike prices of x_1 or below and x_m and above, because they would lead to riskless options which would be exercised always or never. It does allow a strike price that does not happen to equal any of the x_i. Similarly, denote by $V_{put}(f, p)$ the value of a put option that entitles its owner to sell the asset at price p at the terminal time if he wishes. Our focus will be on seeing how option values change if the underlying asset changes to follow a different distribution $g(x)$ which has the same mean as $f(x)$, so

$$Ex = \sum_{i=1}^{m} f(x_i)x_i = \sum_{i=1}^{m} g(x_i)x_i + \sum_{i=m+1}^{n} g(x_i)x_i, \tag{1}$$

where $x_{m+1} < x_{m+2} < ... < x_n$ are points in the support of g but not f. This allows, for example,

$x_{m+1} < x_1$: g can have positive probability on x values below or above the support of $f(x)$, or values between the x's in $f(x)$'s support. We will denote the cumulative distributions by $F(x)$ and $G(x)$.

Denote the discount factor, the present value of a dollar received at the terminal date, by β. The value of a call option is

$$\begin{aligned} V_{call}(f,p) &= \beta \sum_{i=1}^{m} f(x_i) Max\{0,(x_i - p)\} \\ &= \beta \sum_{i=j}^{m} f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \end{aligned} \tag{2}$$

The value of a put option is

$$\begin{aligned} V_{put}(f,p) &= \beta \sum_{i=1}^{m} f(x_i) Max\{0,(p - x_i)\} \\ &= \beta \sum_{i=1}^{j-1} f(x_i)(p - x_i) \text{ where } j : x_{j-1} < p < x_j \end{aligned} \tag{3}$$

Typically, as the option's maturity increases (the difference between the current date and terminal date), the dispersion of the possible values of the underlying asset also increase. In that case, another way to put the question of this paper (time discounting aside) is whether option value strictly increases with maturity. If the option is a real option where the value of the project becomes known after a certain date, then the riskiness will not increase over time—indeed, it will become zero once the uncertainty is resolved. Other kinds of assets, however, have price distributions which do become riskier in one or more of the senses to be defined below. If maturity increases risk, then the propositions below will apply directly to how an option's value changes with its maturity.

Since the model employs only two dates, the current date and the terminal date, it applies to "European" options, which cannot be exercised early, rather than "American" options, which can. European options are more appropriate here because early exercise occurs only when "option value" (in distinction from the "value of an option security") is unimportant because some other benefit of the option determines its value. Our task is to look at whether option value increases strictly or weakly with riskiness, but early exercise occurs only when small changes in riskiness are irrelevant. For example, two common reasons for early exercise are dividend payments (for calls) and re-investment value (for puts). If a stock is about to pay a dividend, it is possible that a call option should be exercised early to receive that dividend, despite the loss of the option value of waiting. If a put option is "in the money" with a high exercise price and the underlying stock's price close to zero (so it cannot fall much further), the put should be exercised immediately so the profit can be reinvested and earn a positive return. In both situations, the value of the option is equal to the immediate cash gain, and since the distribution of the future price of the underlying asset is not determining the option's value, small changes in riskiness will have no effect. Thus, the question of whether the value of options increases strictly with risk rather than possibly remaining unchanged is only interesting if we rule out early exercise.

Also, we will not be considering exotic options that convey purchase or sale rights over ranges of prices that do not slice the real line in two (e.g., the right to buy if the price is either in the interval [3, 5.6] or in [7, 26]). Neither the intuition nor the propositions extend to that kind of option, since an exotic option such as my parenthetic example can increase in value when probability shifts from the extremes to the middle, a reduction in risk.

Defining Risk

The standard definition of risk is based on the idea of the "mean-preserving spread."[1]

Definition 1a: *A* **mean-preserving spread** *consists of three numbers $s(y_1)$, $s(y_2)$, and $s(y_3)$ for $y_1 < y_2 < y_3$ such that*

$$s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0, \text{ (the mean is preserved)} \tag{4}$$

$$s(y_1) + s(y_2) + s(y_3) = 0, \text{ (the new probabilities sum to zero)} \tag{5}$$

and

$$s(y_1) > 0, \ s(y_2) < 0, \ s(y_3) > 0 \text{ (the probability is spread)} \tag{6}$$

Definition 1b: *Distribution $g(x)$ is* **riskier** *than $f(x)$ iff $g(x)$ can be reached from $f(x)$ by a sequence of mean-preserving spreads.*

This definition of risk has long been conventional, since it is equivalent to saying that the asset becomes less attractive to a risk- averse investor (one with a concave utility function) or that f is like g with noise added, although Definition 1b is only a partial ordering, and many pairs of distributions cannot be ranked by it. In the option context, Bliss (2001) shows the importance of using Definition 1b instead of defining risk as simply higher variance, which is not an equivalent definition. Variance can increase without making an asset less attractive to a

[1]Definition 1a is specialized to discrete probability distributions and uses the 3-point mean-preserving spread of Petrakis & Rasmusen (1994) rather than the 4-point mean-preserving spread of Rothschild & Stiglitz (1970), which has negative probability at two middle points rather than one. The two definitions of spread lead to equivalent definitions of risk. The 3-point definition is simpler, as well as allowing an easy fix of an error in Rothschild & Stiglitz's main proof).

risk-averse investor, and option values do not change in a uniform direction with changes in variance.[2]

A fundamental proposition in the theory of options is Proposition 1: option value is weakly increasing in risk— or, rephrased, optional value does not decline with risk.

Proposition 1 (Merton [1970] Theorem 8, p. 149): *If g is riskier than f, then $V_{call}(g,p) \geq V_{call}(f,p)$ and $V_{put}(g,p) \geq V_{put}(f,p)$ for any p.*

Proof: We will demonstrate the result if f and g differ by a single mean-preserving spread, which by induction implies it is true if they differ by a series of them. From (2), we must prove the following two inequalities:

$$V_{call}(g,p) - V_{call}(f,p) = \beta\{s(y_1)Max(y_1 - p, 0) + s(y_2)Max(y_2 - p, 0) + s(y_3)Max(y_3 - p, 0)\} \geq 0 \tag{7}$$

and

$$V_{put}(g,p) - V_{put}(f,p) = \beta\{s(y_1)Max(p - y_1, 0) + s(y_2)Max(p - y_2, 0) + s(y_3)Max(p - y_3, 0)\} \geq 0. \tag{8}$$

From equation (4), the spread is mean preserving, so $s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0$, and by equation (5) the spread's probabilities add to zero, so $[s(y_1) + s(y_2) + s(y_3)] = 0$. Adding or subtracting two zeroes results in zero, so we obtain an expression to be used later:

$$\sum_{i=1}^{3} s(y_i)(y_i - p) = \sum_{i=1}^{3} s(y_i)(p - y_i) = 0, \tag{9}$$

(i) If $p \leq y_1$, inequality (7) becomes

$$V_{call}(g,p) - V_{call}(f,p) = \beta\{s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p)\} \geq 0, \tag{10}$$

which is true by equation (9), and inequality (8) becomes the obviously true expression,

$$V_{put}(g,p) - V_{put}(f,p) = \beta\{s(y_1)(0) + s(y_2)(0) + s(y_3)(0)\} \geq 0. \tag{11}$$

(ii) If $p \geq y_3$, the reasoning is analogous to case (i). Inequality (7) becomes the obviously true $V_{call}(g,p) - V_{call}(f,p) = \beta \sum_{i=1}^{3} s(y_i)(0) \geq 0$ and inequality (8) becomes $V_{put}(g,p) - V_{put}(f,p) = \beta \sum_{i=1}^{3} s(y_i)(p - y_i) \geq 0$, which is true by equation (9).

[2] An example to show that increased variance can increase utility for a risk-averse person is the following. Let the utility be $U = x$ for $x \leq 10$ $U = 10 + \frac{x}{2}$ for $x \geq 10$, which is weakly concave. Suppose wealth is initially distributed as f: (.8:7, .2:12) , which has mean 8, variance $4(= .8 * 1^2 + .2 * 4^2)$, and utility $7.8(= .8 * 7 + .2 * 11)$. If the distribution is changed to g: (.2:0, .8:10), the mean is still 8, the variance increases to $16(= .2 * 8^2 + .8 * 2^2)$, and utility rises to $8(= .2 * 0 + .8 * 10)$. Kurtosis, which increases when moving weight to the tails of the distribution, is equally unreliable for ranking the riskiness of distributions; it starts as $52(= .8 * 1^4 + .2 * 4^4)$ in this example and rises to $832(= .2 * 8^4 + .8 * 2^4)$. Option value, too, can fall with variance: in this example, $V_{Call}(f, 11) = .2(12 - 11) = .2$ but $V_{Call}(g, 11) = 0$.

(iii) If $p \in (y_1, y_3)$ the options have positive value. Then, since $Max(y_1 - p, 0) = 0$ and $Max(y_3 - p, 0) = y_3 - p$, we can rewrite expression (7) as

$$V_{call}(g,p) - V_{call}(f,p) = \beta\{0 + s(y_2)Max(y_2 - p, 0) + s(y_3)(y_3 - p)\} \geq 0. \tag{12}$$

The last term of (12) is positive, and the middle term is either zero (if $y_2 \leq p$) or negative (if $y_2 > p$, since $s(y_2) < 0$). Equation (9) tells us that $s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p) = 0$, so since $s(y_1) > 0$, when $y_1 - p < 0$, as in case (iii), it must be that $s(y_2)(y_2 - p) + s(y_3)(y_3 - p) > 0$, and (12) is true even if $y_2 > p$.

Analogously, we can rewrite inequality (8) as

$$V_{put}(g,p) - V_{put}(f,p) = \beta\{s(y_1)(p - y_1) + s(y_2)Max(p - y_2, 0) + s(y_3)(0)\} \geq 0. \tag{13}$$

The first term of (13) is positive, and the middle term is either zero (if $y_2 \geq p$) or negative (if $y_2 < p$). Equation (9) tells us that $s(y_1)(p - y_1) + s(y_2)(p - y_2) + s(y_3)(p - y_3) = 0$, so since $s(y_3) > 0$, when $p - y_3 < 0$, as in case (iii), it must be that $s(y_2)(p - y_2) + s(y_3)(p - y_3) > 0$, and (13) is true even if $y_2 < p$. Thus the call and put options either increase in value after the spread or are unchanged. Q. E. D.

Compare Proposition 1 with Proposition 1a, which differs only in the strength of the inequality.

Proposition 1a (false): *If g is riskier than f, then $V_{call}(g,p) > V_{call}(f,p)$ and $V_{put}(g,p) < V_{put}(f,p)$ for any strike price p.*

Disproof. Consider a call option with an exercise price of 4.5 and the asset price distributions shown in Figure 1. $V_{call}(f, 4.5) = V_{call}(g, 4.5)$, even though g is riskier than f. The increase in risk has no effect because only changes in the probabilities of terminal values greater than 4.5 matter to the value of the call, and there are no such changes in the example. (Similarly, $V_{put}(f, 4.5) = V_{put}(g, 4.5)$.)

Propositions 1 and 1a differ only in the weakness of the inequality. That is enough, however, for "Proposition 1a: Option value increases with risk" to be false. Instead, we are left with "Proposition 1: Option value does not fall with risk," which although true, is very weak. That kind of statement can be made of any variable outside the model: "Option value does not fall with wealth," or "Option value does not fall with unemployment," or "Option value does not fall with the temperature in Bloomington." The statement "Option value does not fall with risk," however, though it does translate the mathematical notation of Proposition 1, is unnecessarily weak. We can instead say that "Option value does not fall with risk, and for at least one value of the strike price it increases." Propostion 1b expresses this in mathematical notation.

Proposition 1b: *If g is riskier than f, then there exists some exercise price p' such that the associated call and put options are more valuable under g than under f, but no exercise price p'' such that they are more valuable under f:*

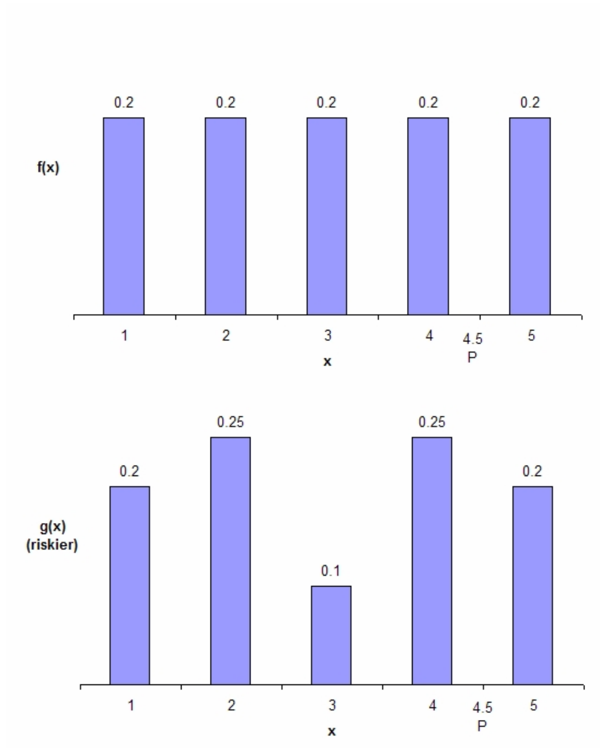

Figure 1: Risk Does Not Increase Option Value

$$\exists\ p' : V_{call}(f, p') < V_{call}(g, p') \textit{ and } V_{put}(f, p') < V_{put}(g, p');$$

but

$$\nexists\ p'' : V_{call}(f, p'') > V_{call}(g, p'') \textit{ or } V_{put}(f, p') > V_{put}(g, p').$$

Proof: Proposition 1's proof showed that if $p \in (y_1, y_3)$, where y_1 and y_3 are from one of the spreads that makes g riskier than f, then the values of the call strictly increases. Thus, simply pick p' from inside (y_1, y_3). That there exists no value p'' for which option value declines is a direct corollary of Proposition 1. QED.

III. New Definitions of Risk

Another approach is to find a definition of risk under which something like the false Proposition 1b becomes true, and the value of the option does strictly increase with "risk" regardless of the strike price.

Definition 2: *Distribution $g(x)$ is* **pointwise riskier** *than $f(x)$ iff f and g have the same mean and there exist points $\underline{x}$ and $\overline{x}$ in (x_1, x_m) such that*

(a) if $x < \underline{x}$, then $g(x) \geq f(x)$ and if $f(x) > 0$ then $g(x) > f(x)$;

(b) if $x \in [\underline{x}, \overline{x}]$, then $g(x) \leq f(x)$ and if $f(x) > 0$ then $g(x) < f(x)$;

(c) if $x > \overline{x}$, then $g(x) \geq f(x)$ and if $f(x) > 0$ then $g(x) > f(x)$.

Definition 2 says that $g(x)$ is pointwise riskier than $f(x)$ if it takes probability away from *each point* in the middle of the distribution and adds probability to *each point* at the two ends, while preserving the mean. Distribution $g(x)$ in Figure 2 is an example. Definition 2 also allows $g(x)$ to add probability to points outside the interval $[x_1, x_m]$— that is, beyond the two extremes of the support of $f(x)$. Pointwise riskiness captures something of the same intuition as the idea of the mean-preserving spread– that probability is to be moved from the middle to the ends of the distribution. If g is pointwise riskier than f, it is also riskier in the conventional sense. Note that the change from f to g need not be symmetric around the mean of the distribution. Every point on the "sides" must gain probability, but not necessarily the same amount of probability, nor must more extreme points gain more probability than less extreme ones. Note, too, that the definition applies directly to continuous distributions, where it has the additional implication that the densities of f and g cross twice, at $\underline{x}$ and $\overline{x}$, as density f does with g and h in Figure 4 below.

Pointwise riskiness will be sufficient but not necessary for option value to increase with risk for all strike prices, as we will see later once we have derived other results useful in proving sufficiency.[3] Distribution $h(x)$ in Figure 2 is an example in which h is not pointwise riskier than f, but $V_{call}(h, p) > V_{call}(f, p)$ for all p nonetheless. Pointwise riskiness is stronger than the standard risk of Definition 1b; if g is pointwise riskier than f it is riskier too, but g could be riskier without being pointwise riskier.[4]

[3] Since pointwise riskiness and strict second order stochastic dominance both can be defined in terms of functions that cross a limited number of times, the reader may wonder if they are the same. Distribution F strictly second-order stochastically dominates G if $\int_0^t G(x)dx > \int_0^t F(x)dx$ for all values of t such that $G(t) > 0$ and $G(t) < 1$. It could be, however, that G is pointwise riskier than F without F strictly second-order dominating G. Suppose, for example, that F is uniform, with $F(1) = 0.25, F(2) = 0.5, F(3) = 0.75, F(4) = 1$ and G moves weight from the middle to the tails and is pointwise riskier so $G(1) = 0.30, G(2) = 0.5, G(3) = .7, G(4) = 1$. If we define $D_F(t) \equiv \int_0^t F(x)dx$ (and similarly for G) then $D_F(1) = 0.25, D_F(2) = 0.75, D_F(3) = 1.5$ and $D_G(1) = .30, D_G(2) = .8, D_G(3) = 1.5$. Since $D_F(3) = D_G(3)$, F does not strictly dominate G. F does weakly dominate G, as we would expect since G is riskier in the conventional sense.

[4] Other strengthened definitions of riskiness have been proposed in the context of comparative statics— for example, how an agent's investment portfolio responds to an increase in risk. Often the response to an increase in standard risk is ambiguous,and clear predictions can only be made for increases in stronger, nonstandard, forms of riskiness. Gollier (1995) has shown what definition is both necessary and sufficient for making predictions about the behavior of any risk-averse agent when riskiness increases. As one might expect from such a strong result, the definition of riskiness he uses is complicated enough to be difficult to use in applications. Another approach, transforming the random variable, was pioneered in Sandmo (1971). In this approach, the distribution function $F(x)$ is transformed to a new distribution by replacing x with $t(x)$ for a suitable function t that keeps the mean constant but increases risk. Sandmo (1971) uses a transformation in which $t(x) - x$ is linear. Meyer & Ormiston (1989) generalize to the "simple transformation," in which $t(x) - x$ can be any monotonic function. This approach provides a way to think about "stretching out" distributions, but it will increase the support of the distribution, something not necessarily true of pointwise riskiness.

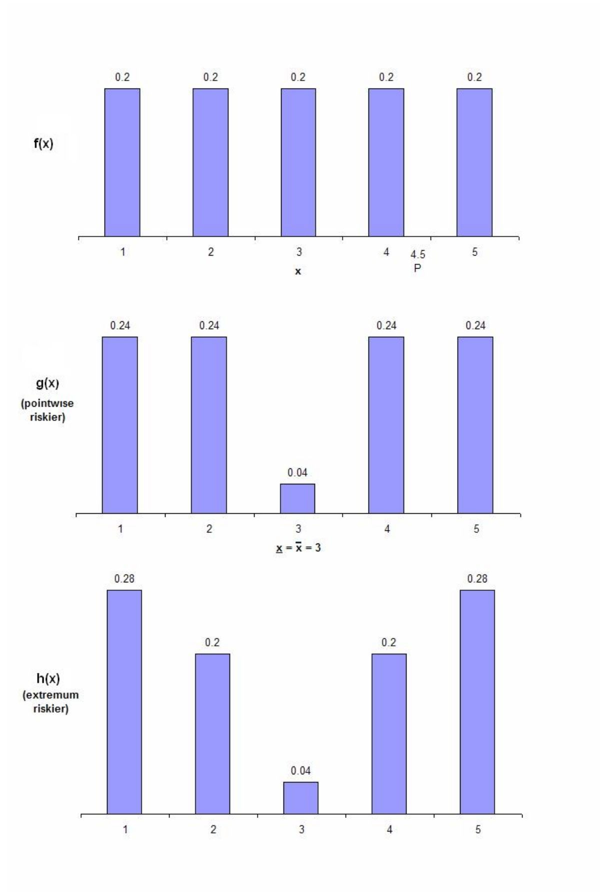

Figure 2: Pointwise and Extremum Riskiness

Our other new definition of risk is one which will be *necessary* for extra risk to increase option value: extremum risk.

Definition 3: *Distribution* $g(x)$ *is* **extremum riskier** *than* $f(x)$ *iff* $G(x_1 + \epsilon) > F(x_1 + \epsilon)$ *and* $G(x_m - \epsilon) < F(x_m - \epsilon)$ *for arbitrarily small* $\epsilon > 0$.

This is stated in terms of the cumulative distributions of f and g, but in discrete distributions it has easy-to-understand implication that (a) either $f(x_1) < g(x_1)$ or $g(x) > 0$ for some $x < x_1$, and (b) either $f(x_m) < g(x_m)$ or $g(x) > 0$ for some $x > x_m$. Definition 3 is slightly stronger than this, however, because it says that if g extends to more extreme values of x it must also increase the total probability of values of x beyond x_1 and x_m.

In Figure 2 (above), distribution $h(x)$ is extremum riskier than $f(x)$ or $g(x)$. In Figure 3 (below), $g(x)$ is both riskier and extremum riskier than $f(x)$, but not riskier, since the probability of both the mean and the extreme values have increased.

The definition of extremum riskiness applies to continuous as well as discrete

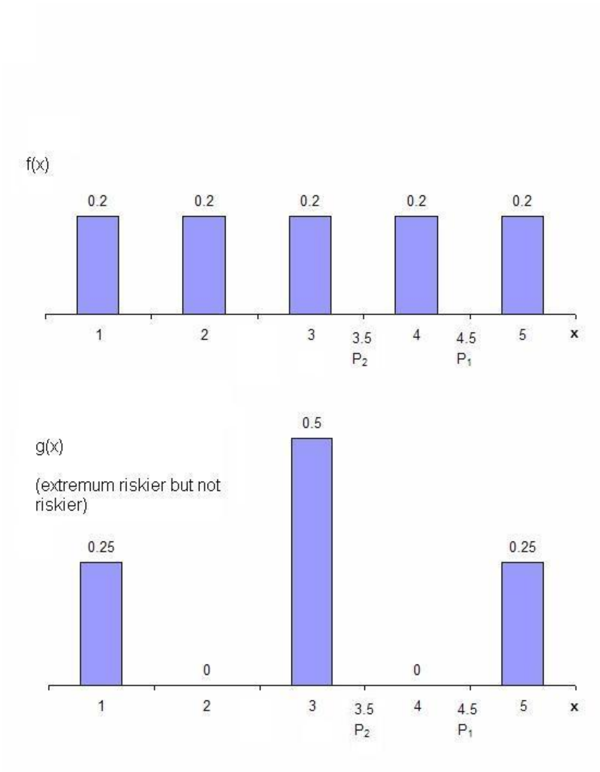

Figure 3: Extremum Riskiness versus Conventional Risk

distributions. Cumulative distributions must be used because if f is a continuous density then each of the extrema has zero probability, even if positive density, and to change the value of an option it is necessary to change probabilities over an interval of f's support, not just over one point. The density g must put more probability on the intervals $[-\infty, x_1 + \epsilon]$ and $[x_m - \epsilon, \infty]$. If the density f's support is unbounded (which cannot happen with a discrete distribution) then the numbers x_1 and x_m are no longer the bounds of f's support. We can continue to define them, however, as the bounds of the interval in which the strike price lies, so $x_1 < p < x_m$, for arbitrarily large and small bounds. The definitions and propositions apply within any such range of strike prices, thus allowing.

Figure 4 shows how the definitions apply to an example of a continuous distribution. All four densities $g(x)$, $h(x)$, $l(x)$, and $m(x)$ are riskier than $f(x)$, which is shown in gray in all five diagrams. Densities $g(x)$ and $h(x)$ are both pointwise riskier and extremum riskier than $f(x)$. Density $l(x)$ is extremum riskier, but not pointwise riskier, since it leaves the density of some points unchanged. Density $m(x)$ is riskier than $f(x)$, since it spreads probability from the center to the two peaks on each side, but it is neither pointwise nor extremum riskier, since it leaves the extreme densities unchanged.

IV. The Effects of Extremum and Pointwise Riskiness on Option Value

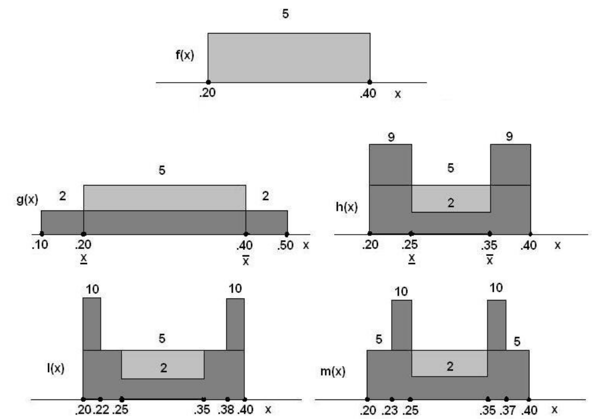

Figure 4: Riskiness in Continuous Distributions

Proposition 2: *Consider two distributions f and g. A necessary condition for it to be true that $V_{call}(g,p) > V_{call}(f,p)$ for any strike price p is that g be extremum-riskier than f. That g be extremum-riskier than f is also a necessary condition for $V_{put}(g,p) > V_{put}(f,p)$ for any strike price p.*

Proof: Let us begin with calls. Since we must show that for any call, $V_{call}(g,p) > V_{call}(f,p)$, that must be true for $p = x_1 + \epsilon$ and $p = x_m - \epsilon$, for arbitrarily small $\epsilon > 0$. We need to show that $V_{call}(g,p) - V_{call}(f,p) > 0$, so from equation (1)'s notation for g's support and equation (2) for call value, we must show that

$$\beta\left(\sum_{i=1}^{m} g(x_i)Max\{0,(x_i-p)\} + \sum_{i=m+1}^{n} g(x_i)Max\{0,(x_i-p)\}\right) - \beta\sum_{i=1}^{m} f(x_i)Max\{0,(x_i-p)\} > 0 \tag{14}$$

where $x_{m+1} < x_{m+2} < ... < x_n$ are points in the support of g but not f. If $p = x_m - \epsilon$, expression (14) becomes

$$\beta\left(g(x_m)(x_m - [x_m-\epsilon]) + \sum_{i=j}^{n} g(x_i)(x_m - [x_m-\epsilon])\right) - \beta f(x_m)(x_m - [x_m-\epsilon]) > 0 \quad \text{or}$$
$$\left(\left[g(x_m) + \sum_{i=j}^{n} g(x_i)\right] - f(x_m)\right)(\epsilon) > 0, \tag{15}$$

where j is chosen so $x_j < x_m - \epsilon < x_{j+1}$: points $x_i \geq x_j$ are the points in g's support but not in f's support that make the option worth exercising (a possibly empty set). Since $f(x_m) = 1 - F(x_m - \epsilon)$ and $\left[g(x_m) + \sum_{i=j}^{n} g(x_i)\right] = 1 - G(x_m - \epsilon)$, the last inequality in (15) is true if and only if $F(x_m - \epsilon) > G(x_m - \epsilon)$, one of the two conditions in Definition 3.

If $p = x_1 + \epsilon$, the call will be exercised except when $x_i \leq x_1$. Thus, we can rewrite the value of the call as the value of the underlying asset minus the discounted exercise price minus the discounted expected value of exercising the option when $x_i \leq x_1$ (which is negative). Then the difference in the call values, $V_{call}(g, x_1 + \epsilon) - V_{call}(f, x_1 + \epsilon)$, is

$$
\begin{aligned}
&\beta \left((Ex - p) - \left(g(x_1)(x_1 - [x_1 + \epsilon]) + \sum_{i=m+1}^{k} g(x_i)(x_1 - [x_1 + \epsilon]) \right) \right) \\
&-\beta \left((Ex - p) - f(x_1)(x_1 - [x_1 + \epsilon]) \right) \text{ or} \\
&\left(\left[g(x_1) + \sum_{i=m+1}^{k} g(x_i) \right] - f(x_1) \right) (\epsilon)
\end{aligned} \tag{16}
$$

where k is chosen so $x_k < x_1 + \epsilon < x_{k+1}$; points $x_i \leq x_k$ are the set of points in g's support but not in f's that make the option not worth exercising (a possibly empty set). Since $f(x_1) = F(x_1 + \epsilon)$ and $\left[g(x_1) + \sum_{i=m+1}^{k} g(x_i) \right] = G(x_1 + \epsilon)$, expression (16) is positive if and only if $F(x_1 + \epsilon) < G(x_1 + \epsilon)$, the second condition in Definition 3.

The proof for the put option is similar. If $p = x_1 + \epsilon$, the put will only be exercised if $x \leq x_1$, so, using the same definition of k as in inequality (16) to indicate small values of x, $V_{put}(g, p) - V_{put}(f, p) > 0$ if

$$
\begin{aligned}
&\beta \left(g(x_1)([x_1 + \epsilon] - x_1) + \sum_{i=m+1}^{k} g(x_i)([x_1 + \epsilon] - x_1) \right) - \beta f(x_m)([x_1 + \epsilon] - x_1) > 0 \text{ or} \\
&\left(\left[g(x_m) + \sum_{i=m+1}^{k} g(x_i) \right] - f(x_m) \right) (\epsilon) > 0,
\end{aligned} \tag{17}
$$

Inequality (17) is true if and only if $F(x_1 + \epsilon) < G(x_1 + \epsilon)$, the second condition in Definition 3.

If $p = x_m - \epsilon$, on the other hand, the put will always be exercised unless $x \geq x_m$. Thus, using the same definition of j as in inequality (15) to indicate large values of x, $V_{put}(g, p) - V_{put}(f, p) > 0$ if

$$
\begin{aligned}
&\beta \left((p - Ex) - \left(g(x_m)([x_m - \epsilon] - x_m) + \sum_{i=j}^{n} g(x_i)([x_m - \epsilon] - x_m) \right) \right) \\
&-\beta \left((p - Ex) - f(x_m)([x_m - \epsilon] - x_m) \right) > 0 \text{ or} \\
&\left(\left[g(x_m) + \sum_{i=j}^{n} g(x_i) \right] - f(x_m) \right) (\epsilon) > 0
\end{aligned} \tag{18}
$$

Inequality (18) is true if and only if $1 - G(x_m - \epsilon) > 1 - F(x_m - \epsilon)$, which is equivalent to $F(x_m - \epsilon) > G(x_m - \epsilon)$, one of the two conditions in Definition 3. Thus, both of the conditions in Definition 3 are also necessary for all puts to increase in value. QED.

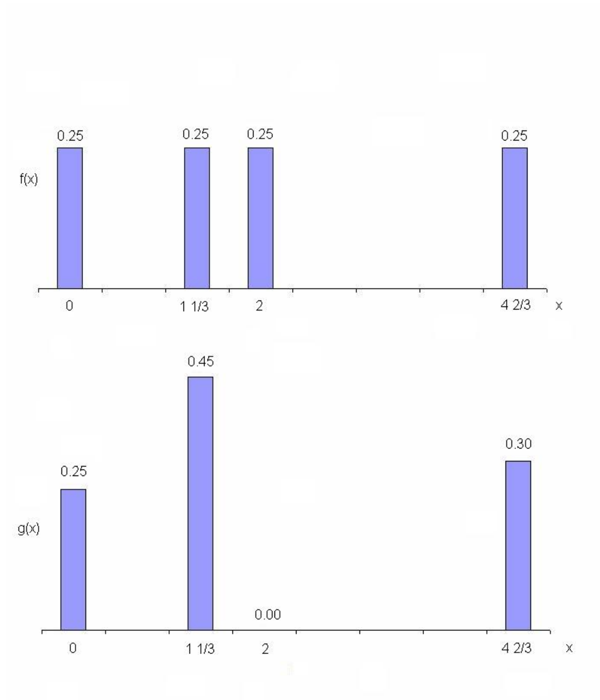

Figure 5: Why Extremum Risk Needs to Spread Probability at Both Ends

Even if Proposition 2 were stated only in terms of increasing the value of calls, not of both calls and puts, both conditions for extremum riskiness in Definition 3 would be necessary. The density g must add probability to f at *both* extremes, not just at the maximum. This was part of the proof of Proposition 2, but the numerical example illustrated in Figure 5 is helpful in understanding why. In Figure 5, g is made riskier than f by shifting probability away from $x = 2$, the mean, to $x = 1\,\frac{1}{3}$ and $x = 4\,\frac{2}{3}$. As a result, g has more probability than f on the maximum, $x = 4\,\frac{2}{3}$, but no more probability on the minimum, $x = 0$. A call with a strike price above $1\,\frac{1}{3}$ is more valuable under g than under f. But think about a call with a strike price of 1. It will have equal value under f and g, because the mean of the distribution conditional on x being greater than 1 has not changed. The probability of the state of the world ($x = 0$) in which the call is *not* exercised is the same with either distribution.

The general problem is that unless both extrema are increased in g, it is possible to find a strike price such that the total amount of probability on prices above the strike price is unchanged. If the minimum does not increase, as in Figure 5, then choose the strike price to be very low, just above the minimum. The call is then a bet that the price will exceed the minimum, and the probability of winning that bet is the same for f and g. If, on the other hand, the maximum does not increase, choose the strike price to be very high, just below the maximum.

Why is extremum risk just a necessary, not sufficient? Look back at Figure 3. In Figure 3,

$g(x)$ has more probability at the extremes than $f(x)$ does– the probability of each extreme is .25 instead of .20— but it is not riskier in the conventional sense, because it cannot be reached from $f(x)$ by a sequence of mean-preserving spreads. If the strike price is 4.5, then the call's value is higher under distribution $g(x)$, because the outcome $x = 5$ occurs with probability .25 instead of $f(x)$'s .20. $V_{call}(f, 4.5) = .20(5 - 4.5) = .10 < V_{call}(g, 4.5) = .25(5 - 4.5) = .125$. If the strike price is 3.5, however, the call's value is higher under distribution $f(x)$, because under $g(x)$ the outcomes $x = 4$ and $x = 5$ together occur with probability .25 instead of .40 and $V_{call}(f, 3.5) = .20(4 - 3.5) + .20(5 - 3.5) = .40 > V_{call}(g, 3.5) = .00(4 - 3.5) + .25(5 - 3.5) = .375$.

Extremum riskiness already implies that g is not less risky than f, since more weight is in the far tail of the distribution in g, but it might be that f and g are not ordered by risk. Although neither conventional nor extremum riskiness is by itself sufficient to make calls more valuable, in combination they do yield a sufficient condition, as stated in Proposition 3.[5]

Proposition 3: *Consider two distributions f and g. A sufficient condition for it to be true that $V_{call}(g, p) > V_{call}(f, p)$ for any strike price p is that*
(a) g is extremum-riskier than f; and
(b) g is riskier than f.
This is also a sufficient condition for it to be true that $V_{put}(g, p) > V_{put}(f, p)$.

Proof: From Proposition 1 we know that if condition (b) is true, then $V_{call}(g, p) \geq V_{call}(f, p)$ and $V_{put}(g, p) \geq V_{put}(f, p)$, that is, Proposition 3's inequalities are true at least weakly. Thus, all that we need to show is that condition (a) makes the inequalities strict. The proof of Proposition 1 showed that if a mean-preserving spread that made g riskier than f changed probability on three points $y_1 < y_2 < y_3$, then if the option's strike price were $p \leq y_1$ or $p \geq y_3$, the option's value would be the same under f as under g.

Since g may be derived from f by a series of mean-preserving spreads, let y_1^* be the lowest x value that is changed and y_3^* the highest. Consider either a call or a put. If the option's strike price were $p \leq y_1^*$ or $p \geq y_3^*$, an option's value would be the same under f as under g. That is the possibility we are trying to rule out. But condition (a) says that g is extremum riskier. That implies that the probability of x_i less than or equal to x_1 increases, so $y_1^* \leq x_1$, and that the probability of x_i greater than or equal to x_m increases, and $y_3^* \geq x_m$. Thus, it is impossible (since we rule out the riskless options with $p = x_1$ or $p = x_m$) that $p \leq y_1^*$ or $p \geq y_3^*$. As a result, the option values cannot be equal for any p and it must be that both $V_{call}(g, p) > V_{call}(f, p)$ and $V_{put}(g, p) > V_{put}(f, p)$. Q.E.D.

You might ask why I did not write Proposition 3 to say that conditions (a) and (b) are

[5]This combination of conventional risk and extremum riskiness is the same idea as the "strong increase in risk" of Meyer & Ormiston (1985), although their formal statement allows a strong increase to be no change at all— they use weak inequalities in their definition. Thus, loosely speaking, a "strong increase in risk" strictly increases the value of an option.

jointly necessary and sufficient, rather than just sufficient. If options on g are to be always more valuable than options on f, isn't it necessary that g be both riskier and extremum- riskier than f? No, as we will see by demonstrating the falsity of Proposition 3a.

Proposition 3a: (False) *Consider two distributions f and g. The following two conditions are necessary and sufficient for it to be true that $V_{call}(f,p) < V_{call}(g,p)$ or $V_{put}(f,p) < V_{put}(g,p)$ for any strike price p:*
(a) g is extremum-riskier than f; and
(b) g is riskier than f.

Disproof: Proposition 3 tells us that Conditions (a) and (b) are jointly sufficient for options on f to be less valuable. Proposition 2 tells us that Condition (a) by itself is necessary for options on f to be less valuable. Thus, what we need to show to prove Proposition 3a is false is that there exist distributions f and g such that Condition (b) is violated but nonetheless $V_{call}(f,p) < V_{call}(g,p)$ for any p— i.e., that g's options are always more valuable but g is not riskier than f. The counterexample in Figure 6 will do this. Distribution g is extremum riskier than f, but it is not riskier, because it has more probability at the mean, $x = 5$. The distributions f and g both have mean $Ex = 5$ and cannot be ordered by risk, yet we will see that all options on g are more valuable.

The value of the options on an asset with distribution f and strike price $p \in (2,8)$ are, from equation (2),

$$\begin{aligned} V_{call}(f,p) &= Max\{0, .25(2-p)\} + Max\{0, .25(4-p)\} + Max\{0, .25(6-p)\} \\ &\quad + Max\{0, .25(8-p)\} \\ V_{put}(f,p) &= Max\{0, .25(p-2)\} + Max\{0, .25(p-4)\} + Max\{0, .25(p-6)\} \\ &\quad + Max\{0, .25(p-8)\} \end{aligned} \tag{19}$$

and if the distribution is g they are

$$\begin{aligned} V_{call}(g,p) &= Max\{0, .30(1-p)\} + Max\{0, .40(5-p)\} + Max\{0, .30(9-p)\} \\ V_{put}(f,p) &= Max\{0, .30(p-1)\} + Max\{0, .40(p-5)\} + Max\{0, .30(p-9)\} \end{aligned} \tag{20}$$

The possible values of p for our comparison go from $p = 2$ to $p = 8$, where the endpoints are not possible (as the option would then be always or never exercised). We will split this up into four intervals and examine each in turn.

$p \in (2,4]$. In this case, $V_{call}(f,p) = .25(4-p) + .25(6-p) + .25(8-p) = 4.5 - .75p < V_{call}(g,p) = .40(5-p) + .30(9-p) = 4.7 - .70p$. The put values are $V_{put}(f,p) = .25(p-2) = .25p - .5 < V_{put}(g,p) = .3(p-1) = .3p - .3$. Thus, g has the more valuable options.

$p \in (4,5]$. In this case, $V_{call}(f,p) = .25(6-p) + .25(8-p) = 3.5 - .50p$, while $V_{call}(g,p) = .40(5-p) + .30(9-p) = 4.7 - .70p$. It is true that $3.5 - .50p < 4.7 - .70p$ if

$.20p < 1.2$, which is true if $p < 6$, and in particular for $p \in [4,5]$. The put values are $V_{put}(f,p) = .25(p-2) + .25(p-4) = .5p - 1.5$, while $V_{put}(g,p) = .3(p-1) = .3p - .3$. The put on g is more valuable if $p < 6$, just as with the call, so g has the more valuable options.

$p \in [5,6]$. In this case, $V_{call}(f,p) = .25(6-p) = .3.5 - .5p$, while $V_{call}(g,p) = .30(9-p) = 2.7 - .30p$. It is true that $3.5 - .5p < 2.7 - .30p$ if $.8 < .2p$, which is true if $p > 4$, and in particular if $p \in [5,6]$. The put values are
$V_{put}(f,p) = .25(p-2) + .25(p-4) = .5p - 1.5$, while
$V_{put}(g,p) = .3(p-1) + .4(p-5) = .7p - 2.3$, which is the more valuable if $p > 4$, just as with the call. Thus, g has the more valuable options.

$p \in [6,8)$. In this case, $V_{call}(f,p) = .25(8-p) = 2 - .25p$, while $V_{call}(g,p) = .30(9-p) = 2.7 - .30p$. It is true that $2 - .25p < 2.7 - .3p$ if $p < 14$, and in particular if $p \in [6,8]$. The put values are
$V_{put}(f,p) = .25(p-2) + .25(p-4) + .25(p-6) = .75p - 3$, while
$V_{put}(g,p) = .3(p-1) + .4(p-5) = .7p - 2.3$, which is the more valuable if $p < 14$, just as with the call. Thus, g has the more valuable options.

Combining all four cases, we see that for any $p \in (2,8)$, g has more valuable options. Q. E. D.

To understand Proposition 3a's falseness, start with the simpler idea that an option with price p can be more valuable under distribution g even if g is not riskier than f. That is true because for some particular p, the call's value is $\sum_{i=j}^{m} f(x_i)(x_i - p)$ for $j : x_{j-1} < p < x_j$, which depends on all of the f distribution for every $x_i > p$ but not on every x_i individually. Thus, it is possible that $g(x_k) < f(x_k)$ for some particular value of $x_k > p$ in a way that makes it impossible to rank f and g by risk, but for that to be outweighed by g's greater weight on most high values of x_i. We can generalize this to the idea that an option can be more valuable for *any* price p even though risk does not rise. We can find a g that puts so much weight on its extrema compared to f that g's expected values over $x_i > p$ will be greater even if it puts more weight on the mean of x too.

Now let us leave extremum riskiness and look back to the second new definition of "riskier": "pointwise riskiness". In applications, it is convenient to specify a simple sufficient condition for options on one distribution to have higher value than those on another. Proposition 4 says that pointwise riskiness is such a condition.

Proposition 4: *If g is pointwise riskier than f, then for any p, $V_{call}(g,p) > V_{call}(f,p)$ and $V_{put}(g,p) > V_{put}(f,p)$.*

Proof: If g is pointwise riskier than f, then it is also riskier and extremum riskier. It is riskier because we can move from f to g by a series of mean-preserving spreads that take probability away from the middle interval $[\underline{x}, \overline{x}]$ and move it to the extremes. It is extremum riskier because $x_1 < \underline{x}$ and $x_m > \overline{x}$, so g puts more probability on x_1 and x_m than f does. It follows from Proposition 3 that calls and puts on g will be more valuable than calls and puts on f. Q.E.D.

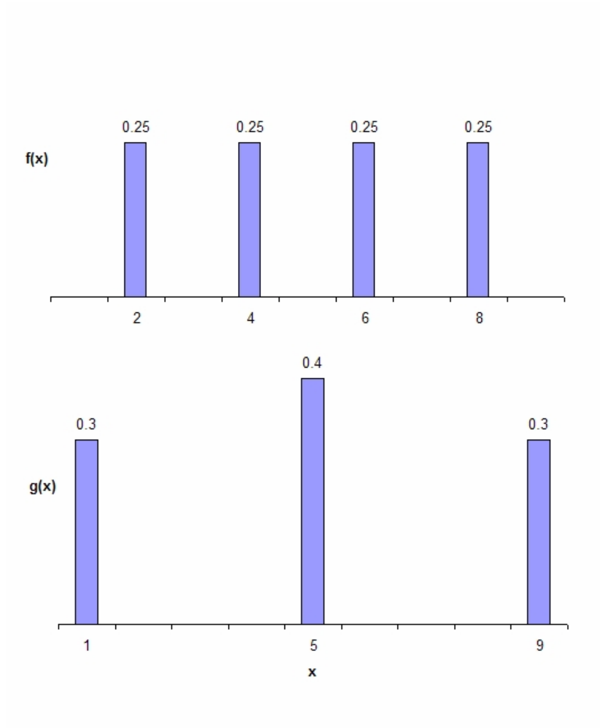

Figure 6: Conventional and Extremum Riskiness Are Not Jointly Necessary to Increase Option Value

We have already found one sufficient condition for options on g to be more valuable than options on f, the combination of riskiness and extremum riskiness in Proposition 3. Proposition 3, in fact, provides a tighter sufficient condition than Proposition 4. If g is pointwise riskier than f it is always both riskier and extremum riskier— but g can be riskier and extremum riskier without being pointwise riskier. Pointwise riskiness is nonetheless a useful concept, because it is simpler and more intuitive than the combined conditions.

V. Concluding Remarks

If distribution g is riskier than distribution f, then any call option on an asset whose value has distribution g will be at least as valuable as the equivalent option on an asset with distribution f. But the option on g might not be *more* valuable, because the values might be equal. This paper has developed a necessary condition for all call options on an asset whose value has distribution g to be strictly more valuable than the equivalent option on an asset with distribution f, and two sufficient conditions for it, differing in strength and convenience. The necessary condition is that g be "extremum riskier": it must put more probability on the extreme values of the asset. One sufficient condition is that g be not only extremum riskier, but also riskier under the conventional definition of risk– that g can be reached from f by a series of mean-preserving spreads. A second sufficient condition, more restrictive but simpler, is that g

be "pointwise riskier": asset values in the middle of g have higher probability than under f, and asset values outside the middle have lower probability.

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