Defining the Mean-Preserving Spread: 3-pt versus 4-pt

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Abstract

The standard way to define a mean-preserving spread is in terms of changes in the probability at four points of a distribution (Rothschild and Stiglitz [1970]). Our alternative definition is in terms of changes in the probability at just three points. Any 4-pt mean-preserving spread can be constructed from two 3-pt mean-preserving spreads, and any 3-pt mean-preserving spread can be constructed from two 4-pt mean-preserving spreads. The 3-pt definition is simpler and more often applicable. It also permits easy rectification of a mistake in the Rothschild-Stiglitz proof that adding a mean-preserving spread is equivalent to other measures of increasing risk.

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The question of what "risky" means is central to information economics. One way to define risk is to say that asset X is riskier than asset Y if every individual with a strictly concave utility function prefers Y to X. By another definition, X is riskier if it is distributed like Y plus an additional asset with zero mean and positive variance. Still another definition says that X is riskier if the distribution of Y has the same mean as X but dominates it in the sense of second-order stochastic dominance or adding a mean-preserving spread. It turns out that all three definitions are equivalent. This is best known to economists from the classic article of Rothschild & Stiglitz (1970), although as the authors themselves pointed out two years later, various components of their key theorem could have been drawn from existing mathematics (Blackwell and Girshick, 1954; Hardy, Littlewood, and Polya, 1953).

The standard way to define a mean-preserving spread is in terms of changes in the probability at four points of a distribution (Rothschild and Stiglitz [1970]). Our alternative definition is in terms of changes in the probability at just three points. Any 4-pt mean-preserving spread can be constructed from two 3-pt mean-preserving spreads, and any 3-pt mean-preserving spread can be constructed from two 4-pt mean-preserving spreads. The 3-pt definition is simpler and more often applicable. It also permits easy rectification of a mistake in the Rothschild-Stiglitz proof that adding a mean-preserving spread is equivalent to other measures of increasing risk.

Let F and G be cumulative density functions of the discrete random variables X and Y, where $Pr(X = a_i) = f_i$, $Pr(Y = a_i) = g_i$, and $\gamma_i = g_i - f_i$. A mean-preserving spread (MPS) is a set of γ's such that if Y differs from X by a single MPS then Y has the same mean as X, but more weight in the tails. Rothschild and Stiglitz use the following 4-pt definition, illustrated in Figure 1a:

A **4pt MPS** is a set of four locations $a_1 < a_2 < a_3 < a_4$ and four probabilities $\gamma_1 \geq 0, \gamma_2 \leq 0, \gamma_3 \leq 0, \gamma_4 \geq 0$ such that $-\gamma_1 = \gamma_2$, $\gamma_3 = -\gamma_4$, and $\sum_i \gamma_i a_i = 0$.

As an alternative, we suggest the 3-pt MPS, illustrated in Figure 1b.

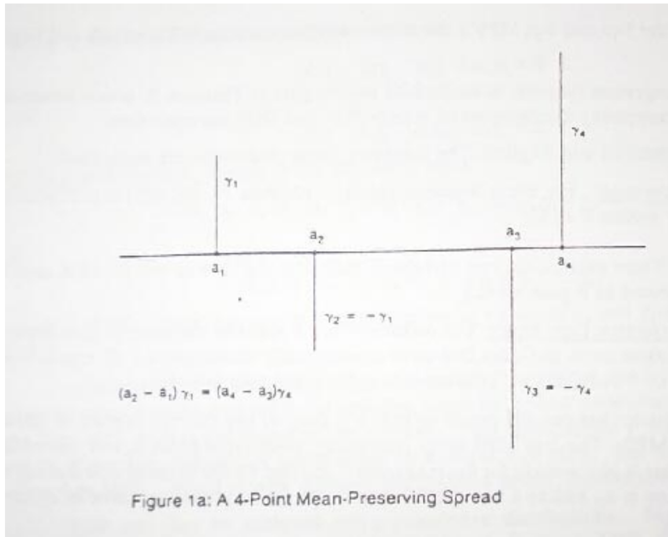

Figure 1:

A **3pt MPS** is a set of three locations $a_1 < a_2 < a_3$ and three probabilities $\gamma_1 \geq 0, \gamma_2 \leq 0, \gamma_3 \geq 0$ such that $\sum_i \gamma_i = 0$ and $\sum_i \gamma_i a_i = 0$.

Compared to the 4-pt MPS, the 3-pt MPS is simpler, more intuitive, and more often applicable (it requires one fewer point of positive probability). Mathematically, the two definitions are equivalent. Proof of this equivalence exists implicitly but not very accessibly in the mathematics literature (the Fishburn [1982] theorem on the properties of the convex cones of certain sets of signed measures in multiple dimensions). We will prove the equivalence more simply here by construction.

THEOREM 1a: *Any 4-pt MPS can be constructed from two 3-pt MPS's.*

Proof. Begin with a 4-pt MPS:

$$MPS^0 = \{a_1, a_2, a_3, a_4; \gamma_1, -\gamma_1, -\gamma_4, \gamma_4\}.$$

Figure 2:

We claim that MPS^0 is the sum of the 3-pt MPS's

$$MPS^1 = \{a_1, a_2, a_3; \gamma_1, -(\gamma_1 + y), y\}$$

and

$$MPS^2 = \{a_2, a_3, a_4; y, -(y + \gamma_4), \gamma_4\},$$

where $y > 0$, and

$$\gamma_1 a_1 - (\gamma_1 + y) a_2 + y a_3 = 0 \tag{1}$$

These definitions make MPS^1 and MPS^2 spreads, and condition (1) makes MPS^1 mean-preserving. When added together, spreads MPS^1 and MPS^2 equal

$$\{a_1, a_2, a_3, a_4; \gamma_1 + 0, -(\gamma_1 + y) + y, y - (y + \gamma_4), 0 + \gamma_4\},$$

which is MPS^0. It must be shown that MPS^2 is mean-preserving. The fact that MPS^0 is mean-preserving implies that

$$\gamma_1 a_1 - \gamma_1 a_2 - \gamma_4 a_3 + \gamma_4 a_4 = 0 \tag{2}$$

Equating (1) and (2) gives

$$\gamma_1 a_1 - \gamma_1 a_2 - \gamma_4 a_3 + \gamma_4 a_4 = \gamma_1 a_1 - (\gamma_1 + y) a_2 + y a_3, \tag{3}$$

which is equivalent to

$$y a_2 + (-y - \gamma_4) a_3 + \gamma_4 a_4 = 0. \tag{4}$$

Equation (4) is the condition that MPS^2 be mean-preserving. Thus, all three spreads are mean-preserving. (Note that the construction can use any of a broad set of different values for y.)
Q.E.D.

THEOREM 1b: *Any 3-pt MPS can be constructed from two 4-pt MPS's.*

Proof. Begin with the 3-pt MPS

$$MPS^0 = (a_1, a_3, a_5; \gamma_1, -(\gamma_1 + \gamma_3), \gamma_3).$$

We claim MPS^0 is the sum of the 4-pt MPS's

$$MPS^1 = (a_1, a_2, a_3, a_4; \gamma_1, -\gamma_1, -\gamma_3, \gamma_3)$$

and

$$MPS^2 = (a_2, a_3, a_4, a_5; \gamma_1, -\gamma_1, -\gamma_3, \gamma_3),$$

where a_2 and a_4 are chosen to satisfy

$$a_1\gamma_1 - a_2\gamma_1 - a_3\gamma_3 + a_4\gamma_2 = 0 \tag{5}$$

and

$$a_2\gamma_1 - a_3\gamma_1 - a_4\gamma_3 + a_5\gamma_3 = 0. \tag{6}$$

When added together, spreads MPS^1 and MPS^2 equal

$$\{a_1, a_2, a_3, a_4, a_5; \gamma_1 + 0, -\gamma_1 + \gamma_1, -\gamma_3 - \gamma_1, \gamma_3 - \gamma_3, 0 + \gamma_3\},$$

or

$$\{a_1, a_2, a_3, a_4, a_5; \gamma_1, 0, -(\gamma_3 + \gamma_1), 0, \gamma_3\},$$

which is MPS^0. (Note that any of a large number of values of a_2 and a_4 satisfy (5) and (6).)
Q.E.D.

Although the two definitions are mathematically equivalent, we suggest that the 3-pt MPS is superior. A good definition does two things: it defines a useful idea, and it does so in a way that is simple and convenient to use. The 3-pt and 4-pt MPS define the same useful idea. But the 3-pt MPS has a distinct, if modest, advantage in simplicity and convenience. It is the simplest possible definition, because two points cannot spread probability while preserving the mean. It matches the intuition of the spread exactly—to take probability from a point and move it to each side of that point in such a way that the mean stays the same, whereas the 4-pt MPS takes probability away from two points. Finally, the 3-pt MPS is more often applicable, since it uses fewer points of the support. Two distributions F and G, each with three points of positive probability, might differ by a single 3-pt MPS, but to move between them using 4-pt MPS's would require negative probabilities in the intermediate step.

Risk can also be analyzed using cumulative distributions. Figure 2 shows that if cumulative distribution G equals cumulative distribution F plus a MPS, then the difference $G-F$ looks like Figure 2a for a 4-pt MPS, and like Figure 2b for a 3-pt MPS.

Rothschild and Stiglitz use the "integral conditions" to look at spreads using cumulative distributions. The first integral condition preserves the mean:

$$\int_0^1 [G(x) - F(x)]dx = 0; \tag{7}$$

and the second integral condition makes the change a spread:

$$\int_0^y [G(x) - F(x)]dx \geq 0, \quad 0 \leq y \leq 1. \tag{8}$$

Condition (8) ensures that F dominates G in the sense of 2nd-order stochastic dominance. If F and G differ by either a 3-pt or a 4-pt MPS, then condition

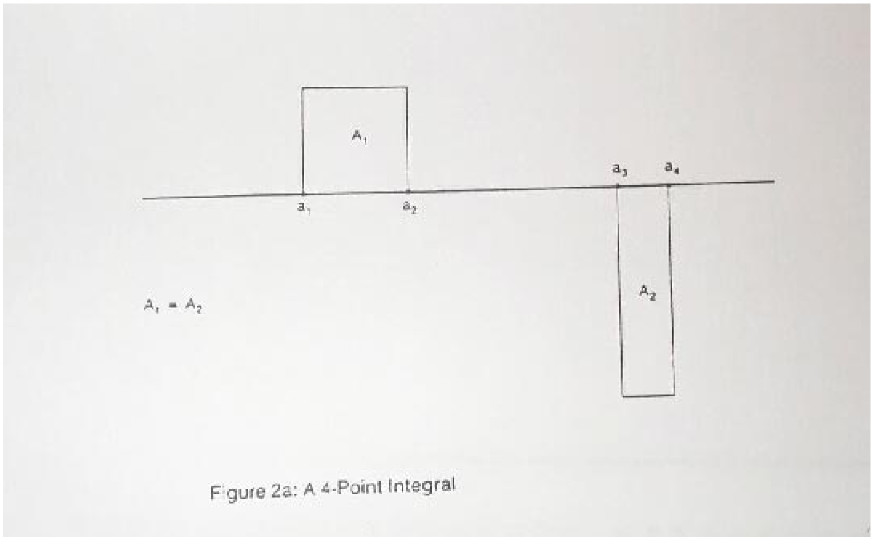

Figure 3:

(8) is satisfied. That the implication runs the other way too can be seen from p. 630 of Fishburn (1982) or by combining Lemma 1 of Rothschild and Stiglitz (1970) with Theorems 1a and 1b of the present article. Lemma 1 of Rothschild and Stiglitz (1970) contains a mistake, but one which can be rectified by using the 3-pt MPS. The first paragraph of its proof says "By (7), $a_2 < a_3$," where it should say, "By (7), $a_2 \leq a_3$." The false step would rule out the $G - F$ shown in Figure 2b, which is clearly an example of stochastic dominance. The proof is thus invalid for the 4-pt MPS, but it can easily be made valid for the 3-pt MPS. Rather than writing a new proof for the 4-pt MPS, it is easier to use the equivalence of the 3-pt and 4-pt MPS to establish that the subsequent propositions in Rothschild and Stiglitz are correct.

The usefulness of the mean-preserving spread lies in its equivalence to other definitions of risk. Since the 3-pt and 4-pt definitions are equivalent, either of them can be used in the theorem below, which says that different ways of comparing distributions of wealth $F(x)$ and $G(x)$ are equivalent.

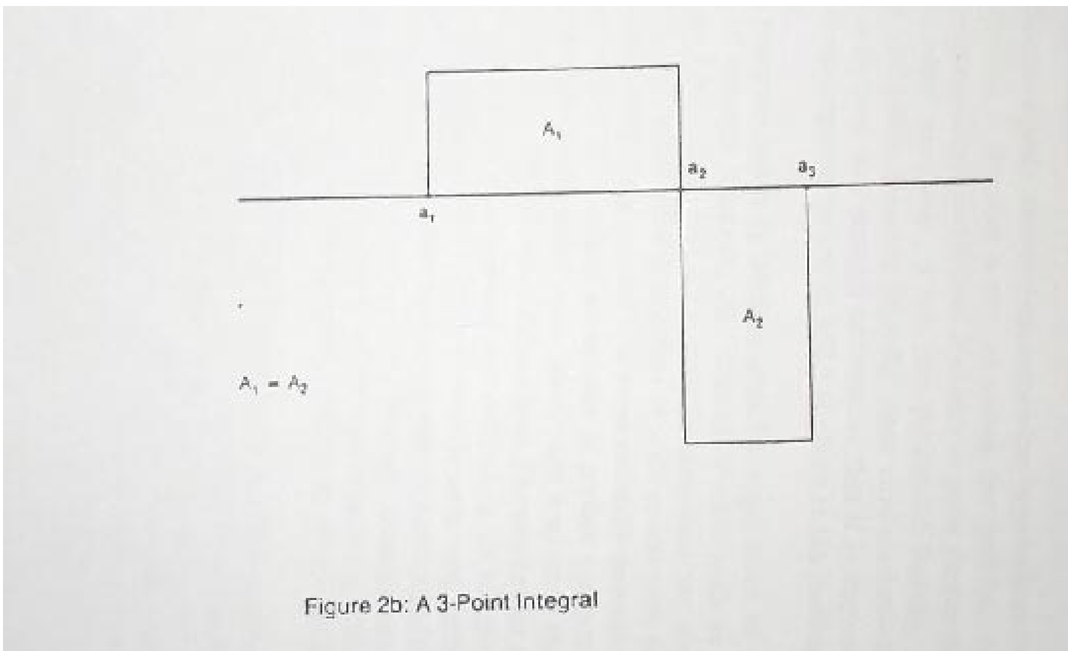

Figure 4:

THEOREM (Rothschild and Stiglitz [1970] Theorem 2): The following three statements are equivalent:

(A) *Risk Aversion.* For every bounded concave function U, $\int U(z)dF(z) \geq \int U(z)dG(z)$. (Every risk averter prefers F to G.)

(B) *Noise.* There exists a random variable Z such that $E(Z|X = x) = 0$ for all x, and $Y \underset{d}{=} X + Z$. (G is distributed as F plus noise.)

(C) *MPS/Stochastic Dominance.* The difference $G - F$ satisfies the integral conditions (7) and (8). (F has the same mean as G, but 2nd-order stochastically dominates it.) (G equals F plus a sequence of 3-pt or 4-pt MPS's.) (G has more weight in the tails than F.)

David Hirshleifer points out that it is easy to see the equivalence of (B) and (C) using the 3-pt MPS. The 3-pt MPS takes probability away from point a_2 and moves it to points a_1 and a_3. This is like waiting for the realization specified by the original distribution, and then, if the realization is a_2, adding a new gamble that either (a) leaves the outcome as a_2, or (b) moves it to a_1 or a_3.

To conclude: risk is so central to information economics that it is important to have its definition be as clear and convenient as possible. There are a number of different ways to define risk that can be shown to be mathematically equivalent, of which the 4-pt MPS is perhaps the best-known. We have suggested that the 4-pt MPS be replaced by the 3-pt MPS, which combines the attractive properties of the 4-pt MPS with additional simplicity and intuitiveness.

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