Lecture Notes on Risk, G601 Industrial Organization

September 15, 2003. Revised 27 September 2006.

Eric Rasmusen

Department of Business Economics and Public Policy, Kelley School of Business,Indiana University, BU 456, 1309 E. 10th Street, Bloomington, Indiana, 47405-1701. Office: (812) 855-9219. Fax: 812-855-3354. Erasmuse@indiana.edu.

1st half: Risk Chapter 9–concavity, Arrow-Pratt, functional forms Chapter 10, Wang notes, MPS, integrals, 4 definitions

2nd half: Options

Risk Aversion and Concavity

DEFINITION: An agent is strictly risk averse if his expected utility from nondeterministic consumption c is less than his utility would be from the expected value of c; i.e., if

$$E[v(c)] < v[E(c)] \qquad (9.1)$$

THEOREM 9.3.1. If an agent is strictly risk averse, his utility function is strictly concave.

MEASURING RISK AVERSION

An agent's **absolute risk aversion** starting at wealth y is

$$A(y) \equiv -\frac{v''(y)}{v'(y)} \qquad (9.9)$$

Why the negative sign?

Why not just use $-v''(y)$?

Think about the units of measurement. Utils per dollar per dollar. and Utils per dollar.

An agent's **risk compensation** (a less common term) for additional risky asset z starting at wealth y is $\rho(y, z)$ such that

$$Ev(y+z) \equiv v(y - \rho(y, z)) \qquad (9.11)$$

An agent's **certainty equivalent** (a more common term) for risky asset z is

$$z - \rho(y, z)$$

If utility function v_1 has higher absolute risk aversion at wealth y than utility function v_2 does, that is equivalent to v_1 having a bigger risk compensation (and thus smaller certainty equivalent) or to v_1 being more concave.

Arrow (1963), Pratt (1964). The Pratt article is very readable–perhaps more so than Leroy-Werner, though not as complete.

ABSOLUTE VERSUS RELATIVE RISK AVERSION

An agent's **absolute risk aversion** starting at wealth y is

$$A(y) \equiv -\frac{v''(y)}{v'(y)} \qquad (9.9)$$

"Absolute risk aversion" is what everyone means when they say "risk aversion".

An agent's **relative risk aversion** starting at wealth y is

$$R(y) \equiv y \cdot A(y) = -\frac{v''(y)}{v'(y)}y \qquad (9.23)$$

Leroy and Werner seem to have forgotten to explain what this concept means, in words. It measures how much compensation the agent needs for risking *a particular fraction of his initial wealth*, as opposed to risking an absolute amount of money. How much compensation would Smith need for risking the loss of 10% of your wealth, as opposed to risking $10,000? If Smith's wealth is $100,000, these are the same question. Suppose, though, that we compare Smith with Jones, who has the same absolute risk aversion, but wealth of $500,000. If their relative risk aversion is the same,because they both need the same compensation for risking 10% of their wealth, then Jones will have much smaller absolute risk aversion, since a loss of $10,000 is only a loss of 2% of his wealth.

In theoretical models, we always use absolute risk aversion, since it is simpler. If you ever measure risk aversion, though, a specification with constant relative risk aversion is more realistic.

I have seen someone criticize Expected Utility theory by showing that a utility function with constant absolute risk aversion implies silly behavior— that a very rich person is unwilling to invest in assets as risky as stocks, for example. That is deceptive, because nobody says constant absolute risk aversion is realistic.

FUNCTIONAL FORMS FOR UTILITY

EXPONENTIAL UTILITY (constant absolute risk aversion, CARA):

$$v(y) = -e^{-\alpha y} \qquad (9.27)$$

Risk aversion is then constant and equals α.

It is less unsettling to use $v(y) = \kappa - e^{-\alpha y}$, because we think of utility as being a positive number.

LOG OR POWER UTILITY (constant relative risk aversion, CRRA):

$$v(y) = \frac{1}{\gamma - 1} (\gamma y)^{1 - \frac{1}{gamma}}, \quad if\ \gamma \neq 1\ and\ \gamma \neq 0 \qquad (9.29)$$

and

$$v(y) = log(y), \quad if\ \gamma = 1 \qquad (9.28)$$

Relative risk aversion is then constant and equals γ. Absolute risk aversion equals $\frac{1}{\gamma y}$.

A neat fact is that as γ approaches 0, a value it is not allowed to take, the utility function approaches exponential, CARA, utility.

QUADRATIC UTILITY

LOG OR POWER UTILITY (constant relative risk aversion, CRRA):

$$v(y) = \frac{1}{\gamma - 1} (\gamma y)^{1-\frac{1}{gamma}}, \quad if\ \gamma \neq 1\ and\ \gamma \neq 0 \qquad (9.29)$$

QUADRATIC UTILITY (Power utility with $y < \alpha \neq 0$, $\gamma = -1$)

$$\begin{aligned} v(y) &= -\tfrac{1}{2} (\alpha - y)^2, \quad y < \alpha \qquad (9.30) \\ &= \alpha y - \tfrac{\alpha^2}{2} - \tfrac{y^2}{2} \end{aligned}$$

This is simple, and so is useful for modelling, but it has *increasing* absolute risk aversion, equal to $\frac{1}{\alpha - y}$, which is most implausible.

All these utility functions have risk tolerance, $1/A$, depending linearly on wealth. Some others do too–that's why LEroy and Warner put $\alpha \neq 0$ into their version.

Four Definitions of Y being Riskier than Z, when EY=EZ

1. Y = Z+ noise

2. Any person with strictly concave utility strictly prefers Z to Y.

3. Y is Z plus a series of mean-preserving spreads (not in Leroy & Werner)

4. The Integral condition (10.7.1)

All Four of the above definitions are equivalent, i.e., if Y is riskier than Z by any one of them, it is for all of them.

NOT

5. $Var(Y) > Var(Z)$. This is a different and a worse definition.

Example 10.6.1: Variance and Uncorrelated Noise

Z = 1, 3,4,6 with probability .25 each.

Y = 2 with probability .5 and 3 or 7 with probability .25 each.

Probability	Z	ϵ	$Y = Z + \epsilon$
.25	1	1	2
.25	3	-1	2
.25	4	-1	3
.25	6	1	7

(1) Think about means. The mean of Z is (1+3+4+6)/4 = 3.5. The mean of ϵ is (1 + -1+-1+1)/4 = 0. The mean of Y is (2 + 2 + 3 + 7)/4 = 3.5. So Z and Y have the same mean.

(2) Think about variance. The variance of Z is $[(1-3.5)^2+(3-3.5)^2+(4-3.5)^2+(6-3.5)^2]/4 = 3.25$. The variance of ϵ is $[1^2 + 1^2 + 1^2 + 1^2]/4 = 1$. The

variance of Y is $[(2-3.5)^2+(2-3.5)^2+(3-3.5)^2+(7-3.5)^2]/4 = 4.25$. So Z has lower variance than Y.

Probability	Z	ϵ	$Y = Z + \epsilon$
.25	1	1	2
.25	3	-1	2
.25	4	-1	3
.25	6	1	7

(3) Think about the correlation between ϵ and Z.

$$\begin{aligned} E(Z\epsilon) &= [1(1)+3(-1)+4(-1)+6(1)]/4 \\ &= [1-3+4+6]/4 = 0. \end{aligned}$$

Thus, $Y = Z + \epsilon$, where Z and ϵ are uncorrelated.

(3′) Think again about the correlation between ϵ and Z.

We found that $E(Z\epsilon) = 0$, so that $Y = Z + \epsilon$, where Z and ϵ are uncorrelated.

But $E(\epsilon|Z) \neq 0$, so Y is not riskier than Z by the standard definition. For example, $E(\epsilon|Z = 4) = -1$, even though $E(\epsilon|Z = 6) = +1$. Z and ϵ are uncorrelated, but they are not independent.

Example 10.6.1, continued—UTILITY

(4) Yet I can find a risk-averse person who would prefer the noisier, higher-variance Y to Z. Consider $u = log(c)$, where c is consumption. The expected utilities are

$$\begin{aligned} Eu(Z) &= \tfrac{1}{4}\left[log(1) + log(3) + log(4) + log(6)\right] \\ &= \tfrac{1}{4}\left[log(1 \cdot 3 \cdot 4 \cdot 6)\right] = \tfrac{1}{4}\left[log(72)\right] \end{aligned}$$

and

$$\begin{aligned} Eu(Y) &= \tfrac{1}{4}\left[log(2) + log(2) + log(3) + log(7)\right] \\ &= \tfrac{1}{4}\left[log(2 \cdot 2 \cdot 3 \cdot 7)\right] = \tfrac{1}{4}\left[log(84)\right]. \end{aligned}$$

Thus, Y has the higher utility for this person.

(5) Other risk-averse people can be found who would prefer Z—quadratic, I think, because variance is what matters there. Let's try that. Now suppose $v = 1000c - c^2$.

$$\begin{aligned} Ev(Z) &= \tfrac{1}{4}\left[1000 - 1^2 + 3000 - 3^2 + 4000 - 4^2 + 6000 - 6^2\right] \\ &= \tfrac{1}{4}\left[14000 - 62\right] \end{aligned}$$

and

$$\begin{aligned} Ev(Y) &= \tfrac{1}{4}\left[2000 - 2^2 + 2000 - 2^2 + 3000 - 3^2 + 7000 - 7^2\right] \\ &= \tfrac{1}{4}\left[14000 - 66\right] \end{aligned}$$

Thus, this second person likes Z better than Y.

The example from my Options paper is better for intuition, though not so neat and I haven't thought about the ϵ angle yet.