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2 Information

Table 1: Ranked Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	$\leftarrow$	$-1,-1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1,-1$	$\rightarrow$	**1,1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

The term **bayesian equilibrium** is used to refer to a Nash equilibrium in which players update their beliefs according to Bayes's Rule. Since Bayes's Rule is the natural and standard way to handle imperfect information, the adjective, "bayesian," is really optional. But the two-step procedure of checking a Nash equilibrium has now become a three-step procedure:

1 Propose a strategy profile.

2 See what beliefs the strategy profile generates when players update their beliefs in response to each others' moves.

3 Check that given those beliefs together with the strategies of the other players each player is choosing a best response for himself.

The rules of the game specify each player's initial beliefs, and Bayes's Rule is the rational way to update beliefs. Suppose, for example, that Jones starts with a particular prior belief, $Prob(Nature\ chose\ (A))$.

In Follow-the- Leader III, this equals 0.7.

He then observes Smith's move — *Large*, perhaps. Seeing *Large* should make Jones update to the **posterior** belief, $Prob(Nature\ chose\ (A))|Smith\ chose\ Large)$, where the symbol "|" denotes "conditional upon" or "given that."

Bayes's Rule shows how to revise the prior belief in the light of new information such as Smith's move. It uses two pieces of information, the likelihood of seeing Smith choose *Large* given that Nature chose state of the world (A), $Prob(Large|(A))$, and the likelihood of seeing Smith choose *Large* given that Nature did not choose state (A), $Prob(Large|(B)\ or\ (C))$.

From these numbers, Jones can calculate

$Prob(Smith\ chooses\ Large)$,

the **marginal likelihood** of seeing *Large* as the result of one or another of the possible states of the world that Nature might choose.

$$\begin{aligned} Prob(Smith\ chooses\ Large) &= Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) \\ &+ Prob(Large|C)Prob(C). \end{aligned} \tag{1}$$

Bayes's Rule is not purely mechanical. It is the only way to rationally update beliefs. The derivation is worth understanding, because Bayes's Rule is hard to memorize but easy to rederive.

$$
\begin{aligned}
Prob(Smith\ chooses\ Large) &= Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) \\
&\quad + Prob(Large|C)Prob(C).
\end{aligned} \tag{2}
$$

To find his posterior,

$Prob(Nature\ chose\ (A))|Smith\ chose\ Large)$,

Jones uses the likelihood and his priors. The joint probability of both seeing Smith choose *Large* and Nature having chosen (A) is

$$
Prob(Large, A) = Prob(A|Large)Prob(Large) = Prob(Large|A)Prob(A). \tag{3}
$$

Since what Jones is trying to calculate is $Prob(A|Large)$, rewrite the last part of (3) as follows:

$$
Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large)}. \tag{4}
$$

Jones needs to calculate his new belief — his posterior — using $Prob(Large)$, which he calculates from his original knowledge using (6). Substituting the expression for $Prob(Large)$ from (6) into equation (4) gives the final result, a version of Bayes's Rule.

$$
Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) + Prob(Large|C)Prob(C)}. \tag{5}
$$

$$
\begin{aligned}
Prob(Smith\ chooses\ Large) &= Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) \\
&\quad + Prob(Large|C)Prob(C).
\end{aligned} \tag{6}
$$

Let us now return to the numbers in Follow-the-Leader III to use the belief-updating rule that was just derived.

Jones has a prior belief that the probability of event "Nature picks state (A)" is 0.7 and he needs to update that belief on seeing the data "Smith picks *Large*". His prior is $Prob(A) = 0.7$, and we wish to calculate $Prob(A|Large)$.

To use Bayes's Rule from equation (5), we need the values of $Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C)$.

These values depend on what Smith does in equilibrium, so Jones's beliefs cannot be calculated independently of the equilibrium. This is the reason for the three-step procedure suggested above.

A candidate for equilibrium is

Smith $(L|A, L|B, S|C$

Jones $(L|L, S|S)$.

Smith $(L|A, L|B, S|C$

Jones $(L|L, S|S)$.

Let us test that this is an equilibrium, starting with the calculation of $Prob(A|Large)$.

If Jones observes $Large$, he can rule out state (C), but he does not know whether the state is (A) or (B).

Bayes's Rule tells him that the posterior probability of state (A) is

$$Prob(A|Large) = \frac{(1)(0.7)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} = 0.875. \tag{7}$$

The posterior probability of state (B) must then be $1-0.875 = 0.125$, which could also be calculated from Bayes's Rule, as follows:

$$(B|Large) = \frac{(1)(0.1)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} = 0.125. \tag{8}$$

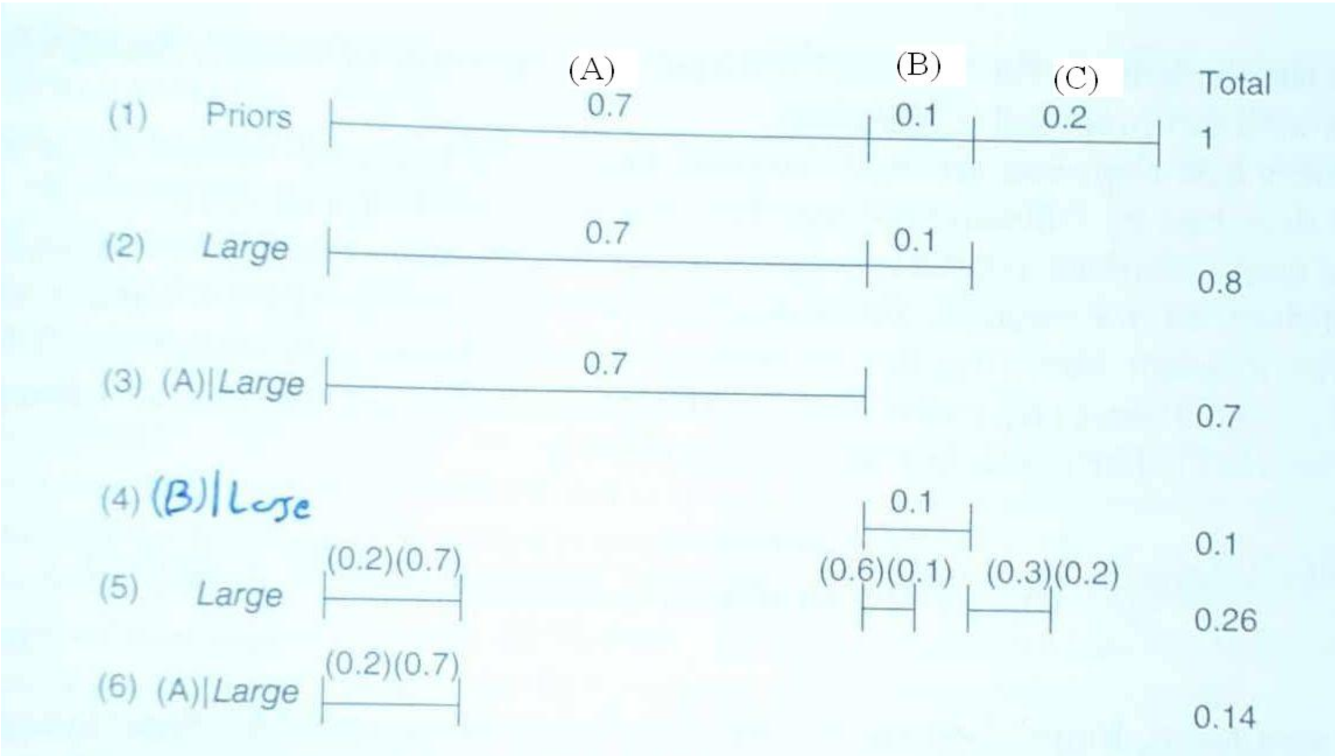

Figure 8: Bayes's Rule

Smith $(L|A, L|B, S|C$

Jones $(L|L, S|S)$.

Jones must use Smith's strategy in the proposed equilibrium to find numbers for

$Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C)$.

Given that Jones believes that the state is (A) with probability 0.875 and state (B) with probability 0.125, his best response is $Large$, even though he knows that if the state were actually (B) the better response would be $Small$:

$E\pi(Small|Large)$ is -0.625 ($= 0.875[-1]+0.125[2]$),

$E\pi(Large|Large)$ is 1.875 ($= 0.875[2] + 0.125[1]$).

A similar calculation can be done for $Prob(A|Small)$.

$$Prob(A|Small) = \frac{(0)(0.7)}{(0)(0.7) + (0)(0.1) + (1)(0.2)} = 0. \tag{9}$$

Given that he believes the state is (C), Jones's best response to $Small$ is $Small$.

Given that Jones will imitate his action, Smith does best by following his equilibrium strategy of $(L|A, L|B, S|C)$.

The calculations are relatively simple because Smith uses a nonrandom strategy in equilibrium, so, for instance, $Prob(Small|A) = 0$ in equation (9).

Consider what happens if Smith uses a random strategy of picking *Large* with probability 0.2 in state (A), 0.6 in state (B), and 0.3 in state (C) (we will analyze such "mixed" strategies in Chapter 3).

$$\begin{aligned} Prob(A|Large) &= \frac{(1)(0.7)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} \\ &= 0.875. \end{aligned} \tag{10}$$

The equivalent of equation (10) is

$$Prob(A|Large) = \frac{(0.2)(0.7)}{(0.2)(0.7) + (0.6)(0.1) + (0.3)(0.2)} = 0.54 \tag{11}$$

If he sees *Large*, Jones's best guess is still that Nature chose state (A), even though in state (A) Smith has the smallest probability of choosing *Large*, but Jones's subjective posterior probability, $Pr(A|Large)$, has fallen to 0.54 from his prior of $Pr(A) = 0.7$.

The last two lines of Figure 8 illustrate this case.