3 Mixing

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	3,2	$\rightarrow$	$-1,3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1,1$	$\leftarrow$	0,0

Payoffs to: (Government, Pauper). Arrows show how a player can increase his payoff.

Each strategy profile must be examined in turn to check for Nash equilibria.

1 I assert that an optimal mixed strategy exists for the government.

2 If the pauper selects *Work* more than 20 percent of the time, the government always selects *Aid*. If the pauper selects *Work* less than 20 percent of the time, the government never selects *Aid*.

3 If a mixed strategy is to be optimal for the government, the pauper must therefore select *Work* with probability exactly 20 percent.

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	3,2	$\rightarrow$	$-1,3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1,1$	$\leftarrow$	0,0

$$\pi(GOV, AID) = \gamma_w(3) + (1-\gamma_w)(-1)$$
$$= \pi(GOV, NO\ AID) = \gamma_w(-1) + (1-\gamma_w)(0)$$

$$3\gamma_w - 1 + \gamma_w = -\gamma_w, \qquad 5\gamma_w = 1, \qquad \gamma_w = .2.$$

$$\pi(Pauper, WORK) = \theta_a(2) + (1-\theta_a)(1) = \pi(Pauper, Loaf) = \theta_a(3)$$

$$2\theta_a + 1 - \theta_a = 3\theta_a, \qquad 1 = 2\theta_a, \qquad \theta_a = .5.$$

The War of Attrition

Two firms are in an industry which is a natural monopoly. The possible actions are to $Exit$ or to $Continue$. In each period that both $Continue$, each earns -1. If a firm exits, its losses cease and the remaining firm obtains 3. The discount rate is r.

The War of Attrition has a continuum of Nash equilibria. One is for Smith to choose ($Continue$ regardless of what Jones does) and for Jones to choose ($Exit$ immediately).

We will solve for a symmetric equilibrium. Let $\theta = Probability(Exit)$, and denote the expected discounted value of Smith's payoffs by V_{stay} if he stays and V_{exit} if he exits. If he exits, he gets $V_{exit} = 0$. If he stays in, his payoff depends on what Jones does. If Jones stays in too, which has probability $(1 - \theta)$, Smith gets -1 currently and his expected value for the following period, which is discounted using r, is unchanged. If Jones exits immediately, which has probability θ, then Smith receives a payment of 3.

$$V_{stay} = \theta \cdot (3) + (1 - \theta) \left(-1 + \left[\frac{V_{stay}}{1 + r} \right] \right), \qquad (1)$$

$$V_{stay} = \left(\frac{1 + r}{r + \theta} \right) (4\theta - 1) . \qquad (2)$$

Since $V_{stay} = V_{exit} = 0$, $\theta = 0.25$ in equilibrium.

The goal of the IRS is to either prevent or catch cheating at minimum cost. The suspects want to cheat only if they will not be caught. Let us assume that the benefit of preventing or catching cheating is 4, the cost of auditing is C, where $C < 4$, the cost to the suspects of obeying the law is 1, and the cost of being caught is the fine $F > 1$.

Table 8: Auditing Game I

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
	Audit (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
IRS:		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	0,0	$\leftarrow$	$4, -1$

Payoffs to: (IRS, Suspects). Arrows show how a player can increase his payoff.

Auditing Game I is a discoordination game, with only a mixed strategy equilibrium.

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
	Audit (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
IRS:		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	0,0	$\leftarrow$	$4, -1$

A second way to model the situation is as a sequential game. The IRS chooses government policy first, and the suspects react to it.

The equilibrium is in pure strategies. The IRS chooses *Audit*, anticipating that the suspect will then choose *Obey*. The payoffs are $(4-C)$ for the IRS and -1 for the suspects, the same for both players as before, although now there is more auditing and less cheating and fine paying.

Suppose the IRS does not have to adopt a policy of auditing or trusting every suspect, but instead can audit a random sample. It chooses α so that

$$\pi_{suspect}(Obey) \geq \pi_{suspect}(Cheat), \tag{3}$$

$$-1 \geq \alpha(-F) + (1-\alpha)(0). \tag{4}$$

In equilibrium, therefore, the IRS chooses $\alpha = 1/F$ and the suspects respond with *Obey*. The IRS payoff is $(4-\alpha C)$, which is better than the $(4-C)$ in the other two games, and the suspect's payoff is -1, exactly the same as before.

The Cournot Game

Players

Firms Apex and Brydox

The Order of Play

Apex and Brydox simultaneously choose quantities q_a and q_b from the set $[0, \infty)$.

Payoffs

Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$, and we will assume it to be linear (for generalization see Chapter 14), and, in fact, will use the following specific function:

$$p(Q) = 120 - q_a - q_b. \tag{5}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_a q_b - q_b^2. \tag{6}$$

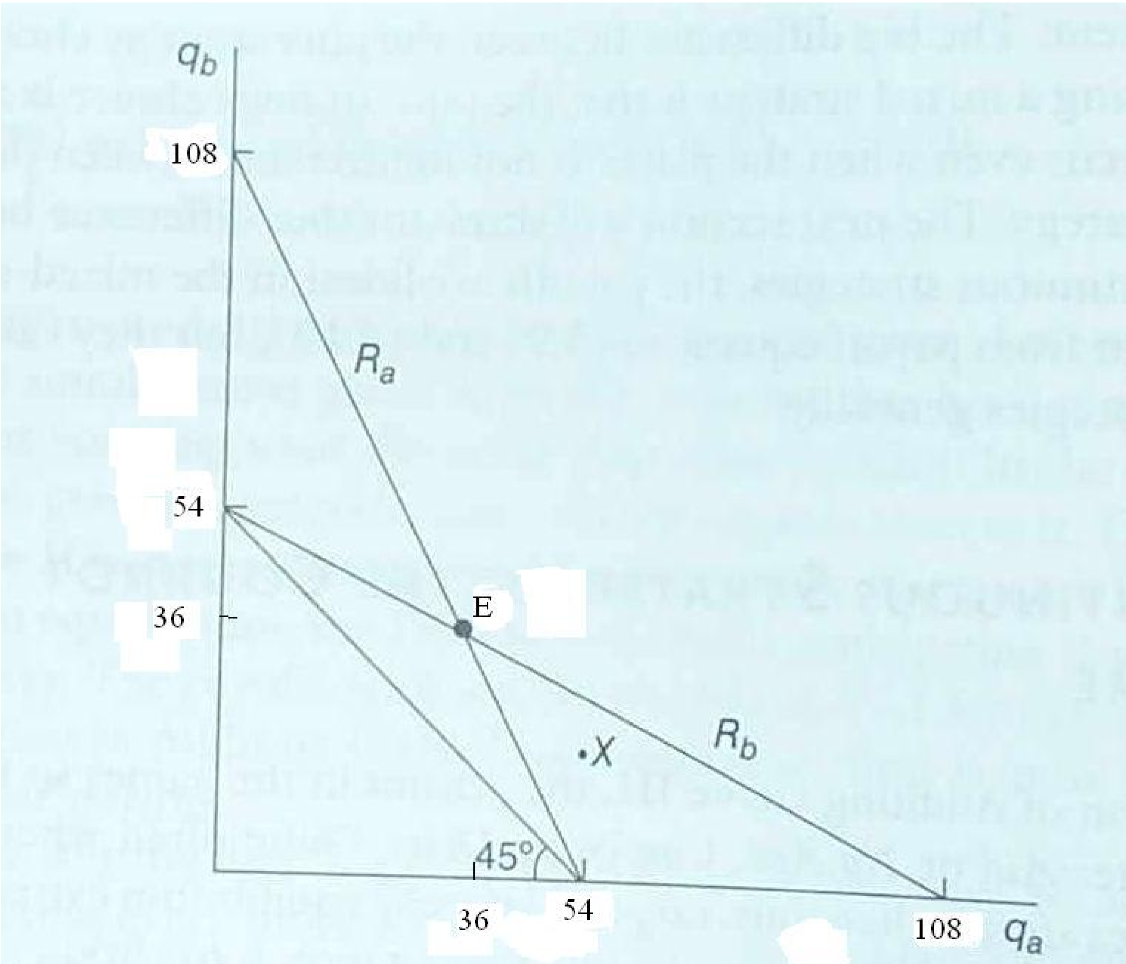

Figure 2: Reaction Curves in the Cournot Game

The monopoly output is 54.

The “Cournot-Nash” equilibrium is found frmo the **best-response functions** for the two players.

If Brydox produced 0, Apex would produce the monopoly output of 54.

If Brydox produced $q_b = 108$ or greater, the market price would fall to 12 and Apex would choose to produce zero.

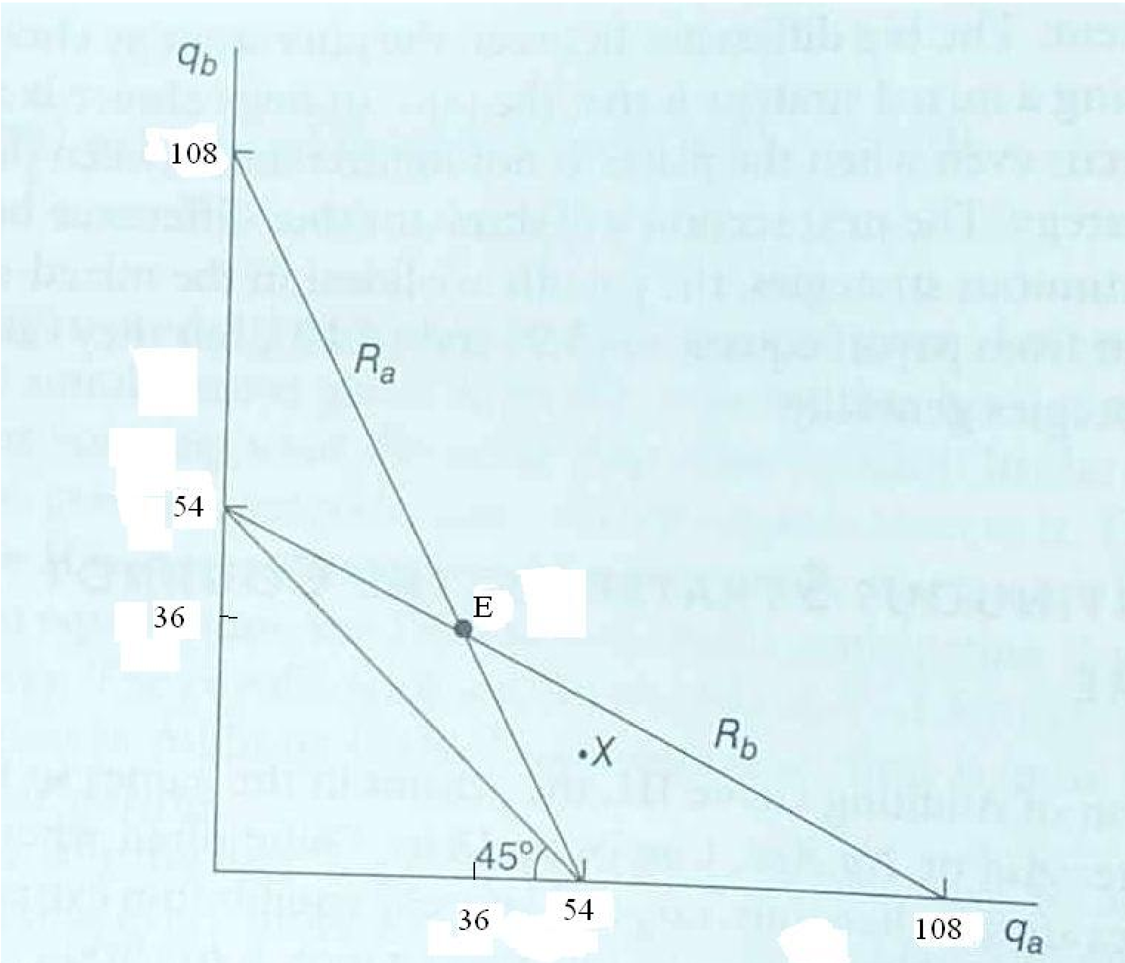

Figure 2: Reaction Curves in the Cournot Game

The best response function is found by maximizing Apex's payoff, given in equation (6), with respect to his strategy, q_a. This generates the first-order condition $120 - c - 2q_a - q_b = 0$, or

$$q_a = 60 - \left(\frac{q_b + c}{2}\right) = 54 - \left(\frac{1}{2}\right) q_b. \tag{7}$$

The unique equilibrium is $q_a = q_b = 40 - c/3 = 36$.

The Stackelberg Game

Players

Firms Apex and Brydox

The Order of Play

1 Apex chooses quantity q_a from the set $[0, \infty)$.
2 . Brydox chooses quantity q_b from the set $[0, \infty)$.

Payoffs

Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$:

$$p(Q) = 120 - q_a - q_b. \tag{8}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_a q_b - q_b^2. \tag{9}$$

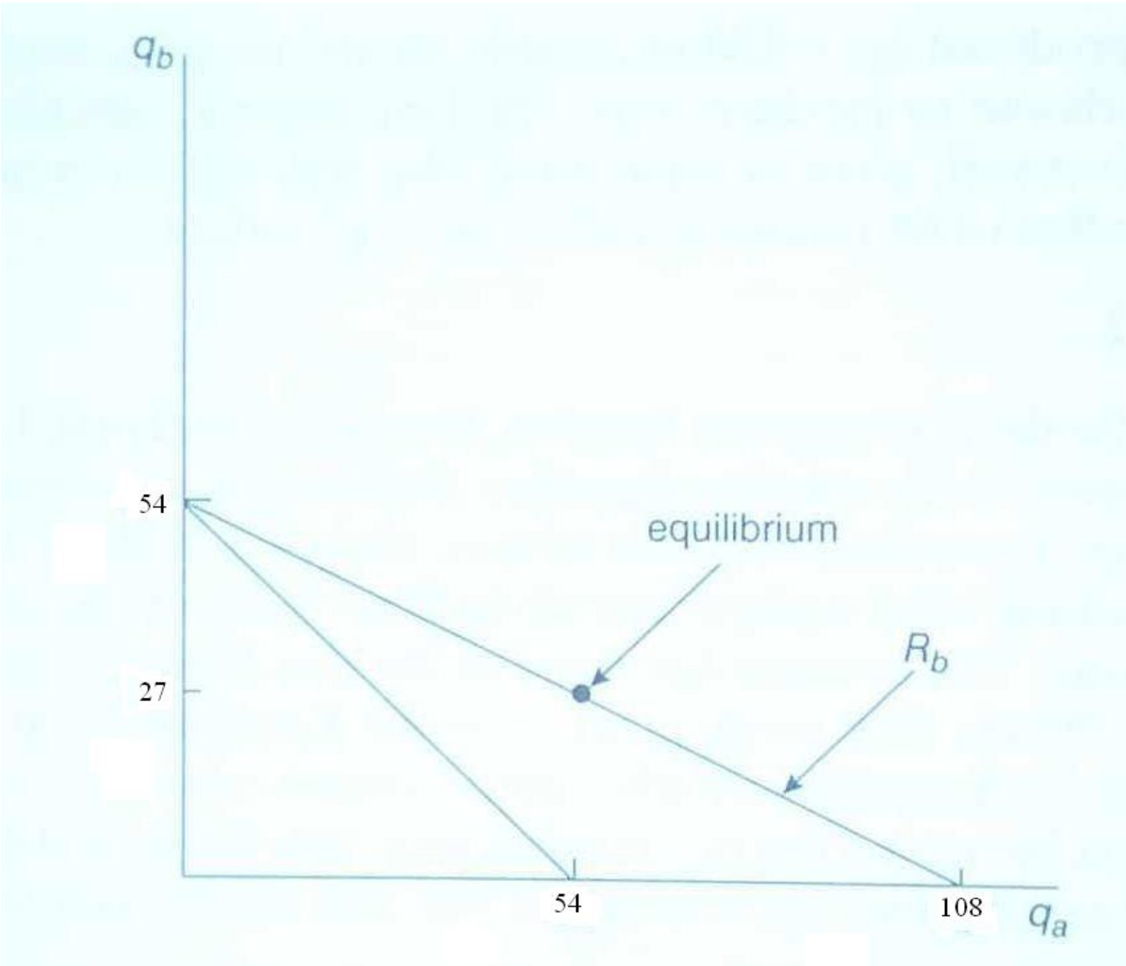

Figure 3: Stackelberg Equilibrium

Since Apex forecasts Brydox's output to be $q_b = 60 - \frac{q_a+c}{2}$ Apex can substitute this into his payoff function:

$$\pi_a = (120 - c)q_a - q_a^2 - q_a(60 - \frac{q_a + c}{2}). \tag{10}$$

Maximizing with respect to q_a yields

$$(120 - c) - 2q_a - 60 + q_a + \frac{c}{2} = 0, \tag{11}$$

which generates Apex's "reaction" function, $q_a = 60 - c/2 = 54$.

Once Apex chooses 54, Brydox reacts with $q_b = 27$.

The Bertrand Game

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q(p) = 120 - p$. The payoff function for Apex (Brydox's would be analogous) is

$$\pi_a = \begin{cases} (120 - p_a)(p_a - c) & \text{if } p_a \leq p_b \\ \frac{(120-p_a)(p_a-c)}{2} & \text{if } p_a = p_b \\ 0 & \text{if } p_a > p_b \end{cases}$$

The Bertrand Game has a unique Nash equilibrium: $p_a = p_b = c = 12$, with $q_a = q_b = 54$. That this is a weak Nash equilibrium is clear: if either firm deviates to a higher price, it loses all its customers and so fails to increase its profits to above zero. In fact, this is an example of a Nash equilibrium in weakly dominated strategies.

*3.7 Four Problems for Existence of Equilibrium

(1) An unbounded strategy space

Let Smith's strategy be $x \in [0, \infty]$, which is the same as saying that $0 \leq x$, and his payoff function be $\pi(x) = x$.

This interval is both closed and unbounded. (Though it is also half-open!)

(2) An open strategy space

Let Smith's strategy be $x \in [0, 1,000)$, which is the same as saying that $0 \leq x < 1,000$, and his payoff function be $\pi(x) = x$.

(3) A discrete strategy space (or, more generally, a nonconvex strategy space)

The Welfare Game. No compromise is possible between a little aid and no aid, until we introduce mixed strategies.

Suppose we had a game in which the government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non- convex– there is gap in it. Without mixed strategies, an equilibrium to the game might well not exist.

(4) A discontinuous reaction function arising from nonconcave or discontinuous payoff functions

For a Nash equilibrium to exist, we need for the reaction functions of the players to intersect.

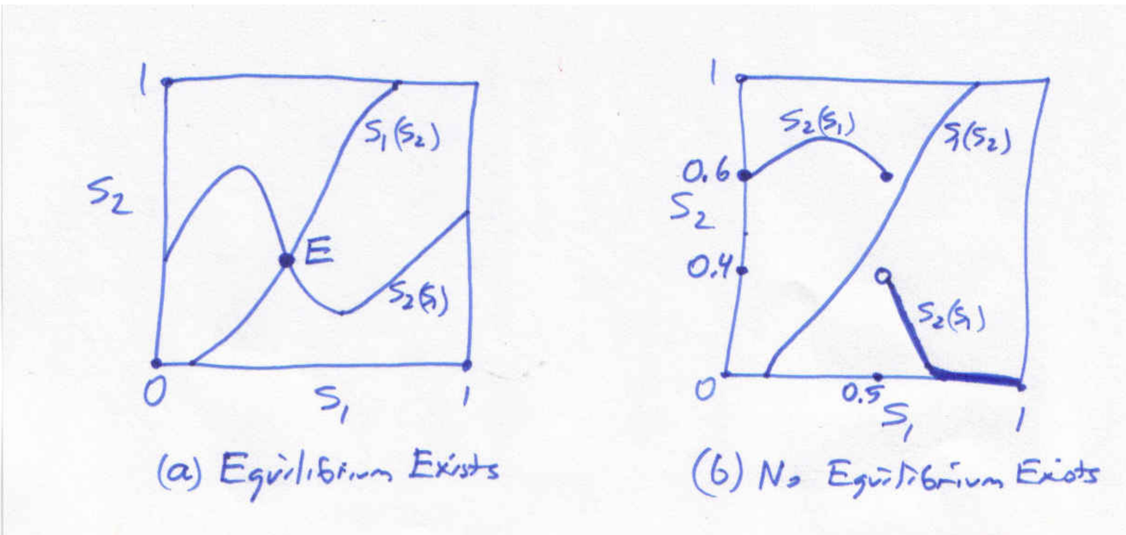

Figure 6: Continuous and Discontinuous Reaction Functions

Patent Race for a New Market

Players

Three identical firms, Apex, Brydox, and Central.

The Order of Play

Each firm simultaneously chooses research spending $x_i \geq 0$, $(i = a, b, c)$.

Payoffs

Firms are risk neutral and the discount rate is zero. Innovation occurs at time $T(x_i)$ where $T' < 0$. The value of the patent is V, and if several players innovate simultaneously they share its value. Let us look at the payoff of firm $i = a, b, c$, with j and k indexing the other two firms:

$$\pi_i = \begin{cases} V - x_i & \text{if } T(x_i) < Min\{T(x_j, T(x_k)\} & \text{(Firm } i \text{ gets the pat} \\ \frac{V}{2} - x_i & \text{if } T(x_i) = Min\{T(x_j), T(x_k)\} & \text{(Firm } i \text{ shares the p} \\ & \quad < Max\{T(x_j), T(x_k)\} & \text{1 other firm)} \\ \frac{V}{3} - x_i & \text{if } T(x_i) = T(x_j = T(x_k) & \text{(Firm } i \text{ shares the p} \\ & & \text{2 other firms)} \\ -x_i & \text{if } T(x_i) > Min\{T(x_j, T(x_k)\} & \text{(Firm } i \text{ does not ge} \end{cases}$$

The game Patent Race for a New Market does not have any pure strategy Nash equilibria, because the payoff functions are discontinuous. If Apex chose any research level x_a less than V, Brydox would respond with $x_a + \varepsilon$ and win the patent. If Apex chose $x_a = V$, then Brydox and Central would respond with $x_b = 0$ and $x_c = 0$, which would make Apex want to switch to $x_a = \varepsilon$.

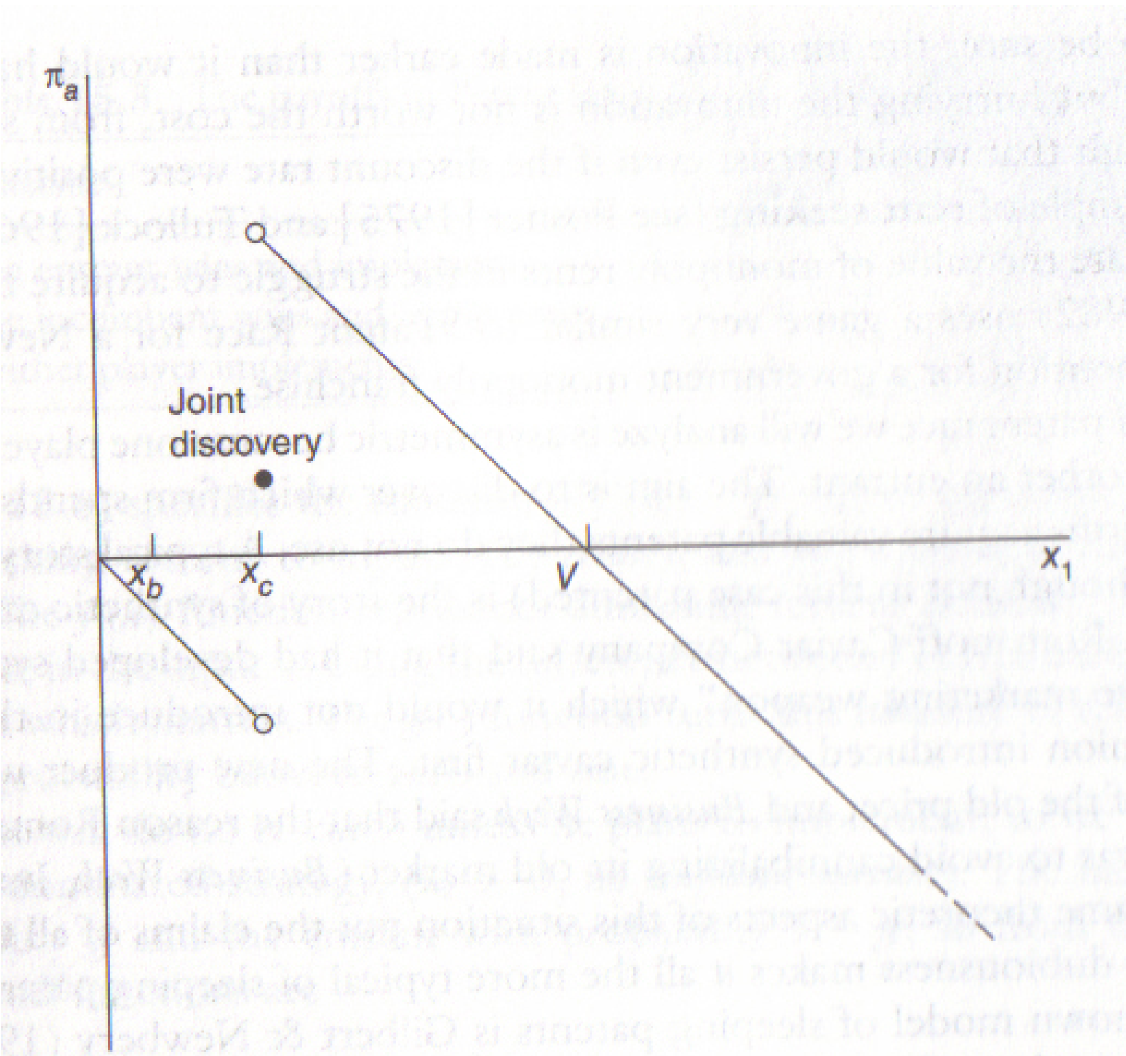

Figure 1: The Payoffs in Patent Race for a New Market

There does exist a symmetric mixed strategy equilibrium. Denote the probability that firm i chooses a research level less than or equal to x as $M_i(x)$.

Since we know that the pure strategies $x_a = 0$ and $x_a = V$ yield zero payoffs, if Apex mixes over the support $[0, V]$ then the expected payoff for every strategy mixed between must also equal zero.

The expected payoff from the pure strategy x_a is the expected value of winning minus the cost of research. Letting x stand for nonrandom and X for random variables, this is

$$\pi_a(x_a) = V \cdot Pr(x_a \geq X_b, x_a \geq X_c) - x_a = 0 = \pi_a(x_a = 0), \tag{12}$$

which can be rewritten as

$$V \cdot Pr(X_b \leq x_a) Pr(X_c \leq x_a) - x_a = 0, \tag{13}$$

or

$$V \cdot M_b(x_a) M_c(x_a) - x_a = 0. \tag{14}$$

We can rearrange equation (14) to obtain

$$M_b(x_a) M_c(x_a) = \frac{x_a}{V}. \tag{15}$$

If all three firms choose the same mixing distribution M, then

$$M(x) = \left(\frac{x}{V}\right)^{1/2} \text{ for } 0 \leq x \leq V. \tag{16}$$