PROBLEM SET 3 ANSWERS

27 September 2006.

Problems: 4.5, 5.4, 6.3, and 6.4.

4.5. Voting Cycles

Uno, Duo, and Tres are three people voting on whether the budget devoted to a project should be Increased, kept the Same, or Reduced. Their payoffs from the different outcomes, given in Table 3, are not monotonic in budget size. Uno thinks the project could be very profitable if its budget were increased, but will fail otherwise. Duo mildly wants a smaller budget. Tres likes the budget as it is now.

	Uno	Duo	Tres
Increase	100	2	4
Same	3	6	9
Reduce	9	8	1

Table 3: Payoffs from Different Policies

Each of the three voters writes down his first choice. If a policy gets a majority of the votes, it wins. Otherwise, *Same* is the chosen policy.

(a) Show that (*Same*, *Same*, *Same*) is a Nash equilibrium. Why does this equilibrium seem unreasonable to us?

Answer. The policy outcome is *Same* regardless of any one player's deviation. Thus, all three players are indifferent about their vote. This seems strange, though, because Uno is voting for his least-preferred alternative. Parts (c) and (d) formalize why this is implausible.

(b) Show that (*Increase*, *Same*, *Same*) is a Nash equilibrium.

Answer. The policy outcome is *Same*, but now only by a bare majority. If Uno deviates, his payoff remains 3, since he is not decisive. If Duo deviates to *Increase*,*Increase* wins and he reduces his payoff from 6 to 2; if Duo deviates to *Reduce*, each policy gets one vote and *Same* wins because of the tie, so his payoff remains 6. If Tres deviates to Increase, *Increase* wins and he reduces his payoff from 9 to 4; if Tres deviates to *Reduce*, each policy gets one vote and *Same* wins because of the tie, so his payoff remains 9.

(c) Show that if each player has an independent small probability ϵ of "trembling" and choosing each possible wrong action by mistake, (*Same*, *Same*, *Same*) and (*Increase*, *Same*, *Same*) are no longer equilibria.

Answer. Now there is positive probability that each player's vote is decisive. As a result, Uno deviates to *Increase*. Suppose Uno himself does not tremble. With positive probability Duo mistakenly chooses *Increase* while Tres chooses *Same*, in which case Uno's choice of *Increase* is decisive for *Increase* winning and will raise his payoff from 3 to 100. Similarly, it can happen that Tres mistakenly chooses *Increase* while Duo chooses *Same*. Again, Uno's choice of *Increase* is decisive for *Increase* winning. Thus, (*Same*, *Same*, *Same*) is no longer an equilibrium.

It is also possible that both Duo and Tres tremble and choose *Increase* by mistake, but in that case, Uno's vote is not decisive, because*Increase* wins even without his vote.

How about (*Increase*, *Same*, *Same*)? First, note that a player cannot benefit by deviating to his least-preferred policy.

Could Uno benefit by deviating to *Reduce*, his second-preferred policy? No, because he would rather be decisive for *Increase* than for *Reduce*, if a tremble might occur.

Could Duo benefit by deviating to *Reduce*, his most-preferred policy? If no other player trembles, that deviation would leave his payoff unchanged. If, however, one of the two other players trembles to *Reduce* and the other does not, then Duo's voting for *Reduce* would be decisive and *Reduce* would win, raising Duo's payoff from 6 to 8. Thus, (*Increase*, *Same*, *Same*) is no longer an equilibrium.

Just for completeness, think about Tres's possible deviations. He has no reason to deviate from *Same*, since that is his most preferred policy. *Reduce* is his least-preferred policy, and if he deviates to *Increase*, *Increase* will win, in the absence of a tremble, and his payoff will fall from 9 to 4– and since trembles have low probability, this reduction dominates any possibilities resulting from trembles.

(d) Show that (*Reduce*, *Reduce*, *Same*) is a Nash equilibrium that survives each player having an independent small probability ϵ of "trembling" and choosing each possible wrong action by mistake.

<u>*Answer.*</u> If Uno deviates to *Increase* or *Same*, the outcome will be *Same* and his payoff will fall from 9 to 3 If Duo deviates to *Increase* or *Same*, the outcome will be *Same* and his payoff will fall from 8 to 6. Tres's vote is not decisive, so his payoff will not change if he deviates. Thus, (*Reduce*, *Reduce*, *Same*) is a Nash equilibrium

How about trembles? The votes of both Uno and Duo are decisive in equilibrium, so if there are no trembles, each loses by deviating, and the probability of trembles is too small to make up for that. Only if a player's equilibrium strategy is weak could trembles make a difference.

Tres's equilibrium strategy is indeed weak, since he is not decisive unless there is a tremble.

With positive probability, however, just one of the other players trembles and chooses *Same*, in which case Duo's vote for *Same* would be decisive, and with the same probability just one of the other players trembles and chooses *Increase*, in which case Duo's vote for *Increase* would be decisive. Since Tres's payoff from *Same* is bigger than his payoff from *Increase*, he will choose *Same* in the hopes of that tremble.

(e) Part (d) showed that if Uno and Duo are expected to choose *Reduce*, then Tres would choose *Same* if he could hope they might tremble– not *Increase*. Suppose, instead, that Tres votes first, and publicly. Construct a subgame perfect equilibrium in which Tres chooses *Increase*. You need not worry about trembles now.

<u>*Answer.*</u> Tres's strategy is just an action, but Uno and Duo's strategies are actions conditional upon Tres's observed choice.

Tres: *Increase*.
Uno: *Increase*|*Increase*; *Reduce*|*Same*, *Reduce*|*Reduce*.
Duo: *Reduce*|*Increase*; *Reduce*|*Same*, *Reduce*|*Reduce*

Uno's equilibrium payoff is 100. If he deviated to *Same*|*Increase* and Tres chose *Increase*, his payoff would fall to 3; if he deviates to *Reduce*|*Increase* and Tres chose *Increase*, his payoff would fall to 9. Out of equilibrium, if Tres chose *Same*, Uno's payoff if he responds with *Reduce* is 9, but if he responds with *Same* it is 4. Out of equilibrium, if Tres chose *Reduce*, Uno's payoff is 9 regardless of his vote.

Duo's equilibrium payoff is 2. If Tres chooses *Increase*, Uno will choose *Increase* too and Duo's vote does not affect the outcome. If Tres chooses anything else, Uno will choose *Reduce* and Duo can achieve his most preferred outcome by choosing *Reduce*.

(f) Consider the following voting procedure. First, the three voters vote between *Increase* and *Same*. In the second round, they vote between the winning policy and *Reduce*. If, at that point, *Increase* is not the winning policy, the third vote is between *Increase* and whatever policy won in the second round.

What will happen? (watch out for the trick in this question!)

<u>*Answer.*</u> If the players are myopic, not looking ahead to future rounds, this is an illustration of the Condorcet paradox. In the first round, *Same* will beat *Increase*. In the second round, *Reduce* will beat *Same*. In the third round, *Increase* will be *Reduce*. The paradox is that the votes have cycled, and if we kept on holding votes, the process would never end.

The trick is that this procedure does *not* keep on going– it only lasts three rounds. If the players look ahead, they will see that *Increase* will win if they behave myopically. That is fine with Uno, but Duo and Tres will look for a way out. They would both prefer *Same* to win. If the last round puts *Same* to a vote against *Increase*, *Same* will win. Thus, both Duo and Tres want *Same* to win the second round. In particular, Duo will *not* vote for *Reduce* in the second round, because he knows it would lose in the third round.

Rather, in the first round Duo and Tres will vote for *Same* against *Increase*; in the second round they will vote for *Same* against *Reduce*; and in the third round they will vote for *Same* against *Increase* again.

This is an example of how particular procedures make voting deterministic even if voting would cycle endlessly otherwise. It is a little bit like the T-period repeated game versus the infinitely repeated one; having a last round pins things down and lets the players find their optimal strategies by backwards induction.

Arrow's Impossibility Theorem says that social choice functions cannot be found that always reflect individual preferences and satisfy various other axioms. The axiom that fails in this example is that the procedure treat all policies symmetrically– our voting procedure here prescribes a particular order for voting, and the outcome would be different under other orderings.

(g) Speculate about what would happen if the payoffs are in terms of dollar willingness to pay by each player and the players could make binding agreements to buy and sell votes. What, if anything, can you say about which policy would win, and what votes would be bought at what price?

<u>*Answer.*</u>

Uno is willing to pay a lot more than the other two players to achieve his preferred outcome, He would be willing, to deviate from any equilibrium in which *Increase* would lose by offering to pay 20 for Duo's vote. Thus, we know *Increase* will win.

But Uno will not have to pay that much to get the vote. We have just shown that *Increase* will win. The only question is whether it is Duo or Tres that has his payoff increased by a vote payment from Uno. Duo and Tres are thus in a bidding war to sell their vote. Competition will drive the price down to zero! See Ramseyer & Rasmusen (1994).

This voting procedure, with vote purchases, also violates one of Arrow's Impossibility axioms– his "Independence of Irrelevant Alternatives" rules out procedures that, like this one, rely on intensity of preferences.

5.4. Repeated Entry Deterrence

Assume that Entry Deterrence I is repeated an infinite number of times, with a tiny discount rate and with payoffs received at the start of each period. In each period, the entrant chooses *Enter* or *Stay Out*, even if he entered previously.

(a) What is a perfect equilibrium in which the entrant enters each period?

<u>*Answer.*</u> (*Enter*, *Collude*) each period.

(b) Why is (*Stay Out, Fight*) not a perfect equilibrium?[1]

<u>*Answer.*</u> (*Stay out*, *Fight*|*Enter*) gives the incumbent no incentive to choose *Fight*. Given the entrant's strategy, if somehow the game ends up off the equilibrium path with the entrant having entered, the entrant will *Stay Out* in succeeding periods. Hence, the incumbent would deviate by choosing *Collude* and getting 50 instead of 0.

(c) What is a perfect equilibrium in which the entrant never enters?

<u>*Answer.*</u> Entrant: *Stay out* unless the incumbent has chosen *Collude* in some previous period, in which case, *Enter*.

Incumbent: *Fight*|*Enter* unless the incumbent has chosen *Collude* in some previous period, in which case, choose *Collude*|*Enter*.

In this equilibrium, the incumbent suffers a heavy penalty if he ever colludes.

[1] xxx In the next edition, add: What happens if the game starts off the equilibrium path, with the entrant having entered?

(d) What is the maximum discount rate for which your strategy profile in part (c) is still an equilibrium?

Answer. If the discount rate is too high, the Entrant will enter and the Incumbent will prefer to collude. Suppose the Entrant has entered, and the incumbent has never yet colluded. The Incumbent's choice is between

$$\pi(collude) = 50 + \frac{50}{r} \tag{1}$$

and

$$\pi(fight) = 0 + \frac{100}{r} \tag{2}$$

These two payoffs equal each other if $50r + 50 = 100$ so $r = 1$. If the discount rate is anything less, the equilibrium in (c) remains an equilibrium.

6.3. Symmetric Information and Prior Beliefs

In the *Expensive-Talk Game* of Table 1, the *Battle of the Sexes* is preceded by by a communication move in which the man chooses *Silence* or *Talk*. *Talk* costs 1 payoff unit, and consists of a declaration by the man that he is going to the prize fight. This declaration is just talk; it is not binding on him.

Table 1: Subgame Payoffs in the Expensive-Talk Game

		Woman	
		Fight	*Ballet*
Man:	*Fight*	3,1	0, 0
	Ballet	0, 0	1,3

Payoffs to: (Man, Woman).

(a) Draw the extensive form for this game, putting the man's move first in the simultaneous-move subgame.

Answer. See Figure A6.1.

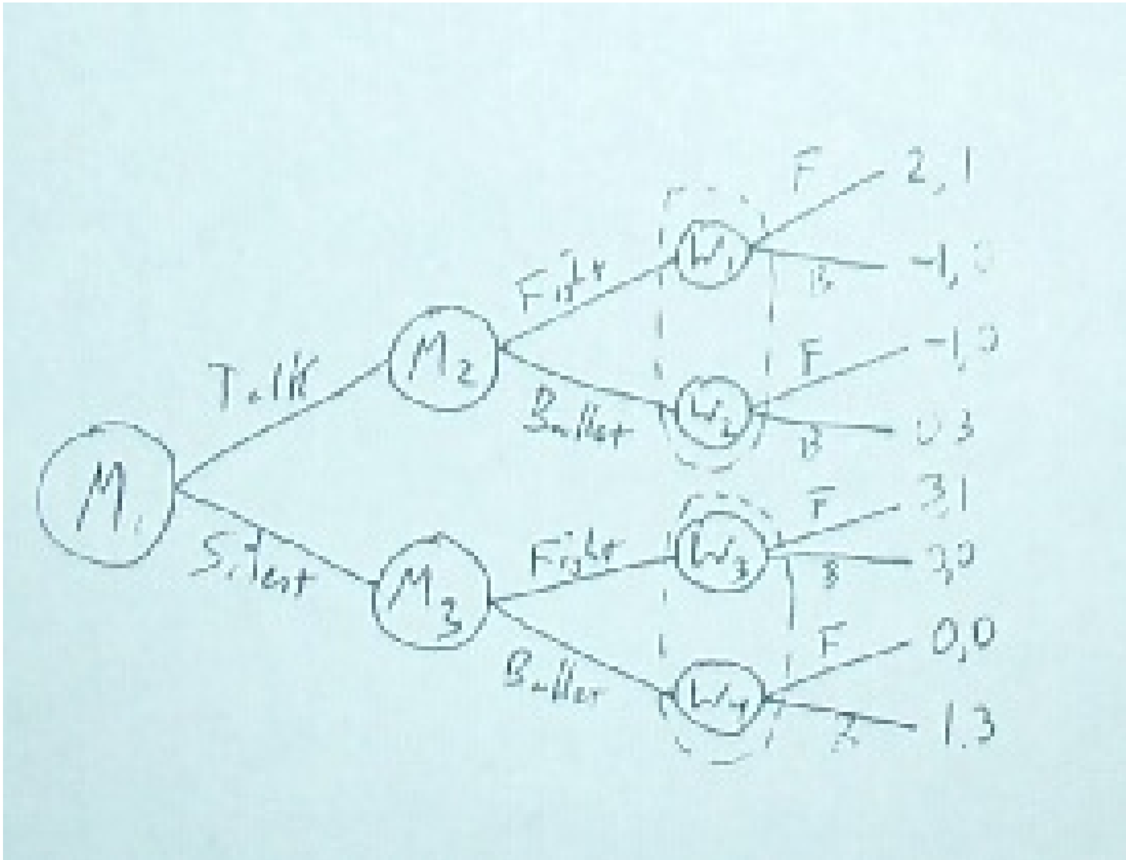

Figure A6.1: The Extensive Form for the Expensive Talk Game

(b) What are the strategy sets for the game? (start with the woman's)

Answer. The woman has two information sets at which to choose moves, and the man has three. Table A6.1 shows the woman's four strategies.

Table A6.1: The Woman's Strategies in "The Expensive Talk Game"

Strategy	W_1, W_2	W_3, W_4
1	F	F
2	F	B
3	B	F
4	B	B

Table A6.2 shows the man's eight strategies, of which only the boldfaced four are important, since the others differ only in portions of the game tree that the man knows he will never reach unless he trembles at M_1.

Table A6.2: The Man's Strategies in the Expensive Talk Game

Strategy	M_1	M_2	M_3
1	**T**	**F**	F
2	T	F	B
3	**T**	**B**	B
4	T	B	F
5	**S**	F	**F**
6	S	B	F
7	**S**	B	**B**
8	S	F	B

(c) What are the three perfect pure-strategy equilibrium outcomes in terms of observed actions? (Remember: strategies are not the same thing as outcomes.)

Answer. SFF, SBB, TFF.

The equilibrium that supports SBB is $[(S, B|S, B|T),\ (B|S, B|T)]$.

TBB is not an equilibrium outcome. That is because the Man would deviate to Silence, saving 1 payoff unit without changing the actions each player took.

(d) Describe the equilibrium strategies for a perfect equilibrium in which the man chooses to talk.

Answer. Woman: $(F|T, B|S)$ and Man: $(T, F|T, B|S)$.

(e) The idea of "forward induction" says that an equilibrium should remain an equilibrium even if strategies dominated in that equilibrium are removed from the game and the procedure is iterated. Show that this procedure rules out SBB as an equilibrium outcome.

See Van Damme (1989). In fact, this procedure rules out TFF *(Talk, Fight, Fight)* also.

Answer. First delete the man's strategy of (T, B), which is dominated by (S, B) whatever the woman's strategy may be. Without this strategy in the game, if the woman sees the man deviate and choose $Talk$, she knows that the man must choose $Fight$. Her strategies of $(B|T, F|S)$ and $(B|T, B|S)$ are now dominated, so let us drop those. But then the man's strategy of (S, B) is dominated by $(T, F|T, B|S)$. The man will therefore choose to $Talk$, and the SBB equilibrium is broken.

This is a strange result. More intuitively: if the equilibrium is SBB, but the man chooses $Talk$, the argument is that the woman should think that the man would not do anything purposeless, so it must be that he intends to choose $Fight$. She therefore will choose $Fight$ herself, and the man is quite happy to choose $Talk$ in anticipation of her response. Taking forward induction one step further: TFF is not an equilibrium, because now that SBB has been ruled out, if the man chooses $Silence$, the woman should conclude it is because he thinks he can thereby get the SFF payoff. She decides that he will choose $Fight$, and so she will choose it herself. This makes it profitable for the man to deviate to SFF from TFF.

6.4. Lack of Common Knowledge

This problem looks at what happens if the parameter values in *Entry Deterrence V* are changed.

(a) Why does $Pr(Strong|Enter, Nature\ said\ nothing) = 0.95$ not support the equilibrium in Section 6.3?

Answer. Under these beliefs, if the entrant deviates and enters, the incumbent's expected payoff from $Fight$ is 15 $(= 0.95(0) + 0.05(300))$, which is less than the 50 he can get from $Collude$.

(b) Why is the equilibrium in Section 6.3 not an equilibrium if 0.7 is the probability that Nature tells the incumbent?

Answer. The entrant would deviate to $Enter|Strong$. If the entrant is strong, he expects the incumbent to fight with probability 0.3 and collude with probability 0.7. The payoff from entry is then 25 $(= 0.3(-10) + 0.7(40))$, which is greater than the 0 from staying out.

(c) Describe the equilibrium if 0.7 is the probability that Nature tells the incumbent. For what out-of-equilibrium beliefs does this remain the equilibrium?

Answer. The equilibrium when Nature tells with probability 0.7 is in mixed strategies, because in a pure-strategy equilibrium the incumbent could deduce the entrant's type from whether the entrant enters or not. If only strong entrants entered, the incumbent would never fight entry, and weak entrants would also enter. The equilibrium is

Entrant: $Enter|Strong$
$Enter\ with\ probability\ \theta = 0.2|Weak$

Incumbent: *Collude|(Enter, Nature said "Strong")*
Fight|(Enter, Nature said "Weak"),
$Collude\ with\ probability\ \gamma = 17/22|(Enter, Nature\ said\ nothing)$

The strong entrant enters because his expected payoff is

$$\begin{aligned} \pi_e(Enter|Strong) &= 0.7(40) + 0.3(\gamma(40) + (1-\gamma)(-10)) \\ &= 28 + 12(17/22) - 3(5/22) \\ &> 0. \end{aligned} \tag{3}$$

The weak entrant must be indifferent between entering and staying out, so

$$\pi_e(Enter|Weak) = 0.7(-10) + 0.3(\gamma(40) + (1-\gamma)(-10)) = \pi_e(Stay\ out|Weak) = 0, \tag{4}$$

which when solved yields $\gamma = 17/22$.

If the incumbent observes that the entrant has entered, he knows that the entrant might be either strong (probability 0.5) or weak (probability 0.5θ). Using Bayes's Rule and equating the incumbent's payoffs from fighting and colluding gives

$$\pi_i(Fight) = 50 = \pi_i(Collude) = \left(\frac{0.5}{0.5+0.5\theta}\right)(0) + \left(\frac{0.5\theta}{0.5+0.5\theta}\right)(300). \tag{5}$$

Solving equation (5) yields $\theta = 0.2$.

Since there is no behavior that could never be observed in equilibrium, no out-of-equilibrium beliefs need be specified.