G604 Midterm, October 22, 2003: ANSWERS

This is a closed-book test, except that you may use one single-sided page of notes. Answer each question as best you can. If you get lost in solving equations, write down in words what you are trying to do and what you think would come out of the mathematical analysis.

This is probably a difficult exam–particularly questions 2c and 5, so budget your time carefully.

Scores: 30, 38, 39, 42, 46, 51, 52, 57, 64, 64.

Since performance was low, I'll substitute your final exam grade for your midterm grade if you do better on the final.

General advice for tests:

1. Try to write something on every question. Start with your intuitive guess as to the answer. Later,cross that out if you decide it is a wrong guess.

2. Understand the basic principle of working back from the end of the game (see question 5).

3. Understand the basic principle of expected utility– that if there is a 70-30 gamble between X and Y, expected utility is .7EU(x) + .3 EU(y), not U(.7X + .3Y). (questions 1,2,3)

4. Understand the basic principle of maximizing the utility of a series of cash flows over time. In particular, note two ideas:

(a) Induction. If your utility at the start of period 1 is U_1, then $U_1 = V(x) + \frac{1}{1+r}U_2$, where V(x) is the flow value of utility from consumption x and U_2 is utility viewed at the start of period 2.

(b) Perpetuities. The value of x per period at the end of each period from now onwards is x/r, where r is the discount rate. (The value of x each period at the start of a period from now till infinity is $x + x/r$)

1. Firms Apex and Brydox are both thinking of starting stores in a small town which only has demand big enough for one firm to make a profit. In each year in which two stores operate, each store loses 1 million dollars. In each year in which only one store operates, that store earns 3 million dollars. You can think of these cash flows as occurring at the end of the year. The market interest rate is 10%. If a store ever exits the market, it cannot re- enter.

(a) (10 points) Describe an equilibrium in which Apex is certain to be the only firm to start a store.

Answer. Apex operates in each period; Brydox stays out in each period, including the first.

This equilibrium does not require any sort of precommitment– it is not necessary for Apex to choose its strategy first, or to sign a contract with customers, or anything like that. All that is necessary is that both players expect this equilibrium to be played out. It will then be self-fulfilling–neither player has incentive to unilaterally deviate.

(b) (20 points) Describe an equilibrium in which for 1 or more periods each firm has probability γ of operating a store. What is the value of γ? What is the expected payoff of Brydox? What is the probability that Apex acquires a monopoly after one year of competition with Brydox?

Answer. Note that this is not a one-period game. Not only does the problem not say it is restricted to one period, but it mentions the interest rate and the impossibility of re-entry, both of which would be inapplicable in a one-period model (except to the trivial extent that a payment value would be 3/(1+r) instead of 3).

If each firm has probability γ of operating a store, it is using a mixed strategy and must be indifferent between the pure-strategy payoffs of $\pi(out) = 0$ and

$$\pi(in) = \gamma(-1 + \frac{1}{1+r}\pi(in)) + (1-\gamma)\frac{3}{r}. \quad (1)$$

where this is made up of the probability γ that the other player operates a store and the probability $(1-\gamma)$ that it does not, and $\frac{3}{r}$ is the value of a perpetual stream of 3 per period. (Note that the losses and gains are per year–this is not a one-period game.) We can rewrite this as

$$\pi(in) - \gamma\frac{1}{1+r}\pi(in) = -\gamma + \frac{3(1-\gamma)}{r}. \quad (2)$$

so

$$\pi(in) = \frac{1+r}{1+r-\gamma}(-\gamma + \frac{3(1-\gamma)}{r}). \quad (3)$$

Equating $\pi(in)$ to $\pi(out) = 0$, we get

$$\frac{1+r}{1+r-\gamma}(-\gamma + \frac{3(1-\gamma)}{r}) = 0 \quad (4)$$

so

$$(-\gamma + \frac{3(1-\gamma)}{r}) = 0, \quad (5)$$

which implies that $-r\gamma + 3 - 3\gamma = 0$, so $3 = (3+r)\gamma$ and $\gamma = \frac{3}{3+r}$.

A couple of students made the following mistake. First, they set this up as a one-period game. Second, they set up the maximization problem for Apex to maximize by choice of γ_a:

$$\gamma[\gamma(-1) + (1-\gamma)(3)] + [1-\gamma](0), \quad (6)$$

with first order condition

$$-8\gamma + 3 = 0. \quad (7)$$

The fallacy in this (besides confining the payoffs to one period) is that the payoff being maximized really has two separate γ's in it– the variable Apex is maximizing, and the one Brydox is maximizing. In equilibrium, these turn out to be equal, but one can't assume that beforehand. The proper way to write Apex's problem is that he maximizes by choice of γ_a:

$$\gamma_a[\gamma_b(-1) + (1-\gamma_b)(3)] + [1-\gamma_a](0), \quad (8)$$

with first order condition

$$[\gamma_b(-1) + (1-\gamma_b)(3)] = 0 \quad (9)$$

so $-4\gamma_b + 3 = 0$. What this means, as the book explains, is that γ_b must take the value 3/4 for Apex to have an interior solution to his maximization problem.

2. Someone with constant absolute risk aversion is thinking of buying a stock which will have a price of P= \$100 next year with probability .5 and \$200 with probability .5.

(a) (3 points) What bounds can you put on the most this person will pay for the stock?

ANSWER: Greater than 100, less than 150. The expected value is 150, so since the person is risk averse, he will not pay more than 150. But he will not require less than 100 either.

Only one person got this right! Commonly, people found the upper bound of 150, but not the lower bound of 100. If the person is extremely risk averse, the most he will pay will be just a little above 100.

(b) (3 points) The same person is thinking of buying a "put" option on the stock. The option has a strike price of \$150, meaning that it gives him the right to sell the stock at a price of \$150. (Note that if he doesn't own a share of the stock, he can sell the put next year to someone else who does have stock to sell.) The person's plan is to buy two put options, at a price of Z each, and one share of stock, at the current market price of 130. The market interest rate is zero.

What is the most a risk-neutral person would pay for the put?

Answer. 25. This is because with probability .5 the price is 200, and the put is worthless— exercising it, the person would be selling stock for 150 when the market price is higher. With probability .5, however, the price is 100, and the person gets to sell the stock at 150 when the market price is 100, for a profit of 50. The expected value of the profit is thus 25.

The current price is not relevant to this. The current price is 130, so the person could exercise the put immediately and earn 20, but by waiting he gets an expected value of 25, which is better.

Only one person got this right.

(c) (4 points) What is the greatest put price Z such that this risk- averse person will go through with his plan?

Answer. We need to think about the person's payoff if he goes through with the plan. The person pays 2Z+130 immediately for the two puts and the share of stock.

With probability .5, P=100 next year. Then the person can sell his stock for 100, and his two puts for 50 each, for an overall payoff of -(2Z+130)+ 100+2(50)= -2Z +70.

With probability .5, P=200 next year. Then the person can sell his stock for 200, and his two puts for 0 each, for an overall payoff of -(2Z+130)+ 200 = -2Z +70.

Thus, the risk-averse person is perfectly hedged. He bears no risk. And so he is willing to pay Z=35, more than the risk-neutral person. That is because the put option acts as insurance for him.

3. A principal hires an agent using contract $w(q)$. The agent accepts or rejects the contract. If he accepts, he chooses effort e, where e is either 2 or 3, and output is $q = e + u$, where u is random noise, equal to either -1 or +1 with equal probability.

If the agent rejects the contract, then $\pi_{agent} = 1$ and $\pi_{principal} = 0$. If the agent accepts the contract, then $\pi_{agent} = w + log(w) - e$ and $\pi_{principal} = q - w$.

(a) (10 points) What effort level would the agent choose if he owned the firm, so $w = q$?

Answer. The agent's utility is either

$$U(e = 2) = 0.5[2 - 1 + log(2 - 1) - 2] + 0.5[2 + 1 + log(2 + 1) - 2] = .5log(3) \quad (10)$$

or

$$U(e = 3) = 0.5[3 - 1 + log(3 - 1) - 3] + 0.5[3 + 1 + log(3 + 1) - 3] = .5[log(2) + log(3)] = .5log(6). \quad (11)$$

So the agent would choose $e = 3$, for a utility of $.5log(6)$.

People made surprisingly naive mistakes about expected utility. If output is either 1 or 3, with expected value of 2, the expected utility is not the expected utility of a sure output of 2. Rather, it is $.5U(1) + .5U(3)$. If you don't understand that, you don't understand concave utility of risk aversion at all.

(b) (10 points) If the principal could use a contract $w(e)$ instead of having to use a contract $w(q)$, what contract could he use and what would be the agent's effort choice and utility in equilibrium? (You need not solve arithmetically for the value of the wage– just show what equation it must solve).

Answer. The principal would desire high effort, because we found that is efficient in part (a), and when the principal can take over some of the risk, that will increase welfare still further.

To minimize the amount paid, the principal would not want to put any risk on the agent (since the agent would have to be compensated for the risk). Thus, of the infinity of contracts that could be used, the cheapest is the one that pays a constant wage z for effort of $e = 3$.

Incentive compatibility is easy here, since effort is observable. A contract that will work is the forcing contract $w(e = 2) = 0$ and $w(e = 3) = z$ for some value z.

The participation constraint is $U(e = 3) \geq 1$, so

$$U(e = 3) = z + log(z) - 3 \geq 1. \tag{12}$$

This could be solved for z, but you didn't have to solve for it on the test.

The agent's utility would thus be 1 and he would choose $e = 3$.

Note that a boiling in oil contract could be used here if effort was not observed, since q=1 can occur only with e=2.

4. (10 points) Explain, in a sentence or two for each, some idea that can be found in the selections we read by two of the following three economists: (a) Adam Smith, (b) Schumpeter, and (c) Stigler.

Answer. (a) Smith: Division of labor, as in the pin factory, allows cheaper production.

(b) Schumpeter: Thinking new thoughts is a rare skill.

(c) Stigler: Regulation is designed to maximize the regulator's payoff, not the public's.

5. Suppose the Prisoner's Dilemma in Table 1 is repeated T times with no discounting. Player Row is definitely rational. With probability $1 - \delta$, Column is also rational, but with probability δ he simply plays the Grim Strategy: Deny unless some player has ever chosen Confess, in which case choose Confess. Row cannot observe whether Column is rational.

In an IO context, we might replace $Deny$ and $Confess$ with $High$ and Low prices, but I will keep the game theory example here since you are more familiar with it.

Table 1: The Prisoner's Dilemma

		Column		
		Deny		*Confess*
	Deny	-1,-1	→	-10, 0
Row		↓		↓
	Confess	0,-10	→	**- 8,-8**

Payoffs to: (Row,Column)

(a) (10 points) Show that if Row knows Column is irrational, there is some T great enough that Row will start by playing $Deny$ and will only play $Confess$ in the last N periods. Find the value of N.

Answer. Consider *Row*'s decision. Suppose nobody has yet chosen $Confess$. He will definitely choose $Confess$ in period T to get 0 instead of -8 or -8 instead of -10.

In period $T - 1$, his continuation payoff if he chooses $Confess$ is

$$\pi(Confess) = 0 - 8 = -8 \tag{13}$$

whereas if he chooses Deny it is

$$\pi(Deny) = -1 - 0 = -1 \tag{14}$$

Thus, it is better to choose *Deny* in period T-1. But this reasoning just gets stronger as we do to earlier periods, because the payoff from playing Deny is $-1 + -1 + ... + -1 + 0$ versus the payoff from Confess of $0 + -8 + -8 + ... + -8 + -8$. Thus, $N = 1$.

If you tried to answer this question by constructing Row's payoff viewed from the first period, instead of starting from the end and working backwards, you would have a much more difficult time finding the answer, and nobody did that successfully. Working back from the end, on the other hand, the problem is easy.

(b) (10 points) Suppose Row believes that Column is irrational with probability δ and that if he is rational, he will start choosing *Confess* (and with probability one) in period $T-K$ but not before. Show that Row will choose *Deny* in periods 0 through $T-K$ if and only if δ equals some value $\delta_0(K)$ or greater, and that δ_0 varies with K but not with T.

Answer. Suppose Row is deciding on his move for period $T-K$ and nobody has yet played *Confess*.

If Row picks *Deny*, then if Column is an irrational, Grim, player Column will choose *Deny* for all of these last K periods, and Row can look forward to $K-1$ periods of payoffs of -1 and one period (at the end) of a payoff of 0; but if Column is rational, Column will choose *Confess* in period $T-K$, for a payoff of -10 to Row in that period, and after that both will play *Confess*, for $K-1$ periods of payoffs of -8.

Row's continuation payoff from *Deny* is therefore

$$\pi(Deny) = \delta[(K-1)(-1)+0] + (1-\delta)[-10+(K-1)(-8)] \tag{15}$$

If Row picks *Confess*, then if Column is the irrational, Grim, player Column will pick *Deny* in period $T-K$, for a payoff of 0 to Row, and *Confess* thereafter, for $K-1$ periods of payoffs of -8; but if Column is rational, Column will also pick *Confess* in period $T-K$ and thereafter, for a total of K periods of -8 payoffs. Row's continuation payoff from *Confess* is therefore

$$\pi(Confess) = \delta[0+(K-1)(-8)] + (1-\delta)[(K)(-8)] \tag{16}$$

The difference $[\pi(Deny) - \pi(Confess)] = 0$ if

$$\begin{aligned}
&\delta[(K-1)(-1)+0] + (1-\delta)[-10+(K-1)(-8)] = \delta[0+(K-1)(-8)] + (1-\delta)[(K)(-8)] \\
&-\delta(K-1) - (1-\delta)(10) - (K-1)(1-\delta)(8) = -8\delta(K-1) - 8K(1-\delta) \\
&7\delta(K-1) - (1-\delta)(10) - K(1-\delta)(8) + (1)(1-\delta)(8) = -K(1-\delta)(8) \\
&7\delta(K-1) - (1-\delta)(10) + (1)(1-\delta)(8) = 0 \\
&\delta(7)(K-1) - 10 + \delta(10) + 8 - \delta(8) = 0 \\
&\delta[(7)(K-1)+2] = 2
\end{aligned} \tag{17}$$

$\delta = \frac{2}{(7)(K-1)+2}$.

Clearly this δ depends on K but not on T.

If δ is big enough that *Row* would not want to *Confess* in period $T-K$, then he will not want to *Confess* in period $T-K-1$ either, a fortiori, because that would merely extend the period of -8 payoffs replacing -1 payoff by 1 period. And a fortiori he would not choose *Confess* before $T-K-1$.

(c) (0 points) I decided to leave part (c) as an assertion rather than have you prove it.

It follows from (b) that if Row believes that Column is irrational with probability δ and that if he is rational, he will start choosing *Confess* as part of a mixed strategy with positive probability some time after $T - S$ but not before. Row will choose *Deny* in periods 0 through $T - S$ if and only if δ equals some value $\delta_1(S)$ or greater, and that δ_1 varies with S but not with T.

This fact will be useful in part (d).

(d) (10 points) Prove the following proposition. (Note that you can do this even if you did not solve parts (a), (b), and (c), just taking the results from those sections as lemmas for this proof.)

Proposition: In any perfect Bayesian equilibrium, the number of stages in which either player chooses Confess is less than some number M that depends on δ but not on T.

Answer. Result (c) says that Row will not start choosing $Confess$ at least until the last S periods, where S depends on δ but not on T, even if he thinks that a rational Column has some probability of choosing $Confess$ starting in period $T - S$.

It's actually easier, though to go back to result (b) and use it instead, finessing the issue of mixed strategies. Suppose δ is big enough that that Row will never choose $Confess$ until at least period $T-K+1$ (he might then, if he thought a rational Column would start choosing $Confess$ in period $T - K + 1$). In that case Column will not choose $Confess$ until period $T-K$ at the earliest, one period before Row might choose $Confess$. This is true for the reasons we saw in part (a): a player will not choose $Confess$ until one period before he thinks the other player will choose $Confess$.

Hence, for every K, there is a δ such that neither player will choose $Confess$ before $T - K$. Such a value of δ was found in part (b), and it depends on K but not on T. Set $M = K$ and the proof is done.

Note, though, that this proof does not establish that some player will choose $Confess$ exactly at period $T - K$— just some time *at or after* $T - K$. It is a weak bound.