2 Information

24 January 2006 Eric Rasmusen, Erasmuse@indiana.edu. http://www.rasmusen.org/.

The two-step procedure of checking a Nash equilibrium has now become a three-step procedure:

1 Propose a strategy profile.
2 See what beliefs the strategy profile generates when players update their beliefs in response to each others' moves.
3 Check that given those beliefs together with the strategies of the other players each player is choosing a best response for himself.

The rules of the game specify each player's initial beliefs, and Bayes's Rule is the rational way to update beliefs. Suppose, for example, that Jones starts with a particular prior belief, $Prob(Nature\ chose\ (A))$. In Follow-the- Leader III, this equals 0.7. He then observes Smith's move — $Large$, perhaps. Seeing $Large$ should make Jones update to the **posterior** belief,

$$Prob(Nature\ chose\ (A))|Smith\ chose\ Large),$$

where the symbol "|" denotes "conditional upon" or "given that."

Bayes's Rule shows how to revise the prior belief in the light of new information such as Smith's move.

Since what Jones is trying to calculate is $Prob(A|Large)$:

$$Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large)}. \tag{1}$$

Substituting the expression for $Prob(Large)$ gives the final result, a version of Bayes's Rule.

$$Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) + Prob(Large|C)Prob(C)}. \tag{2}$$

More generally, for Nature's move x and the observed data,

$$Prob(x|data) = \frac{Prob(data|x)Prob(x)}{Prob(data)} \tag{3}$$

Equation (4) is a verbal form of Bayes's Rule, which is useful for remembering the terminology.

$$(Posterior\ for\ Nature's\ Move) = \frac{(Likelihood\ of\ Player's\ Move) \cdot (Prior\ for\ Nature's\ Move)}{(Marginal\ likelihood\ of\ Player's\ Move)}. \tag{4}$$

Updating Beliefs in Follow-the-Leader III

Let us now return to the numbers in Follow-the-Leader III to use the belief-updating rule that was just derived. Jones has a prior belief that the probability of event "Nature picks state (A)" is 0.7 and he needs to update that belief on seeing the data "Smith picks *Large*". His prior is $Prob(A) = 0.7$, and we wish to calculate $Prob(A|Large)$.

To use Bayes's Rule from equation (2), we need the values of $Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C$ These values depend on what Smith does in equilibrium, so Jones's beliefs cannot be calculated independently of the equilibrium.

A candidate for equilibrium in Follow-the-Leader III is for Smith to choose *Large* if the state is (A) or (B) and *Small* if it is (C), and for Jones to respond to *Large* with *Large* and to *Small* with *Small*. This can be abbreviated as $(L|A, L|B, S|C; L|L, S|S)$. Let us test that this is an equilibrium, starting with the calculation of $Prob(A|Large)$.

A candidate for equilibrium in Follow-the-Leader III is for Smith to choose *Large* if the state is (A) or (B) and *Small* if it is (C), and for Jones to respond to *Large* with *Large* and to *Small* with *Small*. This can be abbreviated as $(L|A, L|B, S|C; L|L, S|S)$. Let us test that this is an equilibrium, starting with the calculation of $Prob(A|Large)$.

If Jones observes *Large*, he can rule out state (C), but he does not know whether the state is (A) or (B). Bayes's Rule tells him that the posterior probability of state (A) is

$$\begin{aligned} Prob(A|Large) &= \frac{(1)(0.7)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} \\ &= 0.875. \end{aligned} \tag{5}$$

The posterior probability of state (B) must then be $1 - 0.875 = 0.125$, which could also be calculated from Bayes's Rule, as follows:

$$\begin{aligned} (B|Large) &= \frac{(1)(0.1)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} \\ &= 0.125. \end{aligned} \tag{6}$$

Jones must use Smith's strategy in the proposed equilibrium to find numbers for $Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C)$. As always in Nash equilibrium, the modeller assumes that the players know which equilibrium strategies are being played out, even though they do not know which particular actions are being chosen.

Given that Jones believes that the state is (A) with probability 0.875 and state (B) with probability 0.125, his best response is $Large$, even though he knows that if the state were actually (B) the better response would be $Small$. Given that he observes $Large$, Jones's expected payoff from $Small$ is -0.625 ($= 0.875[-1] + 0.125[2]$), but from $Large$ it is 1.875 ($= 0.875[2] + 0.125[1]$). The strategy profile $(L|A, L|B, S|C; L|L, S|S)$ is a bayesian equilibrium.

A similar calculation can be done for $Prob(A|Small)$. Using Bayes's Rule, equation (2) becomes

$$Prob(A|Small) = \frac{(0)(0.7)}{(0)(0.7) + (0)(0.1) + (1)(0.2)} = 0. \tag{7}$$

Given that he believes the state is (C), Jones's best response to $Small$ is $Small$, which agrees with our proposed equilibrium.

Smith's best responses are much simpler. Given that Jones will imitate his action, Smith does best by following his equilibrium strategy of $(L|A, L|B, S|C)$.

The calculations are relatively simple because Smith uses a nonrandom strategy in equilibrium, so, for instance, $Prob(Small|A) = 0$ in equation (7). Consider what happens if Smith uses a random strategy of picking *Large* with probability 0.2 in state (A), 0.6 in state (B), and 0.3 in state (C) (we will analyze such "mixed" strategies in Chapter 3). The equivalent of equation (5) is

$$Prob(A|Large) = \frac{(0.2)(0.7)}{(0.2)(0.7) + (0.6)(0.1) + (0.3)(0.2)} = 0.54 \quad (8)$$

If he sees *Large*, Jones's best guess is still that Nature chose state (A), even though in state (A) Smith has the smallest probability of choosing *Large*, but Jones's subjective posterior probability, $Pr(A|Large)$, has fallen to 0.54 from his prior of $Pr(A) = 0.7$.