9 February 2006. Eric Rasmusen, Erasmuse@indiana.edu.
Http://www.rasmusen.org.

Overheads for Chapter 13 (Auctions) of *Games and Information*. These do not cover the entire chapter, just enough for two lectures.

13.1 Values Private and Common, Continuous and Discrete

In a **private-value auction**, a bidder can learn nothing about his value from knowing the values of the other bidders.

In an **independent private-value auction**, knowing his own value tells him nothing about other bidders' values.

In an **affiliated private-value auction** he might be able to use knowledge of his own value to deduce something about other players' values.

In a **pure common-value auction**, the bidders have identical values, but each bidder forms his own estimate on the basis of his own private information.

An example is bidding for U. S. Treasury bills. A bidder's estimate would change if he could sneak a look at the other bidders' estimates, because they are all trying to estimate the same true value.

It is common for economists to speak of mixed auctions as "common-value" auctions, since their properties are closer to those of common-value auctions. Krishna (2002) has used the term **interdependent value** for the mixed case.

The Ten-Sixteen Auction

Players: One seller and two bidders.

Order of Play:
0. Nature chooses Bidder i's value for the object to be either $v_i = 10$ or $v_i = 16$, with equal probability. (The seller's value is zero.)
1. The seller chooses a mechanism $[G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i})]$ that takes payments t and gives the object with probability G to player i (including the seller) if he announces that his value is $\tilde{v}_i$ and the other players announce $\tilde{v}_{-i}$. He also chooses the procedure in which bidders select $\tilde{v}_i$ (sequentially, simultaneously, etc.).
2. Each bidder simultaneously chooses to participate in the auction or to stay out.
3. The bidders and the seller choose $\tilde{v}$ according to the mechanism procedure.
4. The object is allocated and transfers are paid according to the mechanism.

Payoffs:
The seller's payoff is

$$\pi_s = \sum_{i=1}^{n} t(\tilde{v}_i, \tilde{v}_{-i}) \tag{1}$$

Bidder i's payoff is zero if he does not participate, and otherwise is

$$\pi_i(v_i) = G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i}) \tag{2}$$

The Continuous-Value Auction

Players: One seller and two bidders.

Order of Play:

0. Nature chooses Bidder i's value for the object, v_i, using the strictly positive, atomless density $f(v)$ on the interval $[\underline{v}, \overline{v}]$.
1. The seller chooses a mechanism $[G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i})]$ that takes payments t and gives the object with probability G to player i (including the seller) if he announces that his value is $\tilde{v}_i$ and the other players announce $\tilde{v}_{-i}$. He also chooses the procedure in which bidders select $\tilde{v}_i$ (sequentially, simultaneously, etc.).
2. Each bidder simultaneously chooses to participate in the auction or to stay out.
3. The bidders and the seller choose $\tilde{v}$ according to the mechanism procedure.
4. The object is allocated and transfers are paid according to the mechanism, if it was accepted by all bidders.

Payoffs:

The seller's payoff is

$$\pi_s = \sum_{i=1}^{n} t(\tilde{v}_i, \tilde{v}_{-i}) \tag{3}$$

Bidder i's payoff is zero if he does not participate, and otherwise is

$$\pi_i(v_i) = G(\tilde{v}_i, \tilde{v}_{-i})v_i - t(\tilde{v}_i, \tilde{v}_{-i}) \tag{4}$$

13.2 Optimal Strategies under Different Rules in Private-Value Auctions

Ascending (English, open-cry, open-exit)

Rules

Each bidder is free to revise his bid upwards. When no bidder wishes to revise his bid further, the highest bidder wins the object and pays his bid.

Strategies

A bidder's strategy is his series of bids as a function of (1) his value, (2) his prior estimate of other bidders' values, and (3) the past bids of all the bidders. His bid can therefore be updated as his information set changes.

Payoffs

The winner's payoff is his value minus his highest bid ($t = p$ for him and $t = 0$ for everyone else). The losers' payoffs are zero.

In the Continuous-Value Auction, denote the highest announced value by $\tilde{v}_{(1)}$, the second-highest by $\tilde{v}_{(2)}$, and so forth. The ascending auction can be seen as a mechanism in which the highest bidder gets the object with probability $G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 1$ at price $t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = \tilde{v}_{(2)}$, and for $i \neq 1$, $G(\tilde{v}_{(i)}, \tilde{v}_{-i}) = 0$ and $t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 0$. This is incentive compatible, since a player's value announcement only matters if his value is highest, and he then wants to win if and only if the price is less than or equal to his value. It satisfies the participation constraint because his lowest possible payoff following that strategy is zero, and his payoff is higher if he wins and $\tilde{v}_{(1)} > \tilde{v}_{(2)}$.

Since each bidder's expected payoff is strictly positive, the optimal mechanism for the seller would be more complicated. As we will discuss later, it would include a **reserve price** p^* below which the object would remain unsold, changing the first part of the mechanism to $G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 1$ and $t(\tilde{v}_{(1)}, \tilde{v}_{-1}) = Max\{\tilde{v}_{(2)}, p^*\}$ if $\tilde{v}_{(1)} \geq p^*$ but $G(\tilde{v}_{(1)}, \tilde{v}_{-1}) = 0$ if $\tilde{v}_{(1)} < p^*$. We have seen in Chapter 10 that optimal mechanisms are not always efficient, and this is an example: the object will go unsold if $\tilde{v}_{(1)} < p^*$.

In the Ten-Sixteen Auction, the seller's value is $v_s = 0$ and each of two bidders' private values v_1 and v_2 is either 10 or 16 with equal probability, known only to the bidder himself. A bidder's optimal strategy in the ascending auction would be to set his bid or bid ceiling to $p(v = 10) = 10$ and $p(v = 16) = 16$. His expected payoff would be

$$\begin{aligned} \pi(v = 10) &= 0 \\ \pi(v = 16) &= 0.5(16 - 10) + 0.5(16 - 16) = 3 \end{aligned} \tag{5}$$

The expected price, the payoff to the seller, is

$$\pi_s = 0.5^2(10) + 0.5^2(16) + 2(0.5)^2(10) = 2.5 + 4 + 5 = 11.5 \tag{6}$$

First-Price (first-price sealed-bid)

Rules
Each bidder submits one bid, in ignorance of the other bids. The highest bidder pays his bid and wins the object.

Strategies
A bidder's strategy is his bid as a function of his value.

Payoffs
The winner's payoff is his value minus his bid. The losers' payoffs are zero.

Discussion
In the first-price auction what the winning bidder wants to do is to have submitted a sealed bid just enough higher than the second-highest bid to win.

If all the bidders' values are common knowledge and he can predict the second- highest bid perfectly, this is a simple problem.

If the values are private information, then he has to guess at the second-highest bid, however, and take a gamble.

His tradeoff is between bidding high–thus winning more often–and bidding low–thus benefiting more if the bid wins.

The First-Price Auction with a Continuous Distribution of Values

Suppose Nature independently assigns values to n risk-neutral bidders using the continuous density $f(v) > 0$ (with cumulative probability $F(v)$) on the support $[0, \bar{v}]$.

A bidder's payoff as a function of his value v and his bid function $p(v)$ is, letting $G(p(v))$ denote the probability of winning with a particular $p(v)$:

$$\pi(v, p(v)) = G(p(v))[v - p(v)]. \tag{7}$$

Let us first prove a lemma.

Lemma: If player's equilibrium bid function is differentiable, it is strictly increasing in his value: $p'(v) > 0$.

Proof: The first-order condition from payoff (7) is

$$\frac{d\pi(v)}{dp} = G'(v-p) - G = 0. \tag{8}$$

The optimum is an interior solution because at $p_i = 0$ the payoff is increasing and if p_i becomes large enough, π is negative. Thus, $\frac{d^2\pi(v_i)}{dp_i^2} \leq 0$ at the optimum. Using the implicit function theorem and the fact that $\frac{d^2\pi(v_i)}{dp_i dv_i} = G' \geq 0$ because a higher bid does not yield a lower probability of winning, we can conclude that $\frac{dp_i}{dv_i} \geq 0$, at least if the bid function is differentiable. But it cannot be that $\frac{dp_i}{dv_i} = 0$, because then there would be values v_1 and v_2 such that $p_1 = p_2 = p$ and then

$$\frac{d\pi(v_1)}{dp_1} = G'(p)(v_1 - p) - G(p) = 0 = \frac{d\pi(v_2)}{dp_2} = G'(p)(v_2 - p) - G(p), \tag{9}$$

which cannot be true. So the bidder bids more if his value is higher. Q.E.D.

Now let us try to find an equilibrium bid function. From equation (7), it is

$$p(v) = v - \frac{\pi(v, p(v))}{G(p(v))}. \tag{10}$$

First, tackle $G(p(v))$. Monotonicity of the bid function (from Lemma 1) implies that the bidder with the greatest v will bid highest and win. Thus, the probability $G(p(v))$ that a bidder with price p_i will win is the probability that v_i is the highest value of all n bidders. The probability that a bidder's value v is the highest is $F(v)^{n-1}$, the probability that each of the other $(n-1)$ bidders has a value less than v. Thus,

$$G(p(v)) = F(v)^{n-1}. \tag{11}$$

Next think about $\pi(v, p(v))$. The Envelope Theorem says that if $\pi(v, p(v))$ is the value of a function maximized by choice of $p(v)$ then its total derivative with respect to v equals its partial derivative, because $\frac{\partial \pi}{\partial p} = 0$:

$$\frac{d\pi(v,p(v))}{dv} = \frac{\partial \pi(v,p(v))}{\partial p}\frac{\partial p}{\partial v} + \frac{\partial \pi(v,p(v))}{\partial v} = \frac{\partial \pi(v,p(v))}{\partial v}. \tag{12}$$

We can apply the Envelope Theorem to equation (7) to see how π changes with v assuming $p(v)$ is chosen optimally, which is appropriate because we are characterizing not just any bid function, but the optimal bid function. Thus,

$$\frac{d\pi(v, p(v))}{dv} = G(p(v)). \tag{13}$$

Substituting from equation (11) gives us π's derivative, if not π, as a function of v:

$$\frac{d\pi(v, p(v))}{dv} = F(v)^{n-1}. \tag{14}$$

To get $\pi(v, p(v))$ from its derivative, (14), integrate over all possible values from zero to v and include the a base value of $\pi(0)$ as the constant of integration:

$$\pi(v, p(v)) = \pi(0) + \int_0^v F(x)^{n-1}dx = \int_0^v F(x)^{n-1}dx. \tag{15}$$

The last step is true because a bidder with $v = 0$ will never bid a positive amount and so will have a payoff of $\pi(0, p(0)) = 0$.

We can now return to the bid function in equation (10) and substitute for $G(p(v))$ and $\pi(v, p(v))$ from equations (11) (15):

$$p(v) = v - \frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}}. \tag{16}$$

Suppose $F(v) = v/\bar{v}$, the uniform distribution. Then (16) becomes

$$\begin{aligned} p(v) &= v - \frac{\int_0^v \left(\frac{x}{\bar{v}}\right)^{n-1} dx}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\ &= v - \frac{\Big|_{x=0}^{v} \left(\frac{1}{\bar{v}}\right)^{n-1} \left(\frac{1}{n}\right) x^n}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\ &= v - \frac{\left(\frac{1}{\bar{v}}\right)^{n-1} \left(\frac{1}{n}\right) v^n - 0}{\left(\frac{v}{\bar{v}}\right)^{n-1}} \\ &= v - \frac{v}{n} = \left(\frac{n-1}{n}\right) v. \end{aligned} \tag{17}$$

! If there are two bidders and values are uniform on $[0, 1]$, a bidder should bid $p = v/2$, which since he has probability v of winning yields an expected payoff of $v^2/2$. If $n = 10$ he should bid $\frac{9}{10}v$, which since he has probability v^9 of winning yields him an expected payoff of $v^{10}/10$, quite close to zero if $v < 1$.

The First-Price Auction: A Mixed-Strategy Equilibrium in the Ten-Sixteen Auction

The result in equations (**??**) and (17) depended crucially on the value distribution having a continuous support. When this is not true, the equilibrium in a first-price auction may not even be in pure strategies. Now let each of two bidders' private value v be either 10 or 16 with equal probability and known only to himself.

In a first-price auction, a bidder's optimal strategy is to bid $p(v = 10) = 10$, and if $v = 16$ to use a mixed strategy, mixing over the support $[\underline{p}, \bar{p}]$, where it will turn out that $\underline{p} = 10$ and $\bar{p} = 13$, and the expected payoffs will be:

$$\begin{aligned} \pi(v = 10) &= 0 \\ \pi(v = 16) &= 3 \\ \pi_s &= 11.5. \end{aligned} \tag{18}$$

These are the same payoffs as in the ascending auction, an equivalence we will come back to in a later section.

This will serve as an illustration of how to find an equilibrium mixed strategy when bidders mix over a continuum of pure strategies rather than just between two.

The first step is to see why the equilibrium cannot be in pure strategies.

In any equilibrium, $p(v = 10) = 10$, because if either bidder used the bid $p < 10$, it would cause the other player to deviate to $(p + \epsilon)$, and a bid above 10 exceeds the object's value.

If $v = 16$, however, a player will randomize his bid, as I will now show. Suppose the two bidders are using the pure strategies $p_1(v_1 = 16) = z_1$ and $p_2(v_2 = 16) = z_2$. The values of z_1 and z_2 would lie in $(10, 16]$ because a bid of exactly 10 would lose to the positive-probability bid of $p(v = 10) = 10$ given our tie-breaking assumption and a bid over 16 would exceed the object's value. yield a negative payoff. Either $z_1 = z_2$, or $z_1 \neq z_2$. If $z_1 = z_2$, then each bidder has incentive to deviate to $(z_1 - \epsilon)$ and win with probability one instead of tying. If $z_1 < z_2$, then Bidder 2 will deviate to bid $(z_1 + \epsilon)$. If he does that, however, Bidder 1 would deviate to bid $(z_1 + 2\epsilon)$, so he could win with probability one at trivially higher cost. The same holds true if $z_2 < z_1$. Thus, there is no equilibrium in pure strategies.

The second step is to figure out what pure strategies will be mixed between by a bidder with $v = 16$. It turns out that they form the interval $[10, 13]$. As just explained, the bid $p(v = 16)$ will be no less than 10 (so the bidder can win if his rival's value is 10) and no greater than 16 (which would always win, but unprofitably). The pure strategy of $(p = 10)|(v = 16)$ will win with probability of at least 0.50 (when the other bidder happens to have $v = 10$, given our tie-breaking rule), yielding a payoff of $0.50(16 - 10) = 3$. This rules out bids in $(13, 16]$, since even if they always win, their payoff is less than 3. Thus, the upper bound $\bar{p}$ must be no greater than 13.

The lower bound $\underline{p}$ must be exactly 10. If it were at $(10 + \epsilon)$ then a bid of $(10 - 2\epsilon)$ would have an equal certainty of winning the auction, but would have ϵ higher payoff. Thus, $\underline{p} = 10$.

The upper bound $\bar{p}$ must be exactly 13. If it were any less, then the other player would respond by using the pure strategy of $(\bar{p}+\epsilon)$, which would win with probability one and yield a payoff of greater than the payoff of 3 $(= 0.5(16 - 10))$ from $p = 10$. In a mixed-strategy equilibrium, though, the payoff from any of the strategies mixed between must be equal. Thus, $\bar{p}$ cannot be less than 13.

We are not quite done looking at the strategies mixed between. When a player mixes over a continuum, the modeller must be careful to check for (a) atoms (some particular point which has positive probability, not just positive density), and (b) gaps (intervals within the mixing range with zero probability of bids). Are there any atoms or gaps within the interval [10,13]? No, it turns out.

(a) Bidder 2's mixing density does not have an atom at any point a in $[10, 13]$– no point a has positive probability, as opposed to positive density. An example of such an atom would be if the mixing distribution were the density $m(p) = 1/6$ over the interval $[10, 13]$ plus an atom of probability $1/2$ at $p = 13$, so the cumulative probability would be $M(p) = p/6$ over $[10, 13)$ and $M(13) = 1$. Using $M(p)$, a point such as 11 would have zero probability even though the interval, say, of [10.5, 12.5] would have probability 2/6.

If there were an atom at a, Bidder 1 would respond by putting positive probability on $(a + \epsilon)$ and zero probability on a. But then Bidder 2 would respond by putting zero probability on a and shifting that probability to $(a + 2\epsilon)$.

(b) Bidder 2's mixing density does not have a gap $[g, h]$ anywhere with $g > 10$ and $h < 13$. If it did, then Bidder 1's payoff from bidding g and h would be

$$\pi_1(g) = Prob(p_2 < g)v_1 - g \tag{19}$$

and

$$\pi_1(h) = Prob(p_2 < h)v_1 - h = Prob(p_2 < g)v_1 - h, \tag{20}$$

where the second equality in $\pi_1(h)$ is true because there is zero probability that p_2 is between g and h. Bidder 1 will put zero probability on $p_1 = h$, since its payoff is lower than the payoff from $p_1 = g$ and will put zero probability on slightly larger values of p_1 too, since by continuity their payoffs will also be less than the payoff from $p_1 = g$. This creates a gap $[h, h^*]$ in which $p_1 = 0$. But then Bidder 2 will want to put zero probability on $p_2 = h^*$ and slightly higher values, by the same reasoning, which means that our original hypothesis of only a gap $[g, h]$ is false.

Thus, we can conclude that the mixing density $m(p)$ is positive over the entire interval $[10, 13]$, with no atoms. What will it look like? Let us confine ourselves to looking for a symmetric equilibrium, in which both bidders use the same function $m(p)$. We know the expected payoff from any bid p in the support must equal the payoff from $p = 10$ or $p = 13$, which is 3. Therefore, since if our player has value $v = 16$ there is probability 0.5 of winning because the other player has $v = 10$ and probability $0.5M(p)$ of winning because the other player has $v = 16$ too but bid less than p, the payoff is

$$0.5(16 - p) + 0.5M(p)(16 - p) = 3. \tag{21}$$

This implies that $(16 - p) + M(p)(16 - p) = 6$, so

$$M(p) = \frac{6}{16 - p} - 1, \tag{22}$$

which has the density

$$m(p) = \frac{6}{(16 - p)^2} \tag{23}$$

on the support $[10, 13]$, rising from $m(10) = \frac{1}{6}$ to $m(13) = \frac{4}{6}$.

Since each bidder type has the same expected payoff in this first-price auction as in the ascending auction, and the object is sold with probability one, it must be that the seller's payoff is the same, too, equal to 11.5, as we found in equation (6).

The general continuous-value auction has a pure- strategy equilibrium but our particular discrete-value auction does not. Usually if a game lacks a pure-strategy equilibrium in discrete type space, it also lacks one if we "smooth" the probability distribution by making it continuous but still putting almost all the weight on the old discrete types, as in Figure 1.

This is related to a remarkable feature of private-value auctions with discrete values: the mixed-strategy equilibria do not necessarily block efficiency (and the revenue equivalence we study later).

When players randomize, it would seem that sometimes by chance the highest-valuing player would be unlucky and lose the auction, which would be inefficient. Not so here.

As explained in Riley (1989) and Wolfstetter (1999, p. 204), if the values of each player are distributed discretely over some set $\{0, v_a, v_b, ..., v_w\}$ then in the symmetric equilibrium mixed strategy, the supports of the mixing distributions are $v_a : [0, p_1], .v_b : [p_1, p_2], v_w : [p_{w-1}, p_w]$, where $p_1 < p_2 < ... < p_w$. The supports do not overlap.

Each type of bidder acts as if he was in competition just with his own type (since he will surely win over the lower types and will surely lose to the higher types) and the object is allocated to a bidder who values it most. The mixing only determines who wins when two players happen to have the same type.

Second-Price Auctions (Second-price sealed-bid, Vickrey)

Rules

Each bidder submits one bid, in ignorance of the other bids. The bids are opened, and the highest bidder pays the amount of the second-highest bid and wins the object.

Strategies

A bidder's strategy is his bid as a function of his value.

Payoffs

The winning bidder's payoff is his value minus the second-highest bid. The losing bidders' payoffs are zero. The seller's payoff is the second-highest-bid.

Discussion

Second-price auctions are similar to ascending auctions. Bidding one's value is a weakly dominant strategy: a bidder who bids less is no more likely to win the auction (and probably less likely, depending on $f(v)$), but he pays the same price— the second-highest-valuing player's bid— if he does win.

One difference between ascending and second-price auctions is that second- price auctions have peculiar asymmetric equilibria because the actions in them are simultaneous.

Consider a variant of the Ten-Sixteen Auction, in which each of two bidders' values can be 10 or 16, but where the realized values are common knowledge. Bidding one's value is a **symmetric equilibrium**, meaning that the bid function $p(v)$ is the same for both bidders: $\{p(v = 10) = 10, p(v = 16) = 16\}$. But there are asymmetric equilibria, such as

$$\begin{array}{ll} p_1(v = 10) = 10 & p_1(v = 16) = 16 \\ & \\ p_2(v = 10) = 1 & p_2(v = 16) = 10 \end{array} \tag{24}$$

Since Bidder 1 never bids less than 10, Bidder 2 knows that if $v_2 = 10$ he can never get a positive payoff, so he is willing to choose $p_2(v = 10) = 1$. Doing so results in a sale price of 1, for any $p_1 > 1$, which is better for Bidder 1 and worse for the seller than a price of 10, but Bidder 2 doesn't care about their payoffs. In the same way, if $v_2 = 16$, Bidder 2 knows that if he bids 10 he will win if $v_1 = 10$, but if $v_2 = 16$ he would have to pay 16 to win and would earn a payoff of zero. He might as well bid 10 and earn his zero by losing.

Descending Auctions (Dutch)

Rules

The seller announces a bid, which he continuously lowers until some bidder stops him and takes the object at that price.

Strategies

A bidder's strategy is when to stop the bidding as a function of his value.

Payoffs

The winner's payoff is his value minus his bid. The losers' payoffs are zero.

Discussion

The typical descending auction is **strategically equivalent** to the first-price auction, which means there is a one-to-one mapping between the strategy sets and the equilibria of the two games. The reason for the strategic equivalence is that no relevant information is disclosed in the course of the auction, only at the end, when it is too late to change anybody's behavior. In the first-price auction a bidder's bid is irrelevant unless it is the highest, and in the descending auction a bidder's stopping price is irrelevant unless it is the highest. The equilibrium price is calculated the same way for both auctions.

All-Pay Auctions

Rules

Each bidder places a bid simultaneously. The bidder with the highest bid wins, and each bidder pays the amount he bid.

Strategies

A bidder's strategy is his bid as a function of his value.

Payoffs

The winner's payoff is his value minus his bid. The losers' payoffs are the negative of their bids.

Discussion

The winning bid will be lower in the all-pay auction than under the other rules, because bidders need a bigger payoff when they do win to make up for their negative payoffs when they lose. At the same time, since even the losing bidders pay something to the seller it is not obvious that the seller does badly (and in fact, it turns out to be just as good an auction rule as the others, in this simple risk-neutral context).

The Equal-Value All-Pay Auction

Suppose each of the n bidders has the same value, v. That is not a very interesting game for most of the auction rules, though it is true that for the second-price auction there exists the strange asymmetric equilibrium $\{v, 0, 0, ...0\}$. Under the all-pay auction rule, however, this game is quite interesting. The equilibrium is in mixed strategies. This is easy to see, because in any pure-strategy profile, either the maximum bid is less than v, in which case someone could deviate to $p = v$ and increase his payoff; or one bidder bids v and the rest bid at most $p' < v$, in which case the high bidder will deviate to bid just above p'.

Suppose we have a symmetric equilibrium, so all bidders use the same mixing cumulative distribution $M(p)$. Let us conjecture that $\pi(p) = 0$, which we will later verify.[1] The payoff function for each bidder is the probability of winning times the value of the prize minus the bid, which is paid with probability one, and if we equate that to zero we get

$$M(p)^{n-1}v = p, \tag{25}$$

so

$$M(p) = \sqrt[n-1]{\frac{p}{v}}, \tag{26}$$

as shown in Figure 2. At the extreme bids that a bidder with value v might offer, $M(0) = \sqrt[n-1]{\frac{0}{v}} = 0$ and $M(v) = \sqrt[n-1]{\frac{v}{v}} = 1$, so we have found a valid distribution function $M(p)$. Moreover, since the payoff from one of the strategies between which it mixes, $p = 0$, equals zero, we have verified our conjecture that $\pi(p) = 0$ in the equilibrium.

[1] It turns out there is a continuum of asymmetric equilibria in this game if $n > 2$, but a unique equilibrium if $n = 2$. See Kovenock, Baye & de Vries (1996) for a full characterization of all-pay auctions with complete information.

The Continuous-Value All-Pay Auction

Suppose each of the n bidders picks his value v from the same density $f(v)$. Conjecture that the equilibrium is symmetric, in pure strategies, and that the bid function, $p(v)$, is strictly increasing. The equilibrium payoff function for a bidder with value v who pretends he has value z is

$$\pi(v, z) = F(z)^{n-1}v - p(z), \tag{27}$$

since if our bidder bids $p(z)$, that is the highest bid only if all $(n-1)$ other bidders have $v < z$, a probability of $F(z)$ for each of them.

The function $\pi(v, z)$ is not necessarily concave in z, so satisfaction of the first-order condition will not be a sufficient condition for payoff maximization, but it is a necessary condition since the optimal z is not 0 (unless $v = 0$) or infinity and from (27) $\pi(v, z)$ is differentiable in z in our conjectured equilibrium. Thus, we need to find z such that

$$\frac{\partial \pi(v, z)}{\partial z} = (n-1)F(z)^{n-2}f(z)v - p'(z) = 0 \tag{28}$$

In the equilibrium, our bidder does follow the strategy $p(v)$, so $z = v$ and we can write

$$p'(v) = (n-1)F(v)^{n-2}f(v)v \tag{29}$$

Integrating up, we get

$$p(v) = p(0) + \int_0^v (n-1)F(x)^{n-2}f(x)x dx \tag{30}$$

This is deterministic, symmetric, and strictly increasing in v, so we have verified our conjectures.

We can verify that truthelling is a symmetric equilibrium strategy by substituting for $p(z)$ from (30) into payoff equation (27).

$$\begin{aligned}\pi(v,z) &= F(z)^{n-1}v - p(z) \\ &= F(z)^{n-1}v - p(0) - \int_0^z (n-1)F(x)^{n-2}f(x)x\,dx \qquad (31) \\ &= F(z)^{n-1}v - p(0) - F(z)^{n-1}z + \int_0^z F(x)^{n-1}dx,\end{aligned}$$

where the last step uses integration by parts ($\int gh' = gh - \int hg'$, where $g = x$ and $h' = (n-1)F(x)^{n-2}f(x)$). Maximizing (31) with respect to z yields

$$\frac{\partial \pi(v,z)}{\partial z} = (n-1)F(z)^{n-2}f(z)(v-z), \qquad (32)$$

which is maximized by setting $z = v$. Thus, if $(n-1)$ of the bidders are using this $p(v)$ function, so will the remaining bidder, and we have a Nash equilibrium.

Suppose values are uniformly distributed over [0,1], so $F(v) = v$. Then equation (30) becomes

$$\begin{aligned} p(v) &= p(0) + \int_0^v (n-1)x^{n-2}(1)x dx \\ &= p(0) + \Big|_{x=0}^{v} (n-1)\frac{x^n}{n} \\ &= 0 + \left(\frac{n-1}{n}\right) v^n, \end{aligned} \quad (33)$$

where we can tell that $p(0) = 0$ because if $p(0) > 0$ a bidder with $v = 0$ would have a negative expected payoff. If there were $n = 2$ bidders, a bidder with value v would bid $v^2/2$, win with probability v, and have expected payoff $\pi = v(v) - v^2/2 = v^2/2$. If there were $n = 10$ bidders, a bidder with value v would bid $(9/10)v^{10}$, win with probability v^9, and have expected payoff $\pi = v(v^9) - (9/10)v^{10} = v^{10}/(10)$. As we will see when we discuss the Revenue Equivalence Theorem, it is no accident that this is the same payoff as for the first-price auction when values were uniformly distributed on [0,1], in equation (**??**).

A GENERAL REVENUE EQUIVALENCE THEOREM. Let all players be risk-neutral with private values drawn from the same distribution $F(v)$ with positive density everywhere on $[\underline{v}, \bar{v}]$. If in both Equilibrium E_1 of Auction Rule A_1 and Equilibrium E_2 of Auction Rule A_2 it is true that:

(a) The lowest type, $v = \underline{v}$ has the same maximized payoff (e.g., his maximized payoff is always zero); and

(b) A bidder of type v has the same probability $G(v)$ of winning the object;

then E_1 and E_2 have the same expected payoffs for each type of bidder and for the seller.

Proof. Think of an auction as a direct mechanism in which a bidder with type v is choosing a type report z, after which he pays amount $p(z)$ and wins the object with probability $G(z)$ (The Revelation Theorem says that we can think of any auction as being like a direct mechanism.) We can write the probability as a function $G(z)$ because assumption (b) says that in equilibrium each type maps to a single probability of winning and in a direct mechanism $z = v$.

Players maximize their payoffs by choice of z given the strategies of the other players. Let us denote the expected maximized equilibrium payoff of a bidder with value v as $\pi(v)$:

$$\pi(v, z(v)) = \overset{Max}{z} \; \{G(z)v - p(z)]\}. \tag{34}$$

There must be incentive compatibility constraints so that this problem is solved by choosing $z = v$, but we will not have to find those constraints for this proof.

The derivative of the bidder's payoff with respect to v is

$$\frac{d\pi(v,z(v))}{dv} = \frac{\partial\pi(v,z(v))}{\partial v} + \frac{\partial\pi(v,z(v))}{\partial z}\frac{\partial z}{\partial v} = \frac{\partial\pi(v,z(v))}{\partial v} = G(z), \tag{35}$$

because from the Envelope Theorem we know that $\frac{\partial\pi(v,z(v))}{\partial z} = 0$; when v rises infinitesimally, the change in the bidder's optimized payoff arising from the change in the bidder's optimal choice of z is second-order and can be ignored. (Note that (36) is not a first-order condition— the derivative is with respect to v, the type, not the report variable z that is under the bidder's control.) Since the auction rule is a direct mechanism, we can rewrite (36) as

$$\frac{d\pi(v,z(v))}{dv} = G(v). \tag{36}$$

Another way besides equation (34) to write $\pi(v,z(v))$ is in terms of its value for the lowest possible v and its derivatives,

$$\pi(v,z(v)) = \pi(\underline{v},z(\underline{v})) + \int_{\underline{v}}^{v} \frac{d\pi(x,z(x))}{dx}dx. \tag{37}$$

To write $\pi(v,z(v))$ this way, we need for $\frac{d\pi(x,z(x))}{dx}$ to exist everywhere on $[\underline{v},\bar{v}]$, which is why we assumed that the density for $F(v)$ was positive.

Substituting for $\frac{d\pi(x)}{dx}$ from equation (36) yields

$$\pi(v,z(v)) = \pi(\underline{v},z(\underline{v})) + \int_{\underline{v}}^{v} G(x)dx, \tag{38}$$

The theorem postulates that $\pi(\underline{v},z(\underline{v}))$ is the same between auction rules and equilibria. It also postulates that $G(v)$ is the same between auction rules and equilibria. Thus, we have proved that $\pi(v,z(v))$ the payoff for a bidder of type v, is the same between auction rules and equilibria. In turn, this implies that the seller's payoff must be the same between them. Q.E.D.

A SPECIFIC REVENUE EQUIVALENCE THEOREM. Let all players be risk-neutral with private values drawn from the same strictly increasing, atomless distribution $F(v)$. The symmetric equilibria of the ascending, first-price, second-price, descending, and all-pay auctions all have the same expected payoffs for each type of bidder and for the seller.

Proof. The General Theorem said that two auctions will have the same expected payoffs if (a) the lowest type expects the same payoff, and (b) two bidders of a given type have the same probability of winning. Condition (a) is clearly true for all five auction types, since the lowest type has zero probability of winning when $f(v)$ is continuous.

The Ascending Auction. In equilibrium, the winner is the highest-value bidder. Thus, the probability of a bidder of type v winning is the probability that v is the highest value present.

The First-Price Auction. In equilibrium, the winner is the highest- value bidder, just as in the ascending auction. Thus, the probability of a type v winning is the same, and the payoffs will be equivalent.

The Second-Price Auction. In equilibrium, the winner is the highest-value bidder, just as in the ascending auction.

The Descending Auction. This is strategically equivalent to the first- price auction, and so has the same payoffs.

The All-Pay Auction. In equilibrium, the winner is again the highest- value bidder, as we saw above.

13.4 Reserve Prices and the Marginal Revenue Approach

A **reserve price** p^* is a bid put in by the seller, secretly or openly, before the auction begins, which commits him not to sell the object if nobody bids more than p^*. The seller will often find that a reserve price can increase his payoff. If he does, it turns out that he will choose a reserve price strictly greater than his own value: $p^* > v_s$. To see this, we will use the **marginal revenue approach** to auctions, an approach developed in Bulow & Roberts (1989) for risk-neutral private values, and Bulow & Klemperer (1996) for common values and risk aversion. This approach compares the seller in an auction to an ordinary monopolist who sells using a posted price. We start with an auction to just one bidder, then extend the idea to an auction with multiple bidders, and finally return to the surprising similarity between an auction with one bidder and a monopoly selling to a continuum of bidders using a posted price.

1. One Bidder. If there is just one bidder, the seller will do badly in any of the auction rules we have discussed so far. The single bidder would bid $p_1 = 0$ and win.

The situation is really better suited to bargaining or simple monopoly than to an auction. The seller could use an auction, but a standard auction yields him zero revenue. so posting a price offer to the bidder makes more sense. If the auction has a reserve price, however, it can be equivalent to posting a price, just as in bargaining the making of a single take-it-or-leave-it offer of p^* is equivalent to posting a price.

What should the offer p^* be? Let the bidder have value distribution $F(v)$ on $[\underline{v}, \bar{v}]$ which is differentiable and strictly increasing, so the density $f(v)$ is always positive. Let the seller value the object at $v_s \geq \underline{v}$. The seller's payoff is

$$\begin{aligned} \pi(p^*) &= Pr(p^* < v)(p^* - v_s) + Pr(p^* > v)(0) \\ &= [1 - F(p^*)](p^* - v_s). \end{aligned} \tag{39}$$

This has first-order-condition

$$\frac{d\pi(p^*)}{dp^*} = [1 - F(p^*)] - f(p^*)[p^* - v_s] = 0. \tag{40}$$

On solving (40) for for p^* we get

$$p^* = v_s + \left(\frac{1 - F(p^*)}{f(p^*)} \right). \tag{41}$$

The optimal take-it-or-leave-it offer, the "reserve price" p^* satisfies equation (41). The reserve price is strictly greater than the seller's value for the object ($p^* > v_s$) unless the solution is such that $F(p^*) = 1$ because the optimal reserve price is the greatest possible bidder value, in which case the object has probability zero of being sold. One reason to use a reserve price is so the seller does not sell an object for a price worth less than its value to him, but that is not all that is going on.

2. Multiple Bidders. Now let there be n bidders, all with values distributed independently by $F(v)$. Denote the bidders with the highest and second-highest values as Bidders 1 and 2. The seller's payoff in a second-price auction is

$$\begin{aligned}\pi(p^*) &= Pr(p^* > v_1)(0) + Pr(v_2 < p^* < v_1)(p^* - v_s) + Pr(p^* < Ev_2 < v_1)(v_2 - v_s) \\ &= \int_{v_1=\underline{v}}^{p^*} f(v_1)(0)dv_1 + \int_{v_1=p^*}^{\bar{v}} \left(\int_{v_2=\underline{v}}^{p^*} (p^* - v_s)f(v_2)dv_2 + \int_{v_2=p^*}^{v_1} (v_2 - v_s)f(v_2)dv_2 \right) f(v_1)dv_1\end{aligned} \tag{42}$$

This expression integrates over two random variables. First, it matters whether v_1 is greater than or less than p^*, the outer integrals. Second, it matters whether v_2 is less than p^* or not, the inner integrals.

Now differentiate equation (42) to find the optimal reserve price p^* using Leibniz's integral rule (given in the Mathematical Appendix):

$$\begin{aligned}\frac{d\pi(p^*)}{dp^*} &= 0 + -f(p^*) \left(\int_{v_2=\underline{v}}^{p^*} (p^* - v_s)f(v_2)dv_2 + \int_{v_2=p^*}^{p^*} (v_2 - v_s)f(v_2)dv_2 \right) \\ &+ \int_{v_1=p^*}^{\bar{v}} \left((p^* - v_s)f(p^*) - (p^* - v_s)f(p^*) + \int_{v_2=\underline{v}}^{p^*} f(v_2)dv_2 \right) f(v_1)dv_1 \\ &= -f(p^*) \left(\int_{v_2=\underline{v}}^{p^*} (p^* - v_s)f(v_2)dv_2 + 0 \right) + \int_{v_1=p^*}^{\bar{v}} \left(\int_{v_2=\underline{v}}^{p^*} f(v_2)dv_2 \right) f(v_1)dv_1 \\ &= -f(p^*)F(p^*)(p^* - v_s) + (1 - F(p^*))F(p^*) = 0\end{aligned} \tag{43}$$

Dividing by F, the last line of expression (43) implies that

$$p^* = v_s + \frac{1 - F(p^*)}{f(p^*)}, \tag{44}$$

just what we found in equation (41) for the one-bidder case. Remarkably, the optimal reserve price is unchanged! Moreover, equation (44) applies to any number of bidders, not just $n = 2$. Only Bidders 1 and 2 show up in the equations we used in the derivation, but that is because they are the only ones to affect the result in a second-price auction.

In fact, only the highest-valuing bidder matters to the optimal reserve price.

3. A Continuum of Bidders: The Marginal Revenue Interpretation Now think of a firm with a constant marginal cost of c facing a continuum of bidders along the same distribution $F(v)$ that we have been using. The quantity of bidders with values above p will be $(1 - F(p))$, so the demand equation is

$$q(p) = 1 - F(p) \tag{45}$$

and

$$Revenue \equiv pq = p(1 - F(p)) \tag{46}$$

The marginal revenue is then (keeping in mind that $\frac{dq}{dp} = -f(p)$)

$$\begin{aligned} Marginal\ Revenue \equiv \tfrac{dR}{dq} &= p + \left(\tfrac{dp}{dq}\right) q \\ &= p + \left(\tfrac{1}{\frac{dq}{dp}}\right) q \\ &= p + \left(\tfrac{1}{-f(p)}\right)(1 - F(p)) \\ &= p - \tfrac{1-F(p)}{f(p)} \end{aligned} \tag{47}$$

Setting marginal revenue to marginal cost, the profit-maximizing monopoly price is the one at which the marginal revenue in (47) equals c.

The mathematics of the problem is identical whether the seller is facing a continuum of bidders on distribution $F(v)$ or one one bidder drawn randomly from the continuum $F(v)$.

The problem is just like that in a take-it-or-leave-it-offer bargaining model where the bidder's type is unknown to the seller.

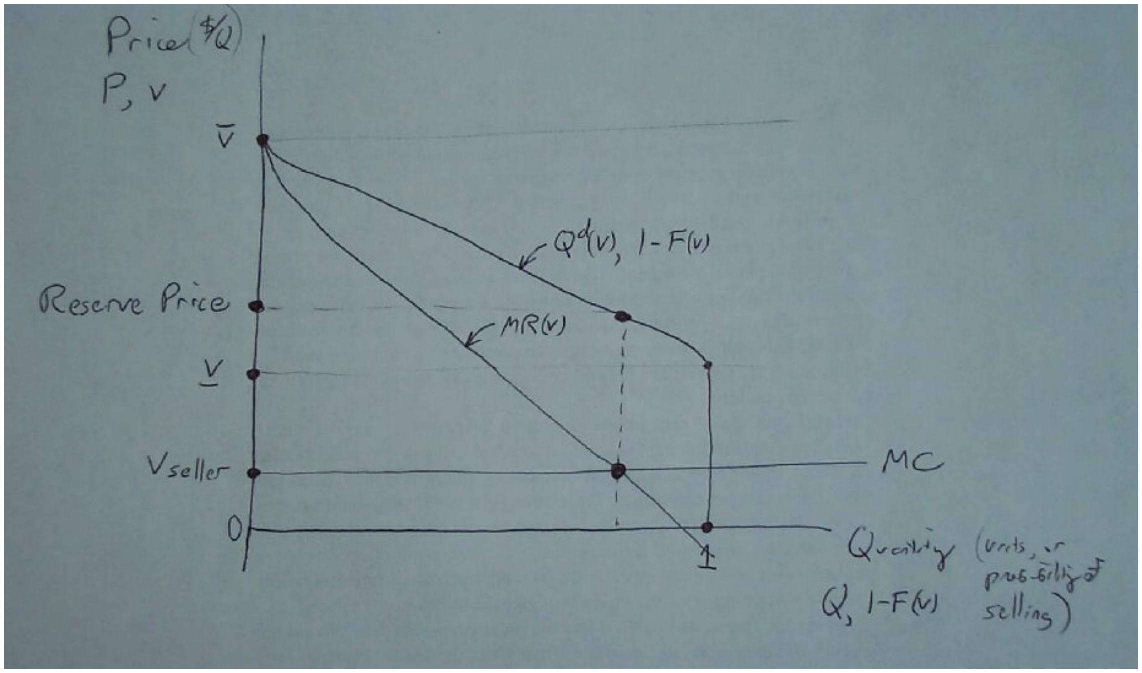

Figure 3a: Auctions and Marginal Revenue: Reserve Price Needed

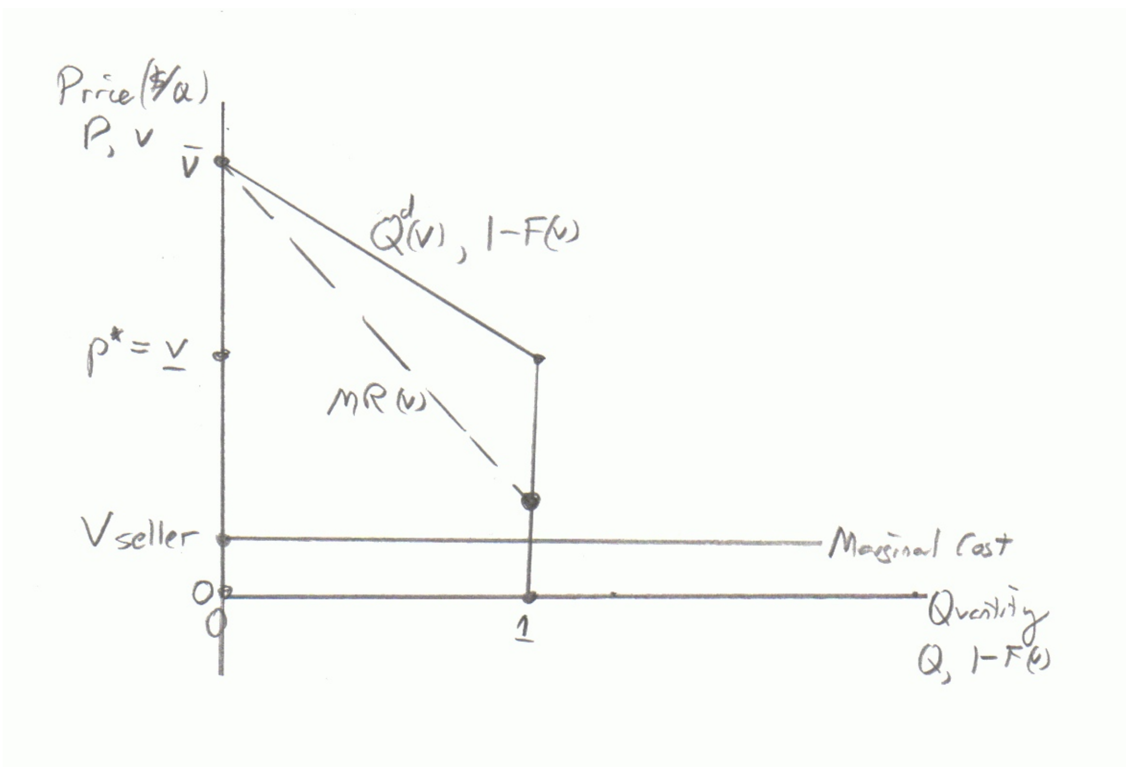

Figure 3b: Auctions and Marginal Revenue: No Reserve Price Needed

13.5 Order Statistics Useful for Common-Value (and other) Auctions

Suppose n signals are independently drawn from the uniform distribution on $[\underline{s}, \overline{s}]$. Denote the j^{th} highest signal by $s_{(j)}$. The expectation of the kth highest value happens to be

$$Es_{(k)} = \underline{s} + \left(\frac{n+1-k}{n+1} \right) (\overline{s} - \underline{s}) \tag{48}$$

This means the expectation of the very highest value is

$$Es_{(1)} \;\; = \underline{s} + \left(\tfrac{n}{n+1} \right) (\overline{s} - \underline{s}) \tag{49}$$

The expectation of the second-highest value is

$$Es_{(2)} \;\; = \underline{s} + \left(\tfrac{n-1}{n+1} \right) (\overline{s} - \underline{s}) \tag{50}$$

The expectation of the lowest value, the n'th highest, is

$$Es_{(n)} \;\; = \underline{s} + \left(\tfrac{1}{n+1} \right) (\overline{s} - \underline{s})\,. \tag{51}$$

Let n risk-neutral bidders, $i = 1, 2, ...n$ each receive a signal s_i independently drawn from the uniform distribution on
$[v - m, v + m]$,
where v is the true value of the object to each of them. Assume that they have "diffuse priors" on v, which means they think any value from $v = -\infty$ to $v = \infty$ is equally likely.

The best estimate of the value given the set of n signals is

$$Ev|(s_1, s_2, ..., s_n) = \frac{s_{(n)} + s_{(1)}}{2}. \tag{52}$$

The estimate depends only on two out of the n signals— a remarkable property of the uniform distribution.

If there were five signals $\{6, 7, 7, 16, 24\}$, the expected value of the object would be 15 (=[6+24]/2), well above the mean of 12 and the median of 7, because only the extremes of 6 and 24 are useful information.

A density that had a peak, like the normal density, would yield a different result, but here all we can tell from the data is that all values of v between $(6 + m)$ and $(24 - m)$ are equally probable.

Someone who saw just signals $s_{(4)}$ and $s_{(1)}$ could deduce that v could not be less than $(s_{(1)} - m)$ or greater than $(s_{(4)} + m)$.

Learning the signals in between— would be unhelpful.

The only information that $s_{(2)}$ conveys is that $v \leq (s_{(2)} + m)$ and $v \geq (s_{(2)} - m)$.

Our observer had already figured that out from $s_{(4)}$ and $s_{(1)}$.

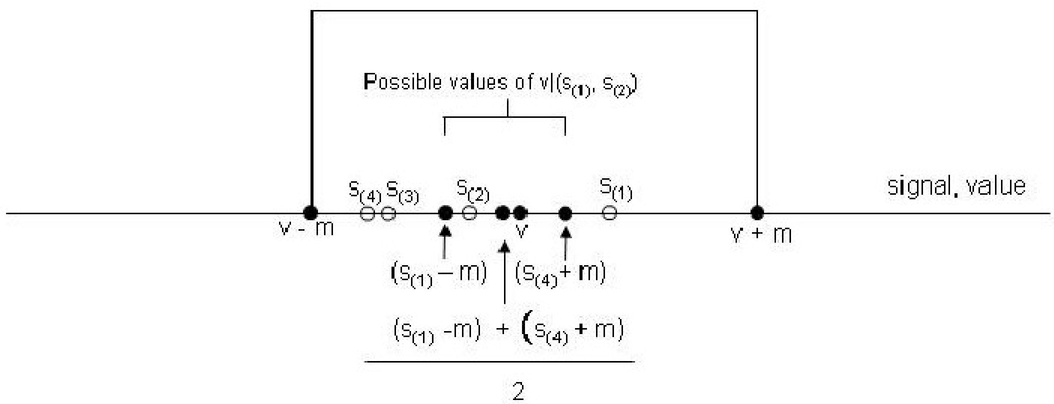

Figure 4: Extracting Information From Uniformly Distributed Signals

The Uniform-Signal Common-Value Auction

Players:
One seller and n bidders.

Order of Play:
0. Nature chooses the common value for the object v using the uniform density on $[-\infty, \infty]$, and sends signal s_i to Bidder i using the uniform distribution on $[v - m, v + m]$.
1. The seller chooses a mechanism that allocated the object and payments based on each player's choice of p. He also chooses the procedure in which bidders select p (sequentially, simultaneously, etc.).
2. Each bidder simultaneously chooses to participate in the auction or to stay out.
3. The bidders and the seller choose value of p according to the mechanism procedure.
4. The object is allocated and transfers are paid according to the mechanism.

Payoffs:
Payoff depends on the particular rules, but if the object is sold, the payoff to the seller is the sum of all the payments and the value to a bidder is the value of the object, if he wins, minus his payments.

The Ascending Auction (open-exit)

Equilibrium: If no bidder has quit yet, Bidder i should drop out when the price rises to s_i.

Otherwise, he should drop out when the price rises to

$$p_i = \frac{p_{(n)} + s_i}{2},$$

where $p_{(n)}$ is the price at which the first dropout occurred.

Explanation: If no other bidder has quit yet, Bidder i is safe in agreeing to pay his signal, s_i. Either
(a) he has the lowest signal,
or (b) everybody else has the same signal value s_i too, and they will all drop out at the same time.

In case (a), having the lowest signal, he will lose anyway. In case (b), the best estimate of the value is s_i, and that is where he should drop out.

Once one bidder has dropped out at $p_{(n)}$, the other bidders can deduce that he had the lowest signal, so they know that signal $s_{(n)}$ must equal $p_{(n)}$. Suppose Bidder i has signal $s_i > s_{(n)}$. Either:

(a) someone else has a higher signal and Bidder i will lose the auction anyway and dropping out too early does not matter or

(b) everybody else who has not yet dropped out has signal s_i too, and they will all drop out at the same time, or

(c) he would be the last to drop out, so he will win.

In cases (b) and (c), his estimate of the value is $p_{(i)} = \frac{p_{(n)}+s_i}{2}$, since $p_{(n)}$ and s_i are the extreme signal values and the signals are uniformly distributed, and that is where he should drop out.

The price paid by the winner will be the price at which the second-highest bidder drops out, which is $\frac{s_{(n)}+s_{(2)}}{2}$. The expected values are, from equations (50) and (51),

$$\begin{aligned} Es_{(n)} &= (v-m) + \left(\frac{n+1-n}{n+1}\right)((v+m)-(v-m)) \\ &= v + \left(\frac{1-n}{n+1}\right)m \end{aligned} \tag{53}$$

and

$$\begin{aligned} Es_{(2)} &= (v-m) + \left(\frac{n+1-2}{n+1}\right)((v+m)-(v-m)) \\ &= v + \left(\frac{n-3}{n+1}\right)m. \end{aligned} \tag{54}$$

Averaging them yields the expected winning price,

$$\begin{aligned} Ep_{(2)} &= \frac{\left[v+\left(\frac{2-n}{n+1}\right)m\right]+\left[v+\left(\frac{n-3}{n+1}\right)m\right]}{2} \\ &= v - \left(\frac{1}{2(n+1)}\right)m. \end{aligned} \tag{55}$$

If $m = 50$ and $n = 4$, then

$$Ep_{(2)} = v - \left(\frac{1}{10}\right)(50) = v - 5. \tag{56}$$

Expected seller revenue increases in n, the number of bidders (and thus of independent signals) and falls in the uncertainty m (the inaccuracy of the signals).

The First-Price Auction

Equilibrium: Bid $(s_i - m)$.

Explanation: Bidder i bids $(s_i - z)$ for some amount z that does not depend on his signal, because given the assumption of diffuse priors, he does not know whether his signal is a high one or a low one.

Define T_i to be how far the signal s_i is above its minimum possible value, $(v - m)$, so

$$T_i \equiv s_i - (v - m) \tag{57}$$

and $s_i \equiv v - m + T_i$.

Bidder i has the highest signal and wins the auction if T_i is big enough, which has probability $\left(\frac{T_i}{2m}\right)^{n-1}$, which we will define as $G(T_i)$ because it is the probability that the $(n-1)$ other signals are all less than $s_i = v - m + T_i$.

He earns v minus his bid of $(s_i - z)$ if he wins, which equals $(z + m - T_i)$.

The equilibrium strategy's payoff is $(z + m - T_i)$.

If Bidder i deviated and bid a small amount ϵ higher, he would win with a higher probability, $G(T_i + \epsilon)$, but he would lose ϵ whenever he would have won with the lower bid. Using a Taylor expansion, $G(T_i + \epsilon) \approx G(T_i) + G'(T_i)\epsilon$, so

$$G(T_i + \epsilon) - G(T_i) \approx (n-1)T_i^{n-2}\left(\frac{1}{2m}\right)^{n-1}\epsilon. \tag{58}$$

The benefit from bidding higher is the higher probability, $[G(T_i + \epsilon) - G(T_i)]$ times the winning surplus $(z + m - T_i)$. The loss from bidding higher is that the bidder would pay an additional ϵ in the $\left(\frac{T_i}{2m}\right)^{n-1}$ cases in which he would have won anyway. In equilibrium, he is indifferent about this infinitesimal deviation, taking the expectation across all possible values of his "signal height" T_i, so

$$\int_{T_i=0}^{2m}\left[\left((n-1)T_i^{n-2}\left(\frac{1}{2m}\right)^{n-1}\epsilon\right)(z+m-T_i) - \epsilon\left(\frac{T_i}{2m}\right)^{n-1}\right]dT_i = 0. \tag{59}$$

This implies that

$$\epsilon\left(\frac{1}{2m}\right)^{n-1}\int_{T_i=0}^{2m}\left[\left((n-1)T_i^{n-2}\right)(z+m) - (n-1)T_i^{n-1} - T_i^{n-1}\right]dT_i = 0. \tag{60}$$

which in turn implies that

$$\epsilon\left(\frac{1}{2m}\right)^{n-1}\Bigg|_{T_i=0}^{2m}\left(T_i^{n-1}(z+m) - T_i^{n}\right) = 0, \tag{61}$$

so $(2m)^{n-1}(z+m) - (2m)^n - 0 + 0 = 0$ and $z = m$. Bidder i's optimal strategy in the symmetric equilibrium is to bid $p_i = s_i - m$.

The winning bid is set by the bidder with the highest signal, and that highest signal's expected value is

$$\begin{aligned} Es_{(1)} &= \underline{s} + \left(\tfrac{n+1-1}{n+1}\right)(\overline{s} - \underline{s}) \\ &= v - m + \left(\tfrac{n}{n+1}\right)((v+m) - (v-m)) \qquad (62) \\ &= v - m + \left(\tfrac{n}{n+1}\right)(2m) \end{aligned}$$

The expected revenue is therefore

$$Ep_{(1)} = v - (1)\left(\frac{1}{n+1}\right) 2m. \qquad (63)$$

If $m = 50$ and $n = 4$, then

$$Ep_{(1)} = v - \left(\frac{1}{5}\right)(100) = v - 20. \qquad (64)$$

Here, the revenue is even lower than in the second-price auction, where it was $(v - 15)$. The revenue is lower even if $n = 2$.

Why the revenue differences? What is the info advantage of the highest valuer under each auction rule?

1. Ascending, open exit. Highest expected revenue. The winning price and who wins depends on the lowest, highest, and second-highest value, and the lowest-value comes to be known by everyone. The highest valuer's only private information advantage is knowing the highest value.

2. 2nd-price sealed bid. Middle expected revenue. The price and who wins depends on the top two values, which are known only to the top two valuers.

3. First-price. Lowest expected revenue. No information except the highest value is incorporated in the price and who wins. Thus, all relevant info is controlled by the highest valuer alone.