ODD

Answers to Odd-Numbered Problems, 4th Edition of Games and Information, Rasmusen

PROBLEMS FOR CHAPTER 11: Signalling

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This contains answers to the odd-numbered problems in the fourth edition of *Games and Information* by Eric Rasmusen.

PROBLEMS FOR CHAPTER 11: Signalling

11.1. Is Lower Ability Better?
Change Education I so that the two possible worker abilities are $a \in \{1, 4\}$.

(a) What are the equilibria of this game? What are the payoffs of the workers (and the payoffs averaged across workers) in each equilibrium?

Answer. The pooling equilibrium is

$$s_L = s_H = 0, w_0 = w_1 = 2.5, Pr(L|s = 1) = 0.5, \tag{1}$$

which uses passive conjectures: . The payoffs are $U_L = U_H = 2.5$, for an average payoff of 2.5.

The separating equilibrium is

$$s_L = 0, s_H = 1, w_0 = 1, w_1 = 4. \tag{2}$$

The payoffs are $U_L = 1$ and $U_H = 2$, for an average payoff of 1.5 . This equilibrium can be justified by the self selection constraints

$$U_L(s = 0) = 1 > U_L(s = 1) = 4 - 8/1 = -4 \tag{3}$$

and

$$U_H(s = 0) = 1 < U_H(s = 1) = 4 - 8/4 = 2. \tag{4}$$

Thus, the payoff averaged across workers is 1.5 (=.5[1] + .5[2]).

(b) Apply the Intuitive Criterion (see N6.2). Are the equilibria the same?

Answer. Yes. The intuitive criterion does not rule out the pooling equilibrium in the game with $a_h = 4$. There is no incentive for *either* type to deviate from $s = 0$ even if the deviation makes the employers think that the deviator is high-ability. The payoff to a persuasive high-ability deviator is only 2, compared the 2.5 that he can get in the pooling equilibrium.

(c) What happens to the equilibrium worker payoffs if the high ability is 5 instead of 4?

Answer. The pooling equilibrium is

$$s_L = s_H = 0, w_0 = w_1 =, Pr(L|s = 1) = 0.5, \tag{5}$$

which uses passive conjectures. The payoffs are $U_L = U_H = 3$, with an average payoff of 3.

The separating equilibrium is

$$s_L = 0, s_H = 1, w_0 = 1, w_1 = 5. \tag{6}$$

The payoffs are $U_L = 1$ and $U_H = 3.4$ with an average payoff of 2.2. The self-selection constraints are

$$U_H(s=0) = 1 < U_H(s=1) = 5 - \frac{8}{5} = 3.4 \tag{7}$$

and

$$U_L(s=0) = 1 > U_L(s=1) = 5 - \frac{8}{1} = -3. \tag{8}$$

(d) Apply the Intuitive Criterion to the new game. Are the equilibria the same?

Answer. No. The strategy of choosing $s = 1$ is dominated for the Lows, since its maximum payoff is -3, even if the employer is persuaded that he is High. So only the separating equilibrium survives.

(e) Could it be that a rise in the maximum ability reduces the average worker's payoff? Can it hurt all the workers?

Answer. Yes. Rising ability would reduce the average worker payoff if the shift was from a pooling equilibrium when $a_h = 4$ to a separating equilibrium when $a_h = 5$. Since the Intuitive Criterion rules out the pooling equilibrium when $a_h = 5$, it is plausible that the equilibrium is separating when $a_h = 5$. Since the pooling equilibrium is pareto-dominant when $a_h = 4$, it is plausible that it is the equilibrium played out. So the average payoff may well fall from 2.5 to 2.2 when the high ability rises from 4 to 5. This cannot make every player worse off, however; the high-ability workers see their payoffs rise from 2.5 to 3.4.

[Below are some additional notes that are in first draft, since they came up in a recent class– be on the lookout for errors]

What if the signal were continuous instead of having to equal either 1 or 4? Then, a lower signal could still induce separation. Would that remove the perverse result of higher ability reducing average payoffs? The basic problem is that any signalling that goes on can't increase output. All the signalling does is to reduce the average payoff. On the other hand, increasing ability does increase the average payoff, as a direct effect. So the question is whether the increase in signalling cost outweighs the increase in output. Let's see what happens here:

The high-ability self-selection constraint would be

$$U_H(s=0) = 1 \leq U_H(s=s^*) = 5 - \frac{8s^*}{5}, \tag{9}$$

which means we would need $s^* \leq 2.5$.

The low-ability self-selection constraint would be

$$U_L(s=0) = 1 \geq U_L(s=1) = 5 - \frac{8s^*}{1}, \tag{10}$$

so s^* must be at least 0.5. If it is, then the separating payoffs are 1 for the low-ability and $5 - \frac{8(0.5)}{5} = 4\ 1/5$ for the high-ability, an average payoff of 2.6.

The Intuitive Criterion results in the minimum necessary signal being used. Thus, use of it would argue for a shift from an average payoff of 2.5 to one of 2.6 when ability increased from 4 to 5 if the equilibrium shifted from pooling to separating. But the Intuitive Criterion would say that the pooling equilibrium would break down even when ability was 4, if the signal can be as low as 0.5. That's because the high-ability worker could deviate to signalling $s = 0.5$ and get payoff $4 - \frac{8*0.5}{4} = 4 - 1 = 3$, which is better than the 2.5 from pooling. In fact, the value of s^* when abilities are 1 and 4 can be even lower: solving $U_L(s = 0) = 1 \geq U_L(s = 1) = 4 - \frac{8s^*}{1}$ yields $s^* = 3/8$. That gives an average payoff of $(3.625 + 1)/2 = 2.3125$. So applying the Intuitive Criterion, higher ability now helps. If we don't apply it, though, a shift from pooling to separating might well reduce average welfare, and could even reduce the welfare of both players.

11.3. Price and Quality

Consumers have prior beliefs that Apex produces low-quality goods with probability 0.4 and high quality with probability 0.6. A unit of output costs 1 to produce in either case, and it is worth 10 to the consumer if it is high- quality and 0 if low-quality. The consumer, who is risk neutral, decides whether to buy in each of two periods, but he does not know the quality until he buys. There is no discounting.

(a) What is Apex' price and profit if it must choose one price, p^*, for both periods?

Answer. A consumer's expected consumer surplus is

$$CS = 0.4(0 - p^*) + 0.6(10 - p^*) + 0.6(10 - p^*) = -1.6p^* + 12. \tag{11}$$

Apex maximizes its profits by setting $CS = 0$, in which case $p^* = 7.5$ and profit is $\pi_H = 13$ (= 2(7.5 - 1)) or $\pi_L = 6.5$ ((=7.5-1).

(b) What is Apex' price and profit if it can choose two prices, p_1 and p_2, for the two periods, but it cannot commit ahead to p_2?

Answer. If Apex is high quality, it will choose $p_2 = 10$, since the consumer, having learned the quality first period, is willing to pay that much. Thus consumer surplus is

$$CS = 0.4(0 - p_1) + 0.6(10 - p_1) + 0.6(10 - 10) = -p_1 + 6, \tag{12}$$

and, setting this equal to zero, $p_1 = 6$, for a profit of $\pi_H = 14$ (= (6-1) + (10-1)) or $\pi_L = 5$ (= 6-1).

(c) What is the answer to part (b) if the discount rate is $r = 0.1$?

Answer. Apex cannot do better than the prices suggested in part (b).

(d) Returning to $r = 0$, what if Apex can commit to p_2?

<u>*Answer.*</u> Commitment makes no difference in this problem, since Apex wants to charge a higher price in the second period anyway if it has high quality— a high price in the first period would benefit the low-quality Apex too, at the expense of the high-quality Apex.

(e) How do the answers to (a) and (b) change if the probability of low quality is 0.95 instead of 0.4? (There is a twist to this question.)

<u>*Answer.*</u> With a constant price, a consumer's expected consumer surplus is

$$CS = 0.95(0 - p^*) + 0.05(10 - p^*) + 0.05(10 - p^*) = -1.05p^* + 0.5 \tag{13}$$

Apex would set $CS = 0$, in which case $p^* = \frac{10}{21}$, but since this is less than cost, Apex in fact would not sell anything at all, and would earn zero profit.

With changing prices, high-quality Apex will choose $p_2 = 10$, since the consumer, having learned the quality first period, is willing to pay that much. Thus consumer surplus is

$$CS = 0.95(0 - p_1) + 0.05(10 - p_1) + 0.05(10 - 10) = -p_1 + 0.5. \tag{14}$$

and, setting this equal to zero, you might think that $p_1 = 0.5$, for a profit of $\pi_H = 8.5 (= (0.5 - 1) + (10 - 1))$. But notice that if the low-quality Apex tries to follow this strategy, his payoff is $\pi_L = 0.5 - 1 < 0$. Hence, only the high-quality Apex will try it. But then the consumers know the product is high-quality, and they are willing to pay 10 even in the first period. What the high-quality Apex can do is charge up to $p_1 = 1$ in the first period, for profits of 9 (=$(1 - 1) + (10 - 1)$).

11.5. Advertising

Brydox introduces a new shampoo which is actually very good, but is believed by consumers to be good with only a probability of 0.5. A consumer would pay 10 for high quality and 0 for low quality, and the shampoo costs 6 per unit to produce. The firm may spend as much as it likes on stupid TV commercials showing happy people washing their hair, but the potential market consists of 100 cold- blooded economists who are not taken in by psychological tricks. The market can be divided into two periods.

(a) If advertising is banned, will Brydox go out of business?

<u>*Answer*</u>. No. It can sell at a price of 5 in the first period and 10 in the second period. This would yield profits of 300 (= (100)(5-6) +(100) (10-6)).

(b) If there are two periods of consumer purchase, and consumers discover the quality of the shampoo if they purchase in the first period, show that Brydox might spend substantial amounts on stupid commercials.

Answer. If the seller produces high quality, it can expect repeat purchases. This makes expenditure on advertising useful if it increases the number of initial purchases, even if the firm earns losses in the first period. If the seller produces low quality, there will be no repeat purchases. Hence, advertising expenditure can act as a signal of quality: consumers can view it as a signal that the seller intends to stay in business two periods.

(c) What is the minimum and maximum that Brydox might spend on advertising, if it spends a positive amount?

Answer. If there is a separating signalling equilibrium, it will be as follows. Brydox would spend nothing on advertising if its shampoo is low quality, and consumers will not buy from any company that advertises less than some amount X, because such a company is believed to produce low quality. Brydox would spend X on advertising if its quality is high, and charge a price of 10 in both periods.

Amount X is between 400 and 500. If a low-quality firm spends X on advertising, consumers do buy from it for one period, and it earns profits of (100)(10-6)-X = 400-X. Thus, the high-quality firm must spend at least 400 to distinguish itself. If a high-quality firm spends X on advertising, consumers buy from it for both periods, and it earns profits of (2) (100)(10-6)-X = 800-X. Since it can make profits of 300 even without advertising, a high-quality firm will spend up to 500 on advertising.

11.7. Salesman Clothing

Suppose a salesman's ability might be either $x = 1$ (with probability θ) or $x = 4$, and that if he dresses well, his output is greater, so that his total output is $x + 2s$ where s equals 1 if he dresses well and 0 if he dresses badly. The utility of the salesman is $U = w - \frac{8s}{x}$, where w is his wage. Employers compete for salesmen.

(a) Under full information, what will the wage be for a salesman with low ability?

Answer. Salesmen with low ability would not dress well. Dressing well would raise their output to 3, but their utility at a wage of 3 would be -5, whereas if they dress poorly their utility is 1. Thus, the wage is 1.

(b) Show the self selection contraints that must be satisfied in a separating equilibrium under incomplete information.

Answer. In a separating equilibrium, the low-ability salemen must be satisfied with a contract in which they dress poorly, so it must be true that

$$\pi_L(poorly) = w(poorly) \geq \pi_L(well) = w(well) - 8.$$

The high-ability salemen must be satisfied with a contract in which they dress well, so it must be true that

$$\pi_H(poorly) = w(poorly) \leq \pi_H(well) = w(well) - 2.$$

(c) Find all the equilibria for this game if information is incomplete.

<u>Answer.</u> In the separating equilibrium, $w(poorly) = 1$ and $w(well) = 6$. This satisfies the self selection constraints of part (b) and yield zero profits to the employers.

In one pooling equilibrium, $w(poorly) = \theta + 4(1-\theta)$ and $w(well) = 3$ and all salesmen dress poorly, where θ is the percentage of low-ability salesmen. This is supported by the out-of-equilibrium belief that anyone who dresses well has low ability.

There is no pooling equilibrium in which everyone dresses well. That would require that $w(poorly) = 1$ and $w(well) = \theta + 4(1-\theta) + 2$, and that

$$\pi_L(poorly) = w(poorly) \leq \pi_L(well) = w(well) - 8,$$

so

$$\pi_L(poorly) = 1 \leq \pi_L(well) = \theta + 4(1-\theta) + 2 - 8,$$

but regardless of how close θ is to 0, this is impossible.

11.9. Crazy Predators (adapted from Gintis [2000], Problem 12.10)

Apex has a monopoly in the market for widgets, earning profits of m per period, but Brydox has just entered the market. There are two periods and no discounting. Apex can either *Prey* on Brydox with a low price or accept *Duopoly* with a high price, resulting in profits to Apex of $-p_a$ or d_a and to Brydox of $-p_b$ or d_b. Brydox must then decide whether to stay in the market for the second period, when Brydox will make the same choices. If, however, Professor Apex, who owns 60 percent of the company's stock, is crazy, he thinks he will earn an amount $p^* > d_a$ from preying on Brydox (and he does not learn from experience). Brydox initially assesses the probability that Apex is crazy at θ.

(a) Show that under the following condition, the equilibrium will be separating, i.e., Apex will behave differently in the first period depending on whether the Professor is crazy or not:

$$-p_a + m < 2d_a \tag{15}$$

<u>Answer.</u> In any equilibrium, Apex will choose *Prey* both periods if the Professor is crazy. In any equilibrium, Apex will choose *Duopoly* in the second period if the Professor is not crazy, by subgame perfectness.

If the equilibrium is separating, Apex will choose *Duopoly* in the first period if the Professor is not crazy, and Brydox will respond by staying in for the second period. This will yield Apex an equilibrium payoff of $2d_a$. The alternative is to deviate to *Prey*. The best this can do is to induce Brydox to exit, leaving Apex an overall payoff of $-p_a + m$ for the two periods, but if $-p_a + m < 2d_a$, deviation is not profitable. (And if Brydox would *not* exit in response to *Prey*, *Prey* is even less profitable.)

(b) Show that under the following condition, the equilibrium can be pooling, i.e., Apex will behave the same in the first period whether the Professor is crazy or not:

$$\theta \geq \frac{d_b}{p_b + d_b} \tag{16}$$

Answer. The only reason for Apex to choose *Prey* in the first period if the Professor is not crazy is to induce Brydox to choose *Exit*. Thus, we should focus on Brydox's decision. Brydox's payoff from *Exit* is 0. Its payoff from staying in is $\theta(-p_b)+(1-\theta)d_b$. Exiting is as profitable as staying in if $0 \geq \theta(-p_b) + (1-\theta)d_b$, which implies that $(p_b + d_b)\theta \geq d_b$, and thus $\theta \geq \frac{d_b}{p_b+d_b}$.

(c) If neither condition (15) nor (16) apply, the equilibrium is hybrid, i.e., Apex will use a mixed strategy and Brydox may or may not be able to tell whether the Professor is crazy at the end of the first period. Let α be the probability that a sane Apex preys on Brydox in the first period, and let β be the probability that Brydox stays in the market in the second period after observing that Apex chose *Prey* in the first period. Show that the equilibrium values of α and β are:

$$\alpha = \frac{\theta p_b}{(1-\theta)d_b} \tag{17}$$

$$\beta = \frac{-p_a + m - 2d_a}{m - d_a} \tag{18}$$

Answer. An equilibrium mixing probability equates the payoffs from its two pure strategy components. First, consider Apex. Apex's two pure-strategy payoffs are:

$$\pi_a(Prey) = -p_a + \beta d_a + (1-\beta)m = d_a + d_a = \pi_a(Duopoly), \tag{19}$$

so $\beta(d_a - m) = -m + p_a + 2d_a$ and we reach equation (18).

Note that we know the numerator of equation (18) is positive, because we have ruled out a separating equilibrium by not having the inequality (15) hold. Also, the mixing probability is less than one because the numerator is less than the denominator.

Now consider Brydox. Brydox's prior that Apex is crazy is θ, but on observing *Prey*, it must modify its beliefs. There was some chance that Apex, if sane (which has probability $(1-\theta)$, would have chosen *Duopoly*, but that didn't happen. That had probability $(1-\alpha)(1-\theta)$ ex ante. Using Bayes' Rule, the posterior probability that Apex is crazy is

$$\frac{\theta}{1-(1-\alpha)(1-\theta)}, \tag{20}$$

and the probability that Apex is sane is

$$\frac{(\alpha)(1-\theta)}{1-(1-\alpha)(1-\theta)}, \tag{21}$$

Brydox's two pure-strategy payoffs after observing *Prey* are therefore

$$\pi_b(Exit) = -p_b = -p_b + \frac{\theta}{1-(1-\alpha)(1-\theta)}(-p_b) + \frac{(\alpha)(1-\theta)}{1-(1-\alpha)(1-\theta)} d_b = \pi_b(Stay\ in), \tag{22}$$

so $0 = \theta(-p_b) + (\alpha)(1-\theta)d_b$ and

$$\alpha = \frac{\theta p_b}{(1-\theta)d_b} \tag{23}$$

If condition (16) is false, then expression (23) is less than 1, a nice check that we have calculated the mixing probability correctly (and it is clearly greater than zero).

(d) Is this behavior related to any of the following phenomenon?– Signalling, Signal Jamming, Reputation, Efficiency Wages.

Answer. This is an example of signal jamming. Apex alters its behavior in the first period so as to avoid conveying information to Brydox. It is not signalling,because Apex is not trying to signal its type. It is not reputation, because this is just a two-period model, not an infinite-period one. In loose language, one might call it reputation, because Apex is trying to avoid acquiring a reputation for sanity, but it has nothing in common with Klein- Leffler reputation models. It is not efficiency wages because no agent is being paid more than his reservation utility so as to maintain incentives, nor is even any firm being rewarded highly under the threat of losing the reward if it behaves badly.

11.11. Monopoly Quality

A consumer faces a monopoly. He initially believes that the probability that the monopoly has a high-quality product is H, and that a high-quality monopoly would be able to send him an advertisement at zero cost. With probability (1-H), though, the monopoly has low quality, and it would cost the firm A to send an ad. The firm does send an ad, offering the product at price P. The consumer's utility from a high-quality product is $X > P$, but from a low quality product it is 0. The production cost is C for the monopolist regardless of quality, where $C < P - A$. If the consumer does not buy the product, the seller does not incur the production cost.

You may assume that the high-quality firm always sends an ad, that the consumer will not buy unless he receives an ad, and that P is exogenous.

(a) Draw the extensive form for this game.

Answer. xxxx Answer unavailable now (the old diagram file is unusable)

(b) What is the equilibrium if H is sufficiently high?

<u>*Answer.*</u> If H is high, then both types of monopoly will advertise, and the consumer will buy the product if he gets an advertisement.

(c) If H is low enough, the equilibrium is in mixed strategies. The high-quality firm always advertises, the low quality firm advertises with probability M, and the consumer buys with probability N. Show using Bayes Rule how the consumer's posterior belief R that the firm is high-quality changes once he receives an ad.

<u>*Answer.*</u> The prior is H. The posterior is

$$R = Prob(High|Advertise) = \frac{Prob(Advertise|High)Prob(High)}{Prob(Advertise)} = \frac{(1)(H)}{(1)(H) + (M)(1-H)}.$$

(d) Explain why the equilibrium is not in pure strategies if H is too low (but H is still positive).

<u>*Answer.*</u> If H is low, then it cannot be an equilibrium for the Low firm always to advertise. Suppose H is close to zero. Then if the Low firm always enters, almost all advertising firms will have low quality, and the consumer will not buy. This would result negative payoffs for the Low firms, so they would not want to advertise.

But neither can it be an equilibrium for no Low firm to advertise. In that case, the consumer would buy, which would make it profitable for the Low firm to advertise.

(e) Find the equilibrium probability of M. (You don't have to figure out N.)

<u>*Answer.*</u> The Low firm's mixing probability M must be such that the consumer is indifferent between buying and not buying. His expected payoff from not buying is 0. From buying, the payoff must be computed using his belief about the probability that the seller has high quality which is the posterior probability R. Thus,

$$R(X-P) + (1-R)(-P) = RX - P = \frac{(X)(H)}{(1)(H) + (M)(1-H)} - P$$

Equating this to the payoff of zero from not buying yields $HX = (H + M - MH)P$, so $HX - HP = MP - MHP$ and $M = \frac{H(X-P)}{P(1-H)}$.

11.x A Continuum of Pooling Equilibria (medium)

Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and that the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the self- selection constraints?

Answer. A pooling equilibrium for any $y^* \in [0, 0.25]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 3 \; if \; y = y^* \end{array} \right. \tag{24}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y = y^*$ for both types.

The self-selection constraints say that neither High nor Low workers want to deviate by acquiring other than y^* education. The most tempting deviation is to zero education, so the constraints are:

$$U_L(y^*) = w(y^*) - 8y^* \geq U_L(0) = w(y \neq y^*) \tag{25}$$

and

$$U_H(y^*) = w(y^*) - \frac{8y^*}{5} \geq U_H(0) = w(y \neq y^*). \tag{26}$$

The constraint on the Lows requires that $y^* \leq 0.25$ for a pooling equilibrium.