ODD

Answers to Odd-Numbered Problems, 4th Edition of Games and Information, Rasmusen

CHAPTER 10: Mechanism Design in Adverse Selection and in Moral Hazard with Hidden Information

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This appendix contains answers to the odd-numbered problems in the fourth edition of *Games and Information* by Eric Rasmusen, which I am working on now and perhaps will come out in 2006. The answers to the even-numbered problems are available to instructors or self-studiers on request to me at Erasmuse@indiana.edu.

Other books which contain exercises with answers include Bierman & Fernandez (1993), Binmore (1992), Fudenberg & Tirole (1991a), J. Hirshleifer & Riley (1992), Moulin (1986), and Gintis (2000). I must ask pardon of any authors from whom I have borrowed without attribution in the problems below; these are the descendants of problems that I wrote for teaching without careful attention to my sources.

PROBLEMS FOR CHAPTER 10: Mechanism Design in Adverse Selection and in Moral Hazard with Hidden Information

10.1. Unravelling
An elderly prospector owns a gold mine worth an amount θ drawn from the uniform distribution $U[0, 100]$ which nobody knows, including himself. He will certainly sell the mine, since he is too old to work it and it has no value to him if he does not sell it. The several prospective buyers are all risk neutral. The prospector can, if he desires, dig deeper into the hill and collect a sample of gold ore that will reveal the value of θ. If he shows the ore to the buyers, however, he must show genuine ore, since an unwritten Law of the West says that fraud is punished by hanging offenders from joshua trees as food for buzzards.

(a) For how much can he sell the mine if he is clearly too feeble to have dug into the hill and examined the ore? What is the price in this situation if, in fact, the true value is $\theta = 70$?

Answer. The price is 50 – the expected value of the uniform distribution from 0 to 100. Even if the mine is actually worth $\theta = 70$, the price remains at 50.

(b) For how much can he sell the mine if he can dig the test tunnel at zero cost? Will he show the ore? What is the price in this situation if, in fact, the true value is $\theta = 70$?

Answer. The expected price is 50. Unravelling occurs, so he will show the ore, and the buyer can discover the true value, which is 50 on average. If the true value is $\theta = 70$, the buyer receives 70.

(c) For how much can he sell the mine if, after digging the tunnel at zero cost and discovering θ, it costs him an additional 10 to verify the results for the buyers? What is his expected payoff?

Answer. He shows the ore iff $\theta \in [20, 100]$. This is because if the minimum quality ore he shows is b, then the price at which he can sell the mine without showing the ore is $\frac{b}{2}$. If $b = 20$ and the true value is 20, then he can sell the mine for 10, and showing the ore to raise the price to 20 would not increase his net profit, given the display cost of 10.

With probability 0.2, his price is 10, and with probability 0.8, it is an average price of 60 but he pays 10 to display the ore. Thus, the prospector's expected payoff is 42 $(= 0.2(10) + 0.8(60 - 10) = 2 + 40 = 42.)$

(d) Suppose that with probability 0.5 digging the test tunnel costs 5 for the prospector, but with probability 0.5 it costs him 120. Keep in mind that the 0-100 value of the mine is net of the buyer's digging cost. Denote the equilibrium price that buyers will pay for the mine after the prospector approaches them without showing ore by P. What is the buyer's posterior belief about the probability it costs 120 to dig the tunnel, as a function of P? Denote this belief by $B(P)$ (Assume, as usual, that all these parameters are common knowledge, although only the prospector learns whether the cost is actually 0 or 120.)

<u>*Answer.*</u> The prospector will decide not to show any ore after digging a test tunnel if the value is less than P. This happens with probability $P/100$ after a tunnel is dug.

The buyers' prior belief that the prospector did not dig a tunnel was .5, but if the prospector approaches them without any ore, that lack of ore is relevant information, to be incorporated into the posterior. Using Bayes's Rule, $Prob(cost = 120|no; sample)$ equals

$$\begin{array}{l} \frac{Prob(cost=120)Prob(no;sample|cost=120)}{Prob(no;sample)} \\ \\ = \frac{Prob(cost=120)Prob(no;sample|cost=120)}{Prob(no;sample|cost=120)Prob(cost=120)+Prob(no;sample|cost=5)Prob(cost=5)} \\ \\ = \frac{.5(1)}{.5(1)+\frac{P}{100}(1)} \\ \\ = \frac{1}{1+\frac{P}{50}} \end{array} \tag{1}$$

(e) What is the prospector's expected payoff in the conditions of part (d) if (i) the tunnel costs him 120, or (ii) the tunnel costs him 5?

<u>*Answer.*</u> The answer depends on the equilibrium value of P. The expected payoff to a buyer if he buys the mine without seeing any sample ore is

$$\begin{array}{ll} \pi_b(no\ sample) & = -P + Prob(cost = 120|no; sample)\left(\frac{100-0}{2}\right) \\ & +Prob(cost = 5|no; sample)\left(\frac{P-0}{2}\right) \\ & \\ & = -P + \left(\frac{1}{1+\frac{P}{50}}\right)(50) + \left(1 - \frac{1}{1+\frac{P}{50}}\right)\left(\frac{P}{2}\right) \\ & \\ & = -P + \left(\frac{1}{1+\frac{P}{50}}\right)(50) + \left(\frac{\frac{P}{50}}{1+\frac{P}{50}}\right)\left(\frac{P}{2}\right) \\ & \\ & = -P + \left(\frac{50}{50+P}\right)(50) + \left(\frac{P}{50+P}\right)\left(\frac{P}{2}\right) \end{array} \tag{2}$$

because with probability $\left(\frac{50}{50+P}\right)$ the prospector did not dig a tunnel and the expected value is 50, and with the remaining probability he did dig a tunnel, but its value was less than P and so he did not show any ore.

Equating the buyer's payoff to 0 because buyers compete profits down to 0 yields

$$
\begin{aligned}
&-P + \left(\frac{50}{50+P}\right)(50) + \left(\frac{P}{50+P}\right)\left(\frac{P}{2}\right) = 0 \\
&-50P - P^2 + 2500 + \frac{P^2}{2} = 0 \\
&-\frac{P^2}{2} - 50P + 2500 = 0 \\
&P = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(-\frac{1}{2})(2500)}}{2(-\frac{1}{2})} \\
&P = -50 \pm \sqrt{2500 + 5000} \\
&P \approx -50 \pm 86.60 = 36.60
\end{aligned}
\tag{3}
$$

(i) With probability .5, the prospector will not dig the tunnel, because the cost of 120 for digging is greater than the greatest possible value of the gold, which is 100. He will show no ore, as a result, but he will still receive the price of 36.60, which will be his payoff.

(ii) With probability .5 he does dig the tunnel at a cost of 5, finding some ore. He then refuses to disclose that ore with probability .3660, for a price of 36.60, and discloses with probability $1 - .3660$, for an average price of $\frac{36.60+100}{2} = 68.30$. His expected payoff from digging the costless tunnel is $-5+(.366)(36.60)+(.634)(68.30) \approx -5 + 13.40 + 43.30 = 51.70$.

The prospector's overall ex ante expected payoff is thus $.5(36.60) + .5(51.70) = 44.15$.

10.3. Agency Law

Mr. Smith is thinking of buying a custom-designed machine from either Mr. Jones or Mr. Brown. This machine costs 5,000 dollars to build, and it is useless to anyone but Smith. It is common knowledge that with 90 percent probability the machine will be worth 10,000 dollars to Smith at the time of delivery, one year from today, and with 10 percent probability it will only be worth 2,000 dollars. Smith owns assets of 1,000 dollars. At the time of contracting, Jones and Brown believe there is there is a 20 percent chance that Smith is actually acting as an "undisclosed agent" for Anderson, who has assets of 50,000 dollars.

Find the price be under the following two legal regimes: (a) An undisclosed principal is not responsible for the debts of his agent; and (b) even an undisclosed principal is responsible for the debts of his agent. Also, explain (as part [c]) which rule a moral hazard model like this would tend to support.

Answer. (a) The zero profit condition, arising from competition between Jones and Brown, is

$$-5000 + .9P + .1(1000) = 0, \tag{4}$$

because Smith will only pay for the machine with probability .9, and otherwise will default and only pay up to his wealth, which is 1,000. This yields $P \approx 5,444$.

(b) If Anderson is responsible for Smith's debts, then Smith will pay the 5,000 dollars. Hence, zero profits require

$$-5000 + .9P + .1(.2)P + .1(.8)(1000) = 0, \tag{5}$$

which yields $P \approx 5,348$.

(c) Moral hazard tends to support rule (b). This is because it reduces bankruptcy and the agent will be more reluctant to order the machine when there is a high chance it is unprofitable. In the model as constructed, this does not arise, because there is only one type of agent, but more generally it would, because there would be a continuum of types of agents, and some who would buy the machine under rule (b) would find it too expensive under rule (a).

Even in the model as it stands, rule (a) leads to the inefficient outcome that a machine worth 2,000 to Smith is not give to Smith. Rather, he pays his wealth and lets the seller keep the machine, which is inefficient since the machine really is worth 2000 to Smith.

This is a question about zero-profit prices. Guessing would have been a good idea here: it is very intuitive that the price would always be above \$5,000, and that it would be higher if the principal never had to cover the agent's debts. You should be able to tell that $P > 10,000$ is impossible, because Smith would never pay it. Also, the sellers compete, so it is their profits that provide a participation constraint, not the benefit to the buyer.

10.5. The Groves Mechanism

A new computer costing 10 million dollars would benefit existing Divisions 1, 2, and 3 of a company with 100 divisions. Each divisional manager knows the benefit to his division (variables $v_i, i = 1, ..., 3$), but nobody else does, including the company CEO. Managers maximize the welfare of their own divisions. What dominant strategy mechanism might the CEO use to induce the managers to tell the truth when they report their valuations? Explain why this mechanism will induce truthful reporting, and denote the reports by $x_i, i = 1, ..., 3$. (You may assume that any budget transfers to and from the divisions in this mechanism are permanent– that the divisions will not get anything back later if the CEO collects more payments than he gives, for example.)

Answer. Let Division 1 pay $(10 - x_2 - x_3)$, Division 2 pay $(10 - x_1 - x_3)$, and Division 3 pay $(10 - x_1 - x_2)$ if the computer is bought, where that payment could be negative, and buy the computer if $x_1 + x_2 + x_3 \geq 10$.

Manager i's report does not affect its payment except by affecting whether the computer is bought. Let us take the case of Manager 1 for concreteness. His payoff is $v_1 - (10 - x_2 - x_3)$ if the computer is bought and 0 otherwise. He therefore wants the computer to be bought if and only if $v_1 + x_2 + x_3 \geq 10$. By reporting $x_1 = v_1$, he achieves exactly that outcome– the computer is bought only when he wants it to be bought. If the other two divisions overreport, he wants the computer to be bought because the mechanism will make him pay less than x_1, and if they underrport, he wants it not to be bought, because the mechanism will make him pay more than x_1.

10.7. Selling Cars

A car dealer must pay $10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between $11,000 and $21,000, Jones's is between $9,000 and $11,000, and Brown's is between $4,000 and $12,000. The dealer's policy is to make a separate take-it-or-leave-it offer to each customer, and he is smart enough to avoid making different offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is R.

(a) What will the offers be?

<u>*Answer.*</u> Let us use units of thousands of dollars. The expected profit from a customer with maximum valuation $\overline{V} > 10$ and range of valuations R is, if price P is charged:

$$\begin{aligned} \pi(P;V,R) &= \int_P^{\overline{V}} \tfrac{P-10}{R} dV \\ &= \left(\tfrac{PV}{R} - \tfrac{10V}{R}\right)\Big|_P^{\overline{V}} \\ &= \tfrac{\overline{V}P}{R} - \tfrac{10\overline{V}}{R} - \tfrac{P^2}{R} + \tfrac{10P}{R}. \end{aligned} \tag{6}$$

Maximizing profit with respect to P yields the first order condition

$$\frac{d\pi(P;V,R)}{dP} = \frac{\overline{V}}{R} - \frac{2P}{R} + \frac{10}{R} = 0, \tag{7}$$

so

$$P^* = \frac{\overline{V}}{2} + 5. \tag{8}$$

Note that the optimal price does not depend on R, the range of possible valuations. That is because what R determines is the probability that the customer's value is greater than $10,000, but if it is greater than $10,000, the seller's optimal price is determined by the possible values between $10,000 and $\overline{v}$, and R is irrelevant.

Applying (8) to the specific customers: Smith will be offered $P = \frac{21}{2} + 5 = \$15,500$, Jones will be offered $P = \frac{11}{2} + 5 = \$10,500$, and Brown will be offered $P = \frac{12}{2} + 5 = \$11,000$. Moreover, Brown probably values the car less than Jones, but because of the higher probability that he values it more than $10,000, he will end up paying more if he buys at all.

(b) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges \$10,000 to each customer?

Answer. Smith will buy with probability 0.55, which is $\frac{21-15.5}{21-11}$. Jones will buy with probability 0.25. Brown will buy with probability 0.125. Thus, Smith is the buyer most likely to buy.

Whether the dealer charges \$10,000 or uses perfect price discrimination, the outcome is the same as far as allocative efficiency: Smith buys with probability 1, Jones buys with probability 0.5, and Brown buys with probability 0.25.

(c) What happens to the equilibrium prices if with probability 0.25 each buyer has a valuation of \$0, but the probability distribution remains otherwise the same? What happens to the equilibrium expected profit?

Answer. The prices are the same as in part (a). If a buyer values the car at less than \$10,000, it is irrelevant what his value may be, since it is unprofitable to sell to him anyway. Only the part of his distribution above \$10,000 matters to the seller's strategy. Note that this has the same flavor as the analysis of auctions, where a bidder's strategy is conditioned on his having the highest valuation, since if he does not, he will generally lose the auction anyway and his bid is irrelevant.

The equilibrium expected profit, however, drops to 0.75 of its former level, since now with probability 0.25 there is no sale for the new reason that $v = 0$.

Another way to look at this is to think about two moves by Nature. In the first move, Nature chooses between $v = 0$ and $v > 0$, with probabilities 0.25 and 0.75. In the second move, if $v > 0$ Nature decides how much greater it is. Suppose the seller observes the first of these moves. If he sees that $v = 0$, he is indifferent among all prices greater than \$10,000, since he will not sell a car anyway. If he sees that $v > 0$, he is in exactly the situation of part (a), so he will choose the same prices as we found there.

(d) What happens to the equilibrium price the seller offers to seller Jones if with probability 0.25 Jones has a valuation of \$30,000, but with probability 0.75 his valuation is uniformly distributed between \$9,000 and \$11,000 as before? Show the relation between price and profit on a rough graph.

Answer. Start by deriving the seller's expected profit from Jones. If the price is below $P = 30$, profit is

$$\begin{aligned} \pi(P; V, R) &= 0.75 \int_P^{\overline{V}} \frac{P-10}{R} dV + 0.25(P-10) \\ &= 0.75 \left(\frac{PV}{R} - \frac{10V}{R} \right)\Big|_P^{\overline{V}} + 0.25(P-10) \\ &= 0.75 \frac{\overline{V}P}{R} - \frac{10\overline{V}}{R} - \frac{P^2}{R} + \frac{10P}{R} + 0.25(P-10). \end{aligned} \tag{9}$$

Maximizing profit with respect to P yields the first order condition

$$\frac{d\pi(P;V,R)}{dP} = 0.75\left(\frac{\overline{V}}{R} - \frac{2P}{R} + \frac{10}{R}\right) + 0.25 = 0, \tag{10}$$

which solves, given that $\overline{V} = 11$ and $R = 2$, to

$$P = 10\,\frac{5}{6}. \tag{11}$$

This, however, is not the profit-maximizing price! The problem is that this is just a local maximum, not a global maximum, for profit. It is a maximum, because the second derivative is $\frac{d^2\pi(P;V,R)}{dP^2} = \frac{0.75(2)}{R} < 0$, but that just means that it is the profit-maximizing price given that the price is no greater than \$11, 000. Suppose, however, that the seller gives up on selling the car during the 75% of the time that Jones has a value between \$9,000 and \$11, 000, and raises his price to $P = \$30,000$. His expected profit will then rise from the roughly \$ (0.26) 11,000 to exactly \$ (0.25) 30,000.

Figure A10.1 is a rough graph of profits:

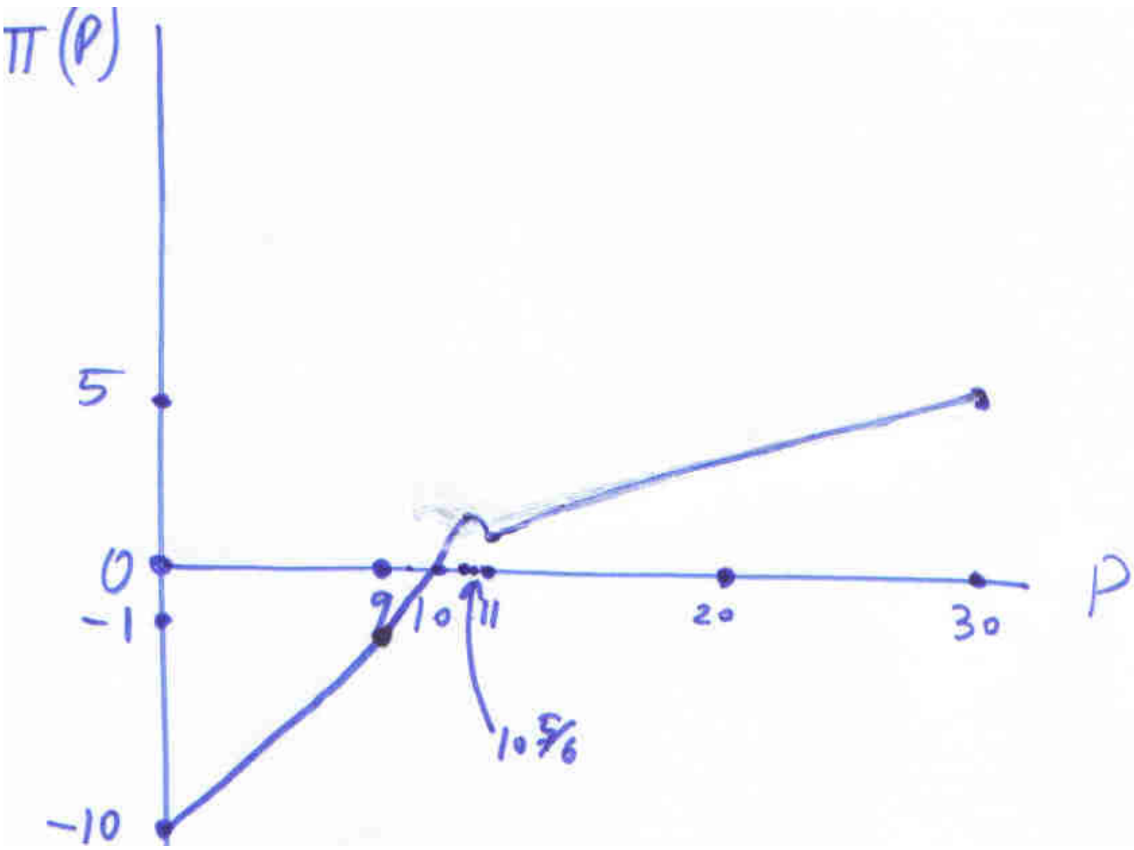

Figure A10.1: Prices and Profits