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The essential elements of a game are **players**, **actions**, **payoffs**, and **information**.

. These are collectively known as the **rules of the game**, and the modeller's objective is to describe a situation in terms of the rules of a game so as to explain what will happen in that situation.

Trying to maximize their payoffs, the players will devise plans known as **strategies** that pick actions depending on the information that has arrived at each moment.

The combination of strategies chosen by each player is known as the **equilibrium**.

Given an equilibrium, the modeller can see what actions come out of the conjunction of all the players' plans, and this tells him the **outcome** of the game.

How would you describe the supply and demand of gasoline in these terms?

A Story to Model

An entrepreneur is trying to decide whether to start a dry cleaning store in a town already served by one dry cleaner.

We will call the two firms "NewCleaner" and "OldCleaner." NewCleaner is uncertain about whether the economy will be in a recession or not, which will affect how much consumers pay for dry cleaning, and must also worry about whether OldCleaner will respond to entry with a price war or by keeping its initial high prices. OldCleaner is a well-established firm, and it would survive any price war, though its profits would fall. NewCleaner must itself decide whether to initiate a price war or to charge high prices, and must also decide what kind of equipment to buy, how many workers to hire, and so forth.

Players *are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.*

An **action** *or* **move** *by player i, denoted a_i, is a choice he can make.*

Player i's **action set**, $A_i = \{a_i\}$, *is the entire set of actions available to him.*

An **action combination** *is a list $a = \{a_i\}$, $(i = 1, \ldots, n)$ of one action for each of the n players in the game.*

Newcleaner's action set: $\{Enter, Stay\ Out\}$. Oldcleaner's action set to be simple: it is to choose price from $\{Low, High\}$.

Nature *is a pseudo-player who takes random actions at specified points in the game with specified probabilities.*

In the Dry Cleaners Game, we will model the possibility of recession as a move by Nature. With probability 0.3, Nature decides that there will be a recession, and with probability 0.7 there will not. Even if the players always took the same actions, this random move means that the model would yield more than just one prediction. We say that there are different **realizations** of a game depending on the results of random moves.

Why is Nature not a real player?

By player i's **payoff** π_i, we mean either:
(1) The utility player i receives after all players and Nature have picked their strategies and the game has been played out; or
(2) His expected utility at the start of the game.

Table 1a: The Dry Cleaners Game: Normal Economy

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-100, -50	100, 100
	Stay Out	0,50	0,300

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Table 1b: The Dry Cleaners Game: Recession

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-160, -110	40, 40
	Stay Out	0,-10	0,240

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Information is modelled using the concept of the **information set.**

The elements of the information set are the different values ofa variable that the player thinks are possible. If the information set has many elements, there are many values the player cannot rule out; if it has one element, he knows the value precisely.

It is convenient to lay out information and actions together in an **order of play**. Here is the order of play we have specified for the Dry Cleaners Game:

1 Newcleaner chooses its entry decision from $\{Enter, Stay\ Out\}$
2 Oldcleaner chooses its price from $\{Low, High\}$.
3 Nature picks demand, D, to be $Recession$ with probability 0.3 or $Normal$ with probability 0.7.

The **outcome** *of the game is a set of interesting elements that the modeller picks from the values of actions, payoffs, and other variables after the game is played out.*

Decision theory is like game theory with just one player.

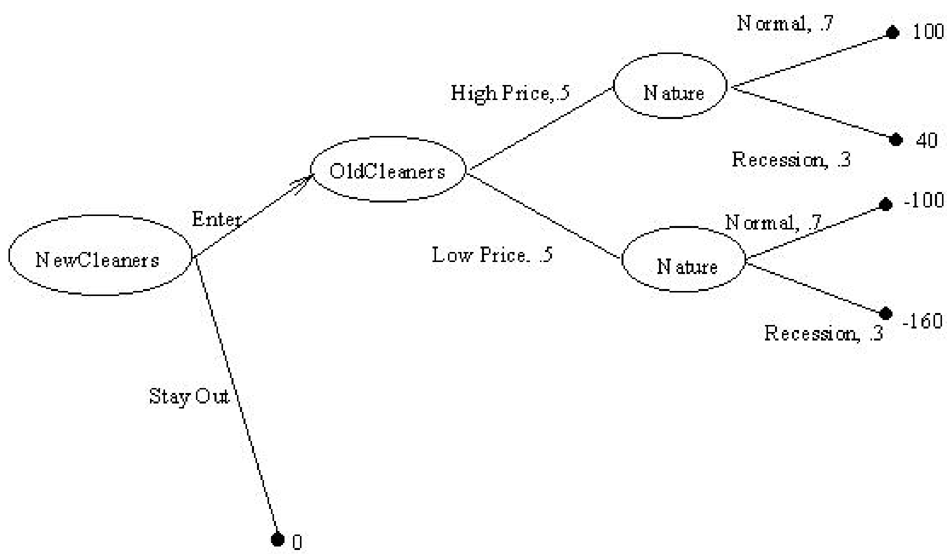

Figure 1: The Dry Cleaners Game as a Decision Tree

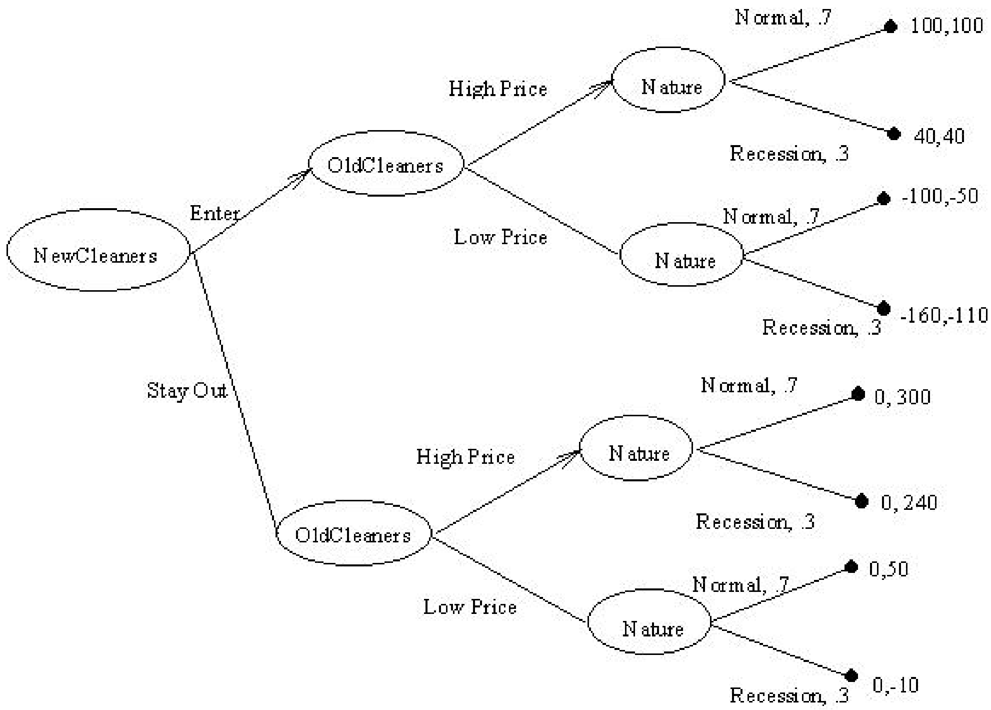

Figure 2: The Dry Cleaners Game as a Game Tree

Player i's **strategy** s_i *is a rule that tells him which action to choose at each instant of the game, given his information set.*

Player i's **strategy set** *or* **strategy space** $S_i = \{s_i\}$ *is the set of strategies available to him.*

A **strategy profile** $s = (s_1, \ldots, s_n)$ *is a list consisting of one strategy for each of the* n *players in the game.*

In The Dry Cleaners Game, the strategy set for NewCleaner is just { *Enter, Stay Out* } , since NewCleaner moves first and is not reacting to any new information. The strategy set for OldCleaner, though, is

$$\left\{ \begin{array}{l} \text{High Price if NewCleaner Entered, Low Price otherwise} \\ \text{Low Price if NewCleaner Entered, High Price otherwise} \\ \text{High Price No Matter What} \\ \text{Low Price No Matter What} \end{array} \right\}$$

Equilibrium

The single outcome of *NewCleaner Enters* would result from either of the following two strategy profiles:

$$\left\{ \begin{array}{l} \textit{High Price if NewCleaner Enters, Low Price otherwise} \\ \textit{Enter} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \textit{Low Price if NewCleaner Enters, High Price if NewClear} \\ \textit{Enter} \end{array} \right.$$

Predicting what happens consists of selecting one or more strategy profiles as being the most rational behavior by the players acting to maximize their payoffs.

An **equilibrium** $s^* = (s_1^*, \ldots, s_n^*)$ *is a strategy profile consisting of a best strategy for each of the* n *players in the game.*

The **equilibrium strategies** are the strategies players pick in trying to maximize their individual payoffs.

Equilibrium= Equilibrium Strategy Profile = set of strategies

Equilibrium outcome = set of values of outcome variables

The equilibrium concept defines "best strategy".

An **equilibrium concept** *or* **solution concept** $F : \{S_1, \ldots, S_n, \pi_1, \ldots, \pi_n\} \to s^*$ *is a rule that defines an equilibrium based on the possible strategy profiles and the payoff functions.*

A given equilibrium concept might lead to no equilibrium existing, or multiple equilibria.

For any vector $y = (y_1, \ldots, y_n)$, denote by y_{-i} the vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$, which is the portion of y not associated with player i.

Using this notation, s_{-Smith}, for instance, is the profile of strategies of every player except player $Smith$. That profile is of great interest to Smith, because he uses it to help choose his own strategy, and the new notation helps define his best response.

Player i's **best response** *or* **best reply** *to the strategies* s_{-i} *chosen by the other players is the strategy* s_i^* *that yields him the greatest payoff; that is,*

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_i' \neq s_i^*. \tag{1}$$

The strategy s_i^d *is a* **dominated strategy** *if it is strictly inferior to some other strategy no matter what strategies the other players choose, in the sense that whatever strategies they pick, his payoff is lower with* s_i^d*. Mathematically,* s_i^d *is dominated if there exists a single* s_i' *such that*

$$\pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}) \;\; \forall s_{-i}. \tag{2}$$

s_i^d is not a dominated strategy if there is no s_{-i} to which it is the best response, but sometimes the better strategy is s_i' and sometimes it is s_i''.

In that case, s_i^d could have the redeeming feature of being a good compromise strategy for a player who cannot predict what the other players are going to do

A dominated strategy is unambiguously inferior to some single other strategy.

The strategy s_i^* *is a* **dominant strategy** *if it is a player's strictly best response to* any *strategies the other players might pick, in the sense that whatever strategies they pick, his payoff is highest with* s_i^*. *Mathematically,*

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \ \ \forall s_{-i}, \ \ \forall s_i' \neq s_i^*. \qquad (3)$$

A **dominant-strategy equilibrium** *is a strategy profile consisting of each player's dominant strategy.*

Table 2: The Prisoner's Dilemma

		Column	
		Silence	*Blame*
Row	*Silence*	-1,-1	-10, 0
	Blame	0,-10	**- 8,-8**

Payoffs to: (Row, Column)

Table 3: ITERATED DOMINANCE— The Battle of the Bismarck Sea

		Imamura	
		North	South
Kenney	North	**2,-2**	$2,-2$
	South	$1,-1$	$3,-3$

Payoffs to: (Kenney, Imamura)

Strategy s'_i is **weakly dominated** if there exists some other strategy s''_i for player i which is possibly better and never worse, yielding a higher payoff in some strategy profile and never yielding a lower payoff. Mathematically, s'_i is weakly dominated if there exists s''_i such that

$$\begin{aligned} &\pi_i(s''_i, s_{-i}) \geq \pi_i(s'_i, s_{-i}) \quad \forall s_{-i}, \text{ and} \\ &\pi_i(s''_i, s_{-i}) > \pi_i(s'_i, s_{-i}) \quad \text{for some } s_{-i}. \end{aligned} \tag{4}$$

An **iterated-dominance equilibrium** *is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until only one strategy remains for each player.*

Table 4: The Iteration Path Game

		Column		
		c_1	c_2	c_3
	r_1	**2,12**	1,10	**1,12**
Row	r_2	0,12	0,10	0,11
	r_3	0,12	1,10	0,13

Payoffs to: (Row, Column)

The strategy profiles (r_1, c_1) and (r_1, c_3) are both iterated dominance equilibria, because each of those strategy profiles can be found by iterated deletion.

The deletion can proceed in the order (r_3, c_3, c_2, r_2), or in the order (r_2, c_2, c_1, r_3).

Zero-Sum Games

A **zero-sum game** is a game in which the sum of the payoffs of all the players is zero whatever strategies they choose.

MATCHING PENNIES (not in chapter 1)

Smith wins the penny if the two pennies match; otherwise Jones wins.

		Jones	
		Heads	*Tails*
Smith	*Heads*	1, -1	$-1, 1$
	Tails	$-1, 1$	1,1

Payoffs to: (Smith, Jones).

Table 5: Boxed Pigs (NASH EQUILIBRIUM)

		Small Pig		
		Press		Wait
	Press	$5,1$	$\rightarrow$	$\boxed{\mathbf{4}}$**,** $\boxed{\mathbf{4}}$
Big Pig		$\downarrow$		$\uparrow$
	Wait	$\boxed{9}, -1$	$\rightarrow$	$0, \boxed{0}$

Payoffs to: (Big Pig, Small Pig). Arrows show how a player can increase his payoff. Best-response payoffs are boxed.

The strategy profile s^* is a **Nash equilibrium** *if no player has incentive to deviate from his strategy given that the other players do not deviate. Formally,*

$$\forall i, \quad \pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \quad \forall s_i'. \tag{5}$$

Table 6: The Modeller's Dilemma

		Column		
		Silence		*Blame*
	Silence	**0, 0**	↔	−10, 0
Row		↕		↓
	Blame	0,-10	→	**-8, -8**

Payoffs to: (Row, Column) . Arrows show how a player can increase his payoff.

What are the equilibria?

What would you predict to happen?

Table 7: Battle of the Sexes

		Woman		
		Prize Fight		*Ballet*
Man	*Prize Fight*	**2,1**	$\leftarrow$	0, 0
		$\uparrow$		$\downarrow$
	Ballet	0, 0	$\rightarrow$	**1,2**

Payoffs to: (Man, Woman). Arrows show how a player can increase his payoff.

Table 8: Ranked Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	$\leftarrow$	$-1,-1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1,-1$	$\rightarrow$	**1,1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Table 9: Dangerous Coordination (ASSURANCE GAME) (STAG HUNT)

		Jones		
		Large		*Small*
	Large	**2,2**	$\leftarrow$	$-1000, -1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1, -1$	$\rightarrow$	**1,1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

(Large, Large) is Nash and Pareto superior (payoff dominant) but not "risk dominant": give a 50-50 chance of the other player choosing each strategy, each player would choose SMALL, as "safer."

Two firms are choosing outputs Q_1 and Q_2 simultaneously. The Nash equilibrium is a pair of numbers (Q_1^*, Q_2^*) such that neither firm would deviate unilaterally.

This troubles the beginner, who says to himself,

> "Sure, Firm 1 will pick Q_1^* if it thinks Firm 2 will pick Q_2^*. But Firm 1 will realize that if it makes Q_1 bigger, then Firm 2 will react by making Q_2 smaller. So the situation is much more complicated, and (Q_1^*, Q_2^*) is not a Nash equilibrium. Or, if it is, Nash equilibrium is a bad equilibrium concept."

Focal Points

1 Circle one of the following numbers: 100, 14, 15, 16, 17, 18.

2 Circle one of the following numbers 7, 100, 13, 261, 99, 666.

3 Name Heads or Tails.

4 Name Tails or Heads.

5 You are to split a pie, and get nothing if your proportions add to more than 100 percent.

6 You are to meet somebody in New York City. When? Where?