3 Mixed and Continuous Strategies

A **pure strategy** *maps each of a player's possible information sets to one action.* $s_i : \omega_i \rightarrow a_i$.

A **mixed strategy** *maps each of a player's possible information sets to a probability distribution over actions.*

$$s_i : \omega_i \rightarrow m(a_i), \quad where \;\; m \geq 0 \;\text{ and }\; \int_{A_i} m(a_i) da_i = 1.$$

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	3,2	$\rightarrow$	$-1,3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1,1$	$\leftarrow$	0,0

Payoffs to: (Government, Pauper). Arrows show how a player can increase his payoff.

If the government plays *Aid* with probability θ_a and the pauper plays *Work* with probability γ_w, the government's expected payoff is

$$\begin{aligned}\pi_{Government} &= \theta_a[3\gamma_w + (-1)(1-\gamma_w)] + [1-\theta_a][-1\gamma_w + 0(1-\gamma_w)] \\ &= \theta_a[3\gamma_w - 1 + \gamma_w] - \gamma_w + \theta_a\gamma_w \\ &= \theta_a[5\gamma_w - 1] - \gamma_w.\end{aligned} \tag{1}$$

Differentiate the payoff function with respect to the choice variable to obtain the first-order condition.

$$\begin{aligned} 0 = \frac{d\pi_{Government}}{d\theta_a} = 5\gamma_w - 1 \\ \Rightarrow \ \gamma_w = 0.2.\end{aligned} \tag{2}$$

We obtained the pauper's strategy by differentiating the government's payoff!

THE LOGIC

1 I assert that an optimal mixed strategy exists for the government.

2 If the pauper selects *Work* more than 20 percent of the time, the government always selects *Aid*. If the pauper selects *Work* less than 20 percent of the time, the government never selects *Aid*.

3 If a mixed strategy is to be optimal for the government, the pauper must therefore select *Work* with probability exactly 20 percent.

To obtain the probability of the government choosing *Aid*:

$$\begin{aligned}
\pi_{Pauper} &= \gamma_w(2\theta_a + 1[1-\theta_a]) + (1-\gamma_w)(3\theta_a + [0][1-\theta_a]) \\
&= 2\gamma_w\theta_a + \gamma_w - \gamma_w\theta_a + 3\theta_a - 3\gamma_w\theta_a \\
&= -\gamma_w(2\theta_a - 1) + 3\theta_a.
\end{aligned} \tag{3}$$

The first-order condition is

$$\begin{aligned}
\frac{d\pi_{Pauper}}{d\gamma_w} &= -(2\theta_a - 1) = 0, \\
&\Rightarrow \theta_a = 1/2.
\end{aligned} \tag{4}$$

The Payoff-Equating Method

In equilibrium, each player is willing to mix only because he is indifferent between the pure strategies he is mixing over. This gives us a better way to find mixed strategies.

First, guess which strategies are being mixed between.

Then, see what mixing probability for the other player makes a given player indifferent.

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	3,2	$\rightarrow$	$-1, 3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1, 1$	$\leftarrow$	0,0

Here,

$\pi_g(Aid) = \gamma_w(3)+(1-\gamma_w)(-1) = \pi_g(No\ aid) = \gamma_w(-1)+(1-\gamma_w)(0)$

So $\gamma_w(3+1+1) = 1$, so $\gamma_w = .2$.

$\pi_p(Work) = \theta_a(2)+(1-\theta_a)(1) = \pi_p(Loaf) = \theta_a(3)+(1-\theta_a)(0)$

so $\theta_a(2-1-3) = -1$ and $\theta_a = .5$.

Interpreting Mixed Strategies

A player who selects a mixed strategy is always indifferent between two pure strategies and an entire continuum of mixed strategies.

What matters is that a player's strategy appear random to other players, not that it really be random.

It could be based on time of day, temperature, etc.

It could be there is a population of identical players, each of whom picks a pure strategy. But each would still be indifferent about his strategy.

Or, mixing could be based on unknown characteristics of the player. Harsanyi (1973).

Let the payoffs not be exactly as in the matrix. Instead, the pauper payoff of 3 is distributed on the continuum [2.9, 3.1] with median 3.

$$
\begin{aligned}
\pi_{Pauper} &= \gamma_w(2\theta_a + 1[1-\theta_a]) + (1-\gamma_w)(X\theta_a + [0][1-\theta_a]) \\
&= 2\gamma_w\theta_a + \gamma_w - \gamma_w\theta_a + X\theta_a - X\gamma_w\theta_a \\
&= (1-X)\gamma_w\theta_a + (1-X)\gamma_w + X\theta_a.
\end{aligned} \tag{5}
$$

The first-order condition is

$$
\begin{aligned}
\frac{d\pi_{Pauper}}{d\gamma_w} &= (1-X)\gamma_w = 0, \\
\Rightarrow \theta_a &= \frac{1}{X-1}.
\end{aligned} \tag{6}
$$

With probability 1, the Government has an strongly optimal pure strategy— either AID or NO AID, $\theta_a = 1$ or $\theta_a = 0$. But to the pauper, it seems there is a 50% chance of the pure strategy AID.

How about if the mixing probability does not come out to .5? Well, let's think about the government having a payoff from (Aid, Loaf) ranging from -.9 to -1.1 with cumulative distribution $F(z)$.

$$
\begin{aligned}
\pi_{Government} &= \theta_a[3\gamma_w + (z)(1-\gamma_w)] + [1-\theta_a][-1\gamma_w + 0(1-\gamma_w)] \\
&= \theta_a[3\gamma_w + z - z\gamma_w + \gamma_w) - \gamma_w \\
&= \theta_a[(4+z)\gamma_w + z] - \gamma_w.
\end{aligned} \tag{7}
$$

The first order condition tells us that the government prefers to make θ_a as big as possible (that is, 1) if $(4 - z)\gamma_w + z > 0$. We need the pauper Su to think there is an

unfinished

Table 2: Pure Strategies Dominated by a Mixed Strategy

		Column	
		North	*South*
	North	0,0	4,-4
Row	*South*	4,-4	0,0
	Defense	1,-1	1,-1

Payoffs to: (Row, Column)

For Row, *Defense* is strictly dominated by (0.5 *North*, 0.5 *South*). In equilibrium, both players choose that.

His expected payoff from this mixed strategy if Column plays *North* with probability N is

$$0.5(N)(0)+0.5(1-N)(4)+0.5(N)(4)+0.5(1-N)(0) = 2, \qquad (8)$$

so whatever response Column picks, Row's expected payoff is higher from the mixed strategy than his payoff of 1 from *Defense*.

Lesson: It is dangerous to assume away mixed strategies. It is better to allow them, and then to say you will only look at pure-strategy equilibria.

Table 3: Chicken

		Jones		
		$Continue\ (\theta)$		$Swerve\ (1-\theta)$
Smith:	$Continue\ (\theta)$	$-3, -3$	$\rightarrow$	**2, 0**
		$\downarrow$		$\uparrow$
	$Swerve\ (1-\theta)$	**0, 2**	$\leftarrow$	1, 1

$$\pi_{Jones}(Swerve) = (\theta_{Smith}) \cdot (0) + (1-\theta_{Smith}) \cdot (1)$$
$$= (\theta_{Smith}) \cdot (-3) + (1-\theta_{Smith}) \cdot (2) = \pi_{Jones}(Cont \tag{9}$$

From equation (9) we can conclude that $1 - \theta_{Smith} = 2 - 5\theta_{Smith}$, so $\theta_{Smith} = 0.25$.

In the symmetric equilibrium, both players choose the same probability, so we can replace θ_{Smith} with simply θ.

The two teenagers will survive with probability $1 - (\theta \cdot \theta) = 0.9375$.

		Jones		
		Continue (θ)		*Swerve* ($1-\theta$)
	Continue (θ)	$-x, -x$	$\rightarrow$	**2, 0**
Smith:		$\downarrow$		$\uparrow$
	Swerve ($1-\theta$)	**0, 2**	$\leftarrow$	1, 1

$$\theta = \frac{1}{1-x}. \tag{10}$$

If $x = -3$, this yields $\theta = 0.25$, as was just calculated.

If $x = -9$, it yields $\theta = 0.10$.

If $x = 0.5$, the equilibrium probability of continuing appears to be $\theta = 2$.

The War of Attrition

The possible actions are $Exit$ and $Continue$. In each period that both $Continue$, each earns -1. If a firm exits, its losses cease and the remaining firm obtains the value of the market's monopoly profit, which we set equal to 3. We will set the discount rate equal to $r > 0$.

(1) Continue in each period, Exit in each period

(2) Each exits with probability θ if it hasn't yet.

Let Smith's payoffs be V_{stay} if he stays and V_{exit} if he exits.

$V_{exit} = 0$.

$$V_{stay} = \theta \cdot (3) + (1-\theta)\left(-1 + \left[\frac{V_{stay}}{1+r}\right]\right), \qquad (11)$$

which, after a little manipulation, becomes

$$V_{stay} = \left(\frac{1+r}{r+\theta}\right)(4\theta - 1). \qquad (12)$$

Thus, $\theta = 0.25$.

Timing games

A **pre-emption game**, in which the reward goes to the player who chooses the move which ends the game, and a cost is paid if both players choose that move, but no cost is incurred in a period when neither player chooses it.

Grab the Dollar. A dollar is placed on the table between Smith and Jones, who each must decide whether to grab for it or not. If both grab, each is fined one dollar. This could be set up as a one-period game, a T period game, or an infinite- period game, but the game definitely ends when someone grabs the dollar.

Table 4: Grab the Dollar

		Jones		
		Grab		*Don't Grab*
	Grab	$-1,-1$	$\rightarrow$	**1,0**
Smith:		$\downarrow$		$\uparrow$
	Don't Grab	**0,1**	$\leftarrow$	$0,0$

A **noisy duel**: if a player shoots and misses, the other player observes the miss and can kill the first player at his leisure.

A **silent duel**: , a player does not know when the other player has fired, and the equilibrium is in mixed strategies.

Patent Race for a New Market (an all-pay auction)

Players

Three identical firms, Apex, Brydox, and Central.

The Order of Play

Each firm simultaneously chooses research spending $x_i \geq 0$, $(i = a, b, c)$.

Payoffs

Firms are risk neutral and the discount rate is zero. Innovation occurs at time $T(x_i)$ where $T' < 0$. The value of the patent is V, and if several players innovate simultaneously they share its value. Let us look at the payoff of firm $i = a, b, c$, with j and k indexing the other two firms:

$$\pi_i = \begin{cases} V - x_i & \text{if } T(x_i) < Min\{T(x_j, T(x_k)\} \quad \text{(wins)} \\ \frac{V}{2} - x_i & \text{if } T(x_i) = Min\{T(x_j), T(x_k)\} \quad \text{(shares with 1)} \\ & \qquad < Max\{T(x_j), T(x_k)\} \\ \frac{V}{3} - x_i & \text{if } T(x_i) = T(x_j = T(x_k) \quad \text{(shares with 2)} \\ & \qquad \text{2 other firms)} \\ -x_i & \text{if } T(x_i) > Min\{T(x_j, T(x_k)\} \quad \text{(loses)} \end{cases}$$

The game Patent Race for a New Market does not have any pure strategy Nash equilibria, because the payoff functions are discontinuous.

A slight difference in research by one player can make a big difference in the payoffs, as shown in Figure 1 for fixed values of x_b and x_c. The research levels shown in Figure 1 are not equilibrium values. If Apex chose any research level x_a less than V, Brydox would respond with $x_a + \varepsilon$ and win the patent. If Apex chose $x_a = V$, then Brydox and Central would respond with $x_b = 0$ and $x_c = 0$, which would make Apex want to switch to $x_a = \varepsilon$.

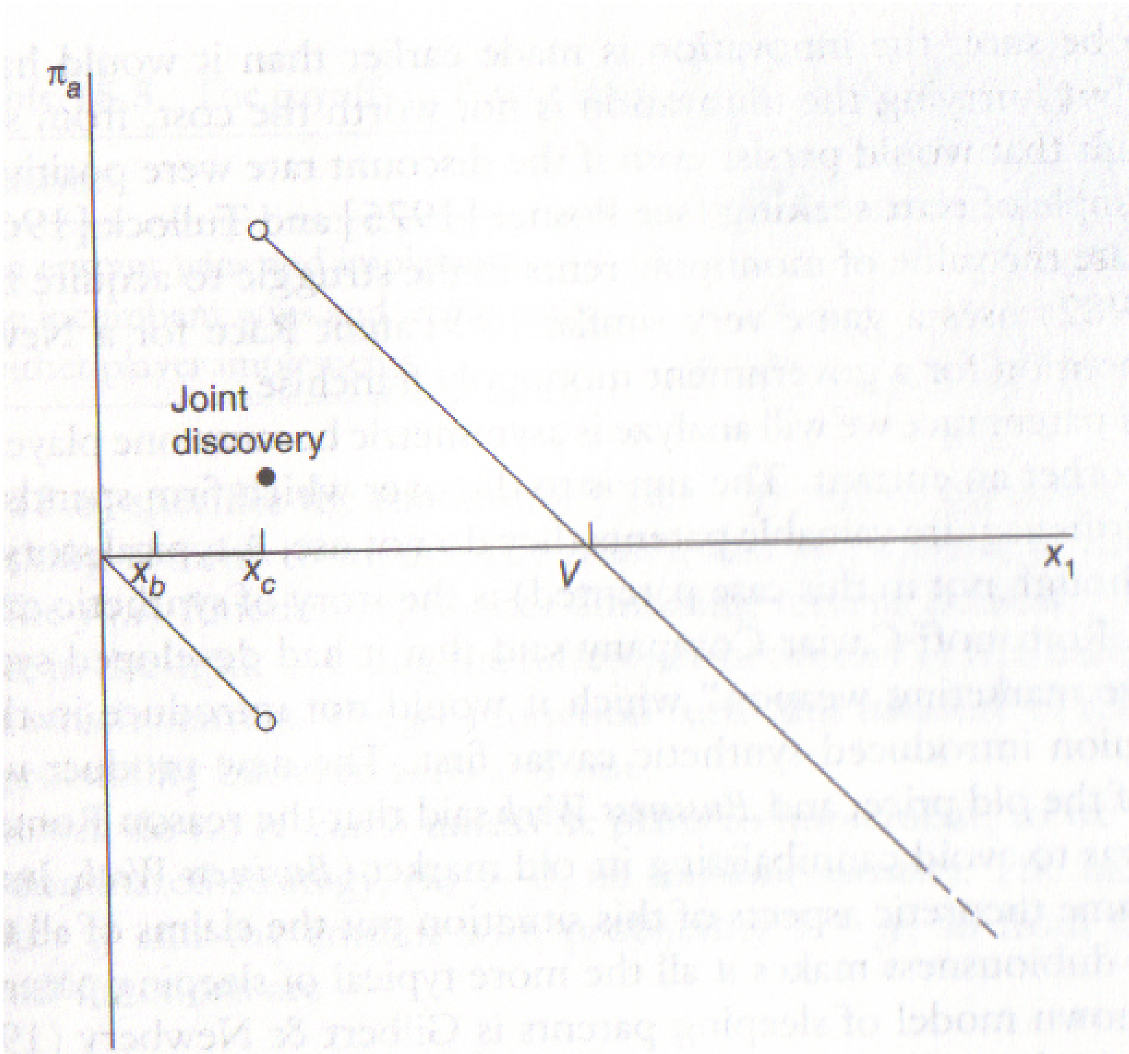

Figure 1: The Payoffs in Patent Race for a New Market

Denote the probability that firm i chooses a research level less than or equal to x as $M_i(x)$. This function describes the firm's mixed strategy.

Since we know that the pure strategies $x_a = 0$ and $x_a = V$ yield zero payoffs, if Apex mixes over $[0, V]$ then the expected payoff for every strategy mixed between must also equal zero.

$$\pi_a(x_a) = V \cdot Pr(x_a \geq X_b, x_a \geq X_c) - x_a = 0 = \pi_a(x_a = 0), \tag{13}$$

which can be rewritten as

$$V \cdot Pr(X_b \leq x_a) Pr(X_c \leq x_a) - x_a = 0, \tag{14}$$

or

$$V \cdot M_b(x_a) M_c(x_a) - x_a = 0. \tag{15}$$

We can rearrange equation (15) to obtain

$$M_b(x_a) M_c(x_a) = \frac{x_a}{V}. \tag{16}$$

If all three firms choose the same mixing distribution M, then

$$M(x) = \left(\frac{x}{V}\right)^{1/2} \text{ for } 0 \leq x \leq V. \tag{17}$$

"all-pay auction", and the techniques and findings of auction theory can be quite useful when modelling this kind of conflict.

Correlated Strategies

Aumann (1974, 1987) has pointed out that it is often important whether players can use the same randomizing device for their mixed strategies. If they can, we refer to the resulting strategies as **correlated strategies**.

Consider Chicken. The only mixed-strategy equilibrium is the symmetric one in which each player chooses *Continue* with probability 0.25 and the expected payoff is 0.75. A correlated equilibrium would be for the two players to flip a coin and for Smith to choose *Continue* if it comes up heads and for Jones to choose *Continue* otherwise. Each player's strategy is a best response to the other's, the probability of each choosing *Continue* is 0.5, and the expected payoff for each is 1.0, which is better than the 0.75 achieved without correlated strategies.

Cheap talk (Crawford & Sobel [1982]). Cheap talk refers to costless communication when players can lie without penalty.

In Ranked Coordination, cheap talk instantly allows the players to make the desirable outcome a focal point, though it does not get rid of the other equilibria.

Table 7: The Civic Duty Game

		Jones		
		Ignore (γ)		*Telephone* ($1-\gamma$)
	Ignore (γ)	0, 0	$\rightarrow$	**10,7**
Smith:		$\downarrow$		$\uparrow$
	Telephone ($1-\gamma$)	**7,10**	$\leftarrow$	7, 7

Payoffs to: (Row, Column). Arrows show how a player can increase his payoff.

In the N-player version of the game, the payoff to Smith is 0 if nobody calls, 7 if he himself calls, and 10 if one or more of the other $N-1$ players calls.

If all players use the same probability γ of $Ignore$, the probability that the other $N-1$ players besides Smith all choose $Ignore$ is γ^{N-1}, so the probability that one or more of them chooses $Telephone$ is $1-\gamma^{N-1}$

. Thus, equating Smith's pure-strategy payoffs using the payoff-equating method of equilibrium calculation yields

$$\pi_{Smith}(Telephone) = 7 = \pi_{Smith}(Ignore) = \gamma^{N-1}(0)+(1-\gamma^{N-1})(10) \tag{18}$$

Equation (18) tells us that

$$\gamma^{N-1} = 0.3 \tag{19}$$

and

$$\gamma^* = 0.3^{\frac{1}{N-1}}. \tag{20}$$

$$\gamma^* = 0.3^{\frac{1}{N-1}}. \tag{21}$$

If $N = 2$, Smith chooses $Ignore$ with a probability of 0.30. As N increases, Smith's expected payoff remains equal to 7 whether $N = 2$ or $N = 38$, since his expected payoff equals his payoff from the pure strategy of $Telephone$. The probability of $Ignore$, γ^*, however, increases with N. If $N = 38$, the value of γ^* is about 0.97.

The probability that nobody calls is γ^{*N}. Equation (19) shows that $\gamma^{*N-1} = 0.3$, so $\gamma^{*N} = 0.3\gamma^*$, which is increasing in N because γ^* is increasing in N. If $N = 2$, the probability that neither player phones the police is $\gamma^{*2} = 0.09$. When there are 38 players, the probability rises to γ^{*38}, about 0.29. The more people that watch a crime, the less likely it is to be reported.

Randomizing Is Not Always Mixing:

Assume that the benefit of preventing or catching cheating is 4, the cost of auditing is C, where $C < 4$, the cost to the suspects of obeying the law is 1, and the cost of being caught is the fine $F > 1$.

Table 8: Auditing Game I

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
	Audit (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
IRS:		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	0,0	$\leftarrow$	$4, -1$

Payoffs to: (IRS, Suspects). Arrows show how a player can increase his payoff.

$$Probability(Cheat) = \theta^* = \frac{4-(4-C)}{(4-(4-C))+((4-C)-0)} = \frac{C}{4} \tag{22}$$

and

$$Probability(Audit) = \gamma^* = \frac{-1-0}{(-1-0)+(-F--1)} = \frac{1}{F}. \tag{23}$$

Table 8: Auditing Game I

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
IRS:	*Audit* (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	0,0	$\leftarrow$	$4, -1$

Payoffs to: (IRS, Suspects). Arrows show how a player can increase his payoff.

The payoffs are

$$\pi_{IRS}(Audit) = \pi_{IRS}(Trust) = \theta^*(0) + (1-\theta^*)(4) = 4 - C. \tag{24}$$

and

$$\pi_{Suspect}(Obey) = \pi_{Suspect}(Cheat) = \gamma^*(-F) + (1-\gamma^*)(0) = -1. \tag{25}$$

Auditing Game II: the sequential game, the IRS chooses government policy first, and the suspects react to it. The equilibrium in Auditing Game II is in pure strategies, a general feature of sequential games of perfect information. In equilibrium, the IRS chooses *Audit*, anticipating that the suspect will then choose *Obey*.

The payoffs are $(4 - C)$ for the IRS and -1 for the suspects, the same for both players as in Auditing Game I, although now there is more auditing and less cheating and fine-paying.

In Auditing Game I, the equilibrium strategy was to audit all suspects with probability $1/F$ and none of them otherwise.

That is different from announcing in advance that the IRS will audit a random sample of $1/F$ of the suspects.

For Auditing Game III, suppose the IRS move first, but let its move consist of the choice of the proportion α of tax returns to be audited.

We know that the IRS is willing to deter the suspects from cheating, since it would be willing to choose $\alpha = 1$ and replicate the result in Auditing Game II if it had to.

It chooses α so that

$$\pi_{suspect}(Obey) \geq \pi_{suspect}(Cheat), \tag{26}$$

i.e.,

$$-1 \geq \alpha(-F) + (1 - \alpha)(0). \tag{27}$$

In equilibrium, therefore, the IRS chooses $\alpha = 1/F$ and the suspects respond with *Obey*. The IRS payoff is $(4 - \alpha C)$, which is better than the $(4 - C)$ in the other two games, and the suspect's payoff is -1, exactly the same as before.

The Cournot Game

Players

Firms Apex and Brydox

The Order of Play

Apex and Brydox simultaneously choose quantities q_a and q_b from the set $[0, \infty)$.

Payoffs

Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$, and we will assume it to be linear (for generalization see Chapter 14), and, in fact, will use the following specific function:

$$p(Q) = 120 - q_a - q_b. \tag{28}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_a q_b - q_b^2. \tag{29}$$

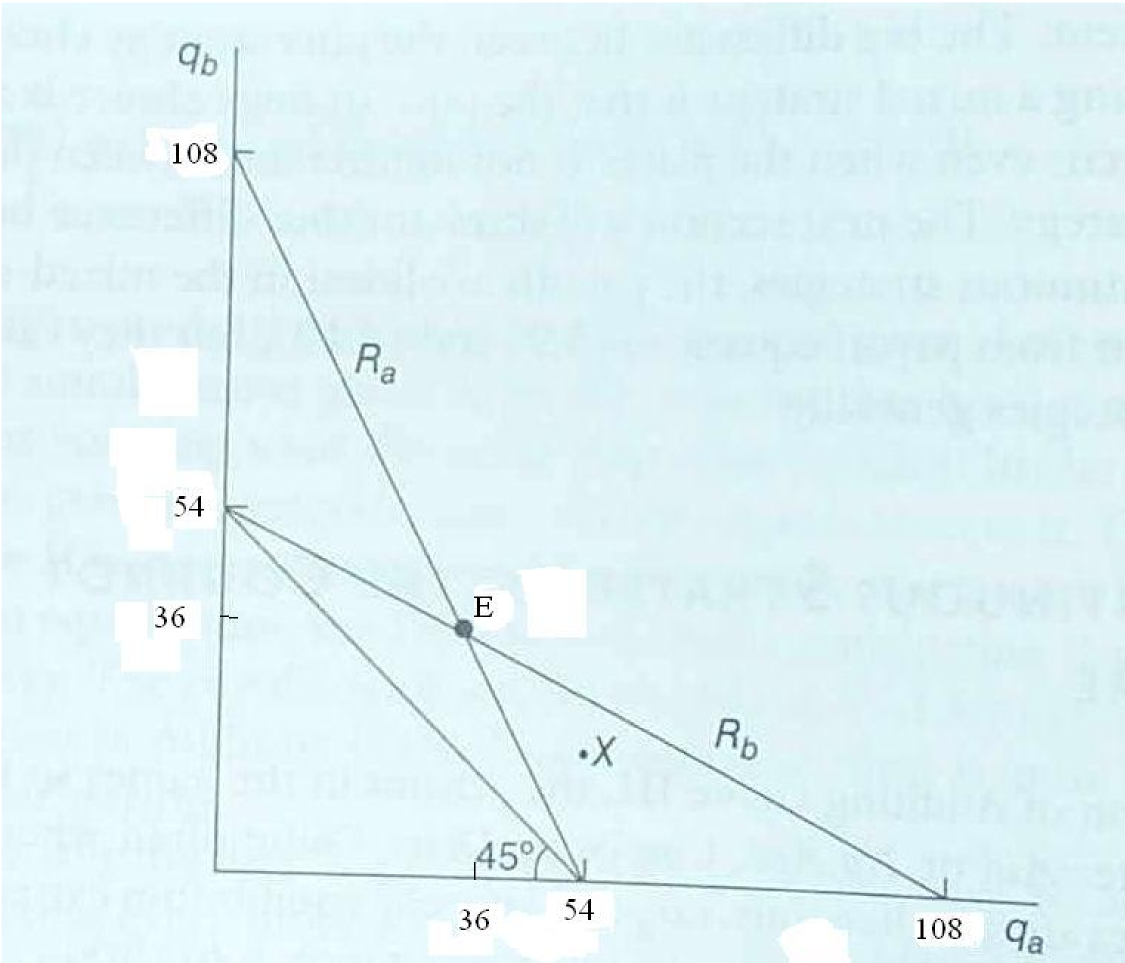

Figure 2: Reaction Curves in the Cournot Game

The monopoly output maximizes $pQ - cQ = (120 - Q - c)Q$ with respect to the total output of Q, resulting in the first-order condition

$$120 - c - 2Q = 0, \tag{30}$$

which implies a total output of $Q = 54$ and a price of 66

To find the "Cournot-Nash" equilibrium, we need to refer to the **best-response functions** or **reaction functions** for the two players. If Brydox produced 0, Apex would produce the monopoly output of 54. If Brydox produced $q_b = 108$ or greater, the market price would fall to 12 and Apex would choose to produce zero. The best response function is found by maximizing Apex's payoff, given in equation (28), with respect to his strategy, q_a. This generates the first-order condition $120 - c - 2q_a - q_b = 0$, or

$$q_a = 60 - \left(\frac{q_b + c}{2}\right) = 54 - \left(\frac{1}{2}\right) q_b. \tag{31}$$

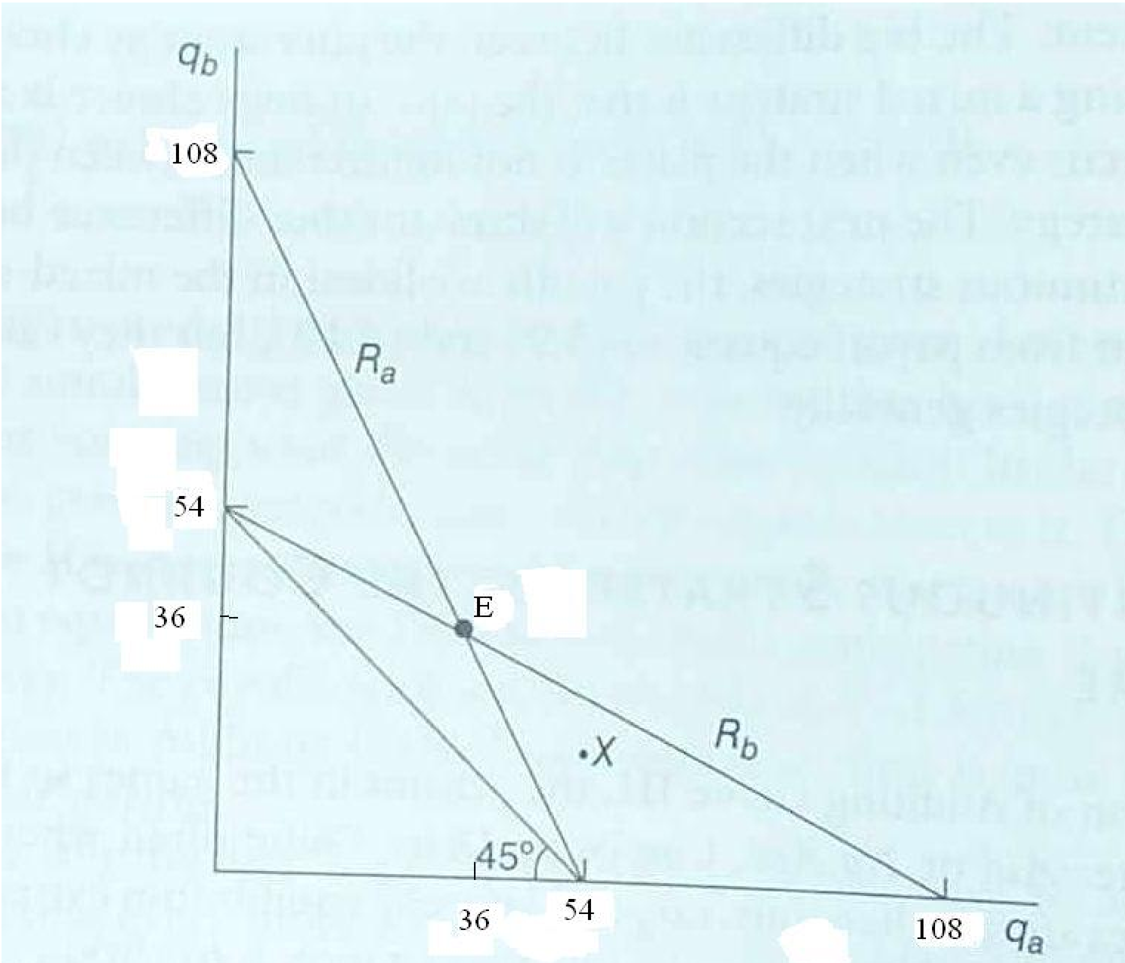

Figure 2: Reaction Curves in the Cournot Game

The reaction functions of the two firms are labelled R_a and R_b in Figure 2. Where they cross, point E, is the **Cournot-Nash equilibrium**, the Nash equilibrium when the strategies consist of quantities.

Algebraically, it is found by solving the two reaction functions for q_a and q_b, which generates the unique equilibrium, $q_a = q_b = 40 - c/3 = 36$. The equilibrium price is then 48 (= 120-36-36).

The Stackelberg Game

Players

Firms Apex and Brydox

The Order of Play

1 Apex chooses quantity q_a from the set $[0, \infty)$.
2 . Brydox chooses quantity q_b from the set $[0, \infty)$.

Payoffs

Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$:

$$p(Q) = 120 - q_a - q_b. \tag{32}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b;$$

$$\pi_{Brydox} = (120 - q_a - q_b)q_b - cq_b = (120 - c)q_b - q_a q_b - q_b^2. \tag{33}$$

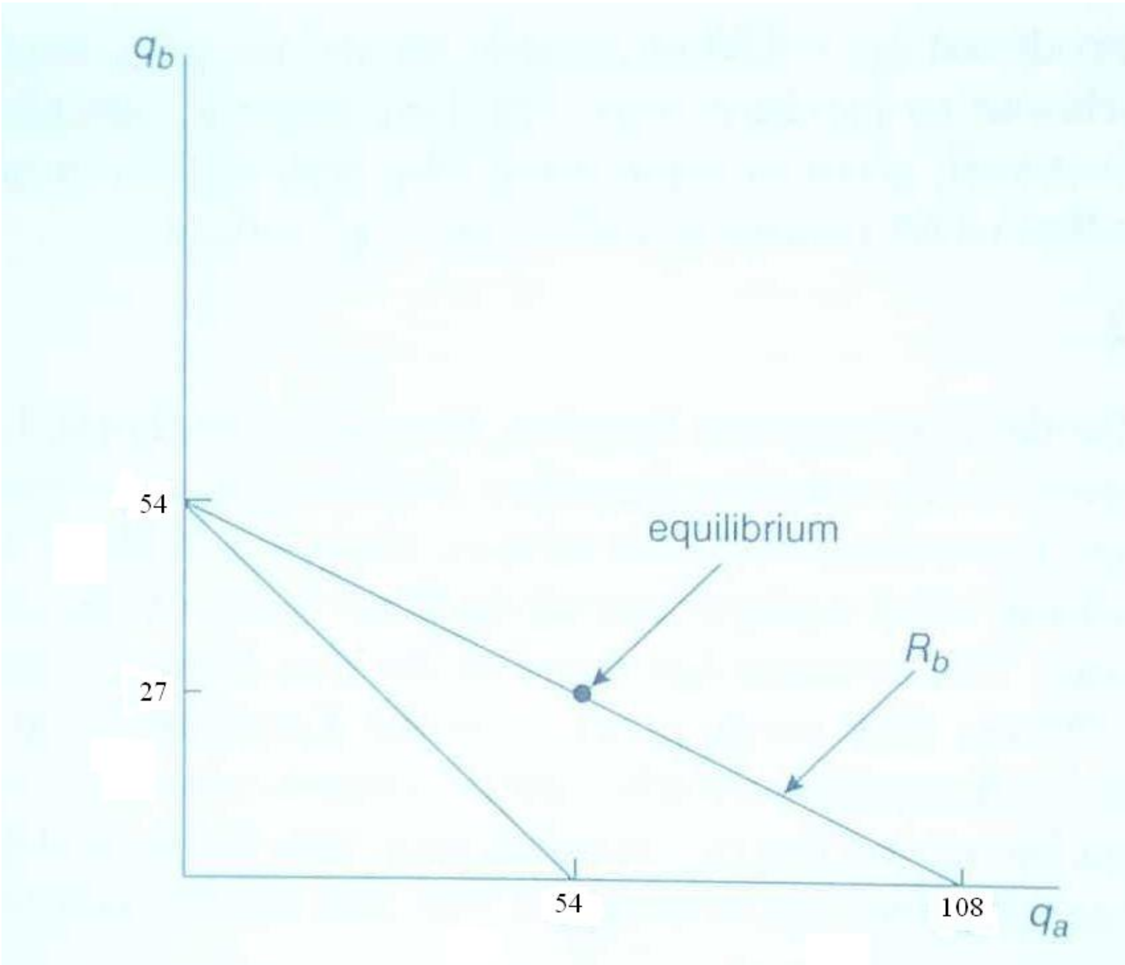

Figure 3: Stackelberg Equilibrium

Since Apex forecasts Brydox's output to be $q_b = 60 - \frac{q_a+c}{2}$, Apex can substitute this into his payoff function in (28) to obtain

$$\pi_a = (120 - c)q_a - q_a^2 - q_a(60 - \frac{q_a + c}{2}). \qquad (34)$$

Maximizing his payoff with respect to q_a yields the first-order condition

$$(120 - c) - 2q_a - 60 + q_a + \frac{c}{2} = 0, \qquad (35)$$

so $q_a = 60 - c/2 = 54$. Once Apex chooses this output, Brydox chooses his output to be $q_b = 27$.

The Bertrand Game

Players

Firms Apex and Brydox

The Order of Play

Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Payoffs

Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q(p) = 120 - p$. The payoff function for Apex (Brydox's would be analogous) is

$$\pi_a = \begin{cases} (120 - p_a)(p_a - c) & \text{if } p_a \leq p_b \\ \frac{(120 - p_a)(p_a - c)}{2} & \text{if } p_a = p_b \\ 0 & \text{if } p_a > p_b \end{cases}$$

The Bertrand Game has a unique Nash equilibrium: $p_a = p_b = c = 12$, with $q_a = q_b = 54$. That this is a weak Nash equilibrium is clear: if either firm deviates to a higher price, it loses all its customers and so fails to increase its profits to above zero. In fact, this is an example of a Nash equilibrium in weakly dominated strategies.

That the equilibrium is unique is less clear. To see why it is, divide the possible strategy profiles into four groups:

$p_a < c$ or $p_b < c$. In either of these cases, the firm with the lowest price will earn negative profits, and could profitably deviate to a price high enough to reduce its demand to zero.

$p_a > p_b > c$ or $p_b > p_a > c$. In either of these cases the firm with the higher price could deviate to a price below its rival and increase its profits from zero to some positive value.

$p_a = p_b > c$. In this case, Apex could deviate to a price ϵ less than Brydox and its profit would rise, because it would go from selling half the market quantity to selling all of it with an infinitesimal decline in profit per unit sale.

$p_a > p_b = c$ or $p_b > p_a = c$. In this case, the firm with the price of c could move from zero profits to positive profits by increasing its price slightly while keeping it below the other firm's price.

The Differentiated Bertrand Game

Let us now move to a different duopoly market, where the demand curves facing Apex and Brydox are

$$q_a = 24 - 2p_a + p_b \tag{36}$$

and

$$q_b = 24 - 2p_b + p_a, \tag{37}$$

and they have constant marginal costs of $c = 3$.

The payoffs are

$$\pi_a = (24 - 2p_a + p_b)(p_a - c) \tag{38}$$

and

$$\pi_b = (24 - 2p_b + p_a)(p_b - c). \tag{39}$$

Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

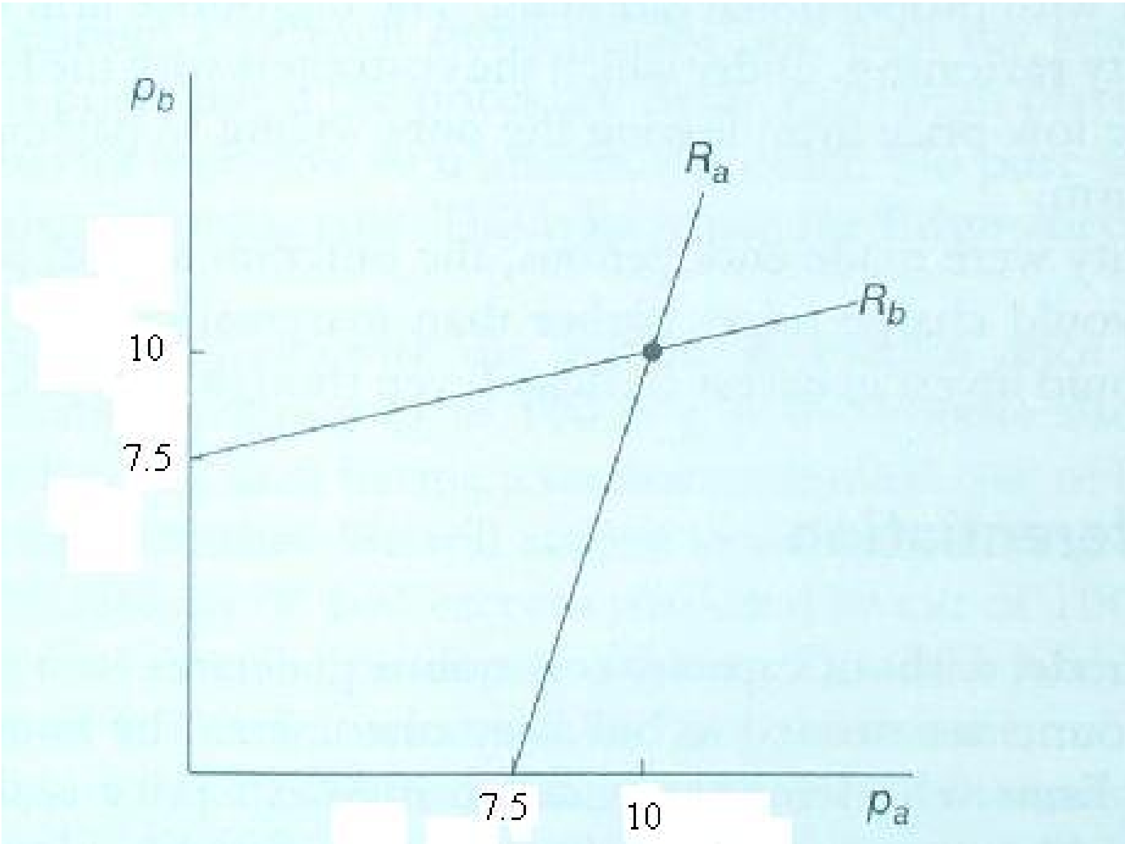

Figure 4: Bertrand Reaction Functions with Differentiated Products

Maximizing Apex's payoff by choice of p_a, we obtain the first- order condition,

$$\frac{d\pi_a}{dp_a} = 24 - 4p_a + p_b + 2c = 0, \tag{40}$$

and the reaction function,

$$p_a = 6 + \left(\frac{1}{2}\right) c + \left(\frac{1}{4}\right) p_b = 7.5 + \left(\frac{1}{4}\right) p_b. \tag{41}$$

Since Brydox has a parallel first-order condition, the equilibrium occurs where $p_a = p_b = 10$. The quantity each firm produces is 14, which is below the 21 each would produce at prices of $p_a = p_b = c = 3$. Figure 4 shows that the reaction functions intersect. Apex's demand curve has the elasticity

$$\left(\frac{\partial q_a}{\partial p_a}\right) \cdot \left(\frac{p_a}{q_a}\right) = -2\left(\frac{p_a}{q_a}\right), \tag{42}$$

which is finite even when $p_a = p_b$, unlike in the undifferentiated-goods Bertrand model.

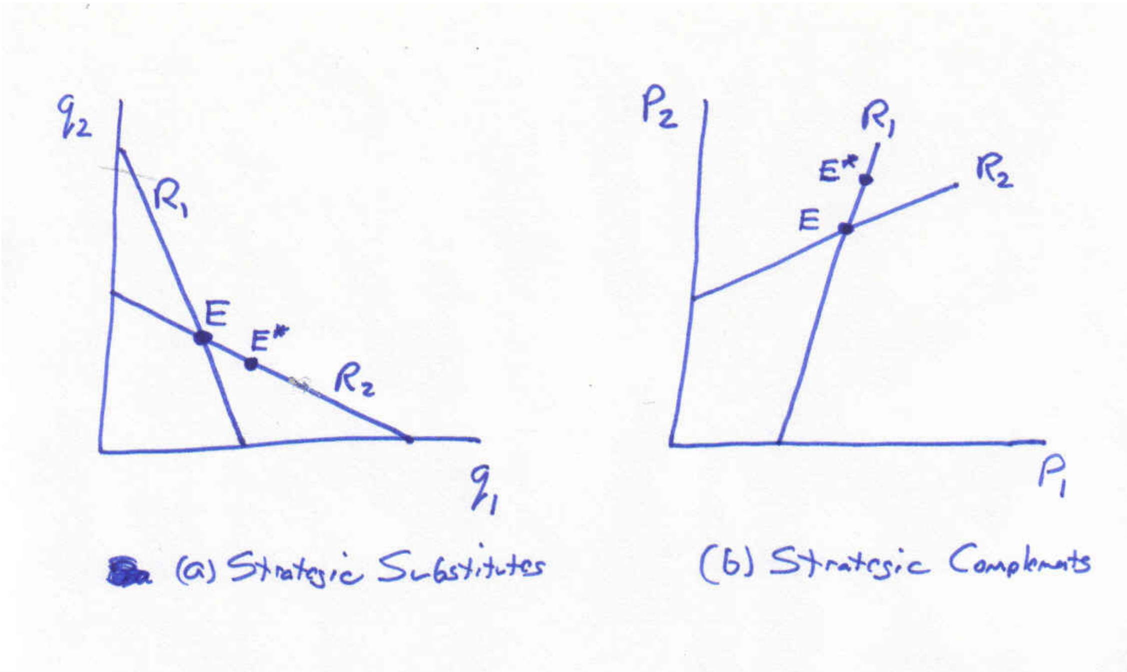

Figure 5: Cournot vs. Differentiated Bertrand Reaction Functions (Strategic Substitutes vs. Strategic Complements)

Esther Gal-Or (1985) notes that if reaction curves slope down (as with strategic substitutes and Cournot) there is a first-mover advantage, whereas if they slope upwards (as with strategic complements and Differentiated Bertrand) there is a second-mover advantage.

Four common reasons why an equilibrium might not exist

(1) An unbounded strategy space

Smith can borrow money and buy as much tin as he wants for \$6/pound. He knows that the price will be \$7/pound tomorrow. What quantity x will he buy, if his borrowing is unlimited?

Choosing x in the strategy set $[0, \infty)$ when his payoff function is $\pi = (1)x$, there is no best strategy.

(2) An open strategy space

Now say that government regulations constrain him to buy less than 1,000 pounds. His strategy is $x \in [0, 1, 000)$, which is bounded by 1000.

(3) A discrete strategy space (or, more generally, a nonconvex strategy space)

Suppose we start with an arbitrary pair of strategies s_1 and s_2 for two players. If the players' strategies are strategic complements, then if player 1 increases his strategy in response to s_2, then player 2 will increase his strategy in response to that. An equilibrium will occur where the players run into diminishing returns or increasing costs.

If the strategies are strategic substitutes, then if player 1 increases his strategy in response to s_2, player 2 will in turn want to reduce his strategy. If the strategy spaces are discrete, player 2 cannot reduce his strategy just a little bit– he has to jump down a discrete level. That could then induce Player 1 to increase his strategy by a discrete amount. This jumping of responses can be never- ending–there is no equilibrium.

This is a problem of "gaps" in the strategy space. Suppose we had a game in which the government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non-convex.

(4) A discontinuous reaction function arising from nonconcave or discontinuous payoff functions

For a Nash equilibrium to exist, we need for the reaction functions of the players to intersect. If the reaction functions are discontinuous, they might not.

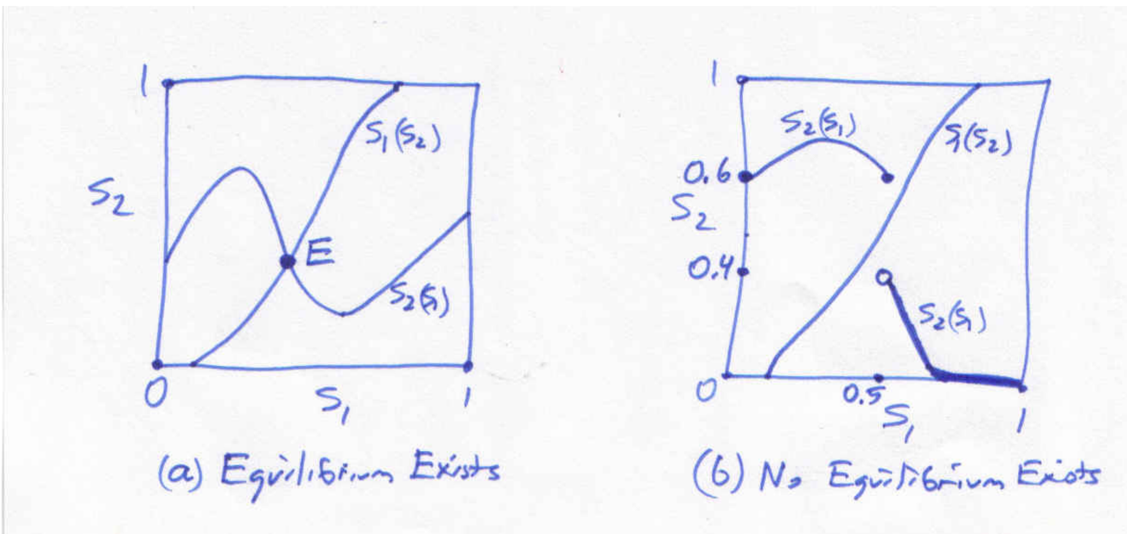

Figure 6: Continuous and Discontinuous Reaction Functions

In Panel (a) a Nash equilibrium exists, at the point, E, where the two reaction functions intersect.

In Panel (b), however, no Nash equilibrium exists. The problem is that Firm 2's reaction function $s_2(s_1)$ is discontinuous at the point $s_1 = 0.5$. It jumps down from $s_2(0.5) = 0.6$ to $s_2(0.50001) = 0.4$. As a result, the reaction curves never intersect, and no equilibrium exists.

If the two players can use mixed strategies, then an equilibrium will exist even for the game in Panel (b).

A first reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is discontinuous in either his own or the other players' strategies. This is what happens in Chapter 14's Hotelling Pricing Game, where if Player 1's price drops enough (or Player 2's price rises high enough), all of Player 2's customers suddenly rush to Player 1.

A second reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is not concave.