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Notes on Common Knowledge (inspired by Prof. Peck's "Knowledge and Common Knowledge" www.econ.ohio-state.edu/jpeck/gametheory/gameL5.pdf)

Suppose the possible states of the world are {1,2,3,4,5,6,7}.

Player Alpha has information partition

{1,2}, {3,4},{5,6,7}.

That means that if the state is 1, he can detect that the state is either 1 or 2, but not which of those.

In a Bayesian game, Player Alpha will have prior beliefs as to the probability of each state, priors which are updated to posteriors as he sees other players' actions. He might put probability .7 on state 1 and .3 on state 2, for example. Here, though, we will just think about when Alpha knows something with certainty– when he can put probability 1 on a state or a set of states.

We will call E an "event" that happens in certain states. An event might be that the state is in the set {1,2,3,7}.The event might be that the quality of a good being auctioned is high, for example. You can think of the four states 1,2,3, and 7 as being special cases of this event.

When does player Alpha "know" that this event E happened? Answer: He knows it in states 1 or 2. Then his information set is {1,2}, both of which result in E happening. If the state is 3, on the other hand, his information set is {3,4}, so he can't rule out that the true state is 4 and E hasn't happened.

We say that the player's knowledge function is $K_\alpha(E) = K_\alpha(1, 2, 3, 7) = \{1, 2\}$. This function maps one set of states to another. It shows the set of states such that Alpha knows E occurred.

We could have a second player, Beta, with information partition

{1,3}, {2,4},{5,6,7}

In states 1 and 3, Beta knows for sure that E happened so $K_\beta(E) = \{1, 3\}$.

Notice that in state 1, both players know that E occurred. Event 1 is in both $K_\alpha(E)$ and $K_\beta(E)$.

We can now define Alpha knowing that Beta knows about event E. We can do that because $K_\beta(E)$ is itself an event! It is a set of states, the same kind of object that an event is.

So what is $K_\alpha(K_\beta(E))$?

If the true state is one of the states in $K_\beta(E)$, when does Alpha know that for sure?

Well, those states are {1,3}. Alpha can tell if the state is 1, but not if it is 3. So $K_\alpha(1,3) = K\alpha(K_\beta(E))$ = the null set. Even if Alpha knows E, he never can be sure that Beta knows it too.

For an event E to be common knowledge between players Alpha and Beta in a given state, every member of the sequence $K_\alpha(K_\beta(K_\alpha(K_\beta(....(E)))))$ has to contain that state.

Thus, event E is not common knowledge between players Alpha and Beta in state 1. Nor is it common knowledge in any state.

Let's introduce a third player, Gamma. Let Gamma have the partition

{1}, {2}, {3},{4}, {5,6,7}

Gamma knows that E occurred in states 1, 2, and 3. $K_\gamma(E) = K_\gamma(1,2,3,7) = \{1,2,3\}$.

When does Alpha know that Gamma knows that E occurred? That is shown by $K_\alpha(K_\gamma(E)) = K_\alpha(1,2,3) = \{1,2\}$.

Now let's iterate. When does Gamma know that Alpha knows that Gamma knows that E occurred? That is shown by $K_\gamma(K_\alpha(K_\gamma(E))) = K_\gamma(1,2) = \{1,2\}$. Clearly, further iterations would make no difference.

We could do the iteration in a different order, and it wouldn't matter as far as finding that E is common knowledge in a given state. When does Gamma know that Alpha knows that E occurred? That is shown by $K_\gamma(K_\alpha(E)) = K_\gamma(1,2) = \{1,2\}$. If we iterate further, we stay with {1,2}.

The Finest Common Coarsening Definition

Aumann used this to formally define event E being common knowledge at state w if E contains the information set of the finest common coarsening of the players' information partitions that contains state w.

A common coarsening is an information partition that weakly coarsens both players' information partitions. The ultimate coarse partition is one common coarsening, so here we look for the finest common coarsening– the one that loses the least information.

The state w is in some information set in that coarsening. We ask whether Event E contains ALL the elements of that information set, so when w occurs, and the players both know some state in that information set has occurred, they both know that E has occurred. (That is, there can't be any states in that information set that WOULDNT be part of E.)

Player Alpha's partition: {1,2}, {3,4},{5,6,7}

Player Beta's partition: {1,3}, {2,4},{5,6,7}

E = {1,2,3,7}

The finest common coarsening between Alpha and Beta is

{1,2,3,4},{5,6,7}

But E doesn't contain {1,2,3,4}. So E is not common knowledge in state 1.

How about common knowledge between Alpha and Gamma?

Player Gamma's partition: {1}, {2}, {3}, {4}, {5,6,7}

The finest common coarsening between Alpha and Gamma is

{1,2} {3,4},{5,6,7}

Thus, E is common knowledge between them in states 1 and 2.