Notes on Discounting and Dynamic Programming

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The present value of a one-time payment of X, received at the end of one year, is, if the discount rate is r per year,

$$V = \frac{X}{1+r} \tag{1}$$

As a result, the value of X received at the end of N years is $\frac{X}{(1+r)^N}$. The value of X received immediately is, of course, X.

The value of a flow of X per year, received at the end of each year, is

$$V(consol) = \frac{X}{r}. \tag{2}$$

The value if the payment is made at the beginning of each year is just the consol value plus the X received immediately, $X + \frac{X}{r}$.

The value r is the discount rate. The discount factor is

$$\delta = \frac{1}{1+r} \tag{3}$$

I remember being irritated by the redundancy of this terminology when I was a student. The reason for having to remember two terms is that sometimes r makes the equations neater and easier to write, sometimes δ.

Here are two ways to think about the consol formula.

(1) It says that if the interest rate is 5 percent, the value of a payment of 1,000 dollars per year is 20,000 dollars. That makes sense, because if you started with a lump sum of 20,000 dollars, you could put it in the bank and earn interest of (.05)(20,000) = 1,000 dollars per year. The flow of 1,000 and the stock of 20,000 have equal value.

(2) Use dynamic programming. Let V be the present value of the stream of X per year. It is equal to the amount X, discounted because it is received at the end of the year, plus at the end of the year you'll still have the value V. Thus,

$$V = \frac{X}{1+r} + \frac{V}{1+r} \tag{4}$$

Thus, $(1+r)V = X + V$, and $rV = X$ and $V = \frac{X}{r}$.

This last formulation is useful in thinking of the value of a growing flow. Suppose the flow starts at X and grows at γ per year. That means the value of the asset rises at γ per year too.

Thus,

$$V = \frac{X}{1+r} + (1+\gamma)\left(\frac{V}{1+r}\right) \tag{5}$$

Then $\frac{1+r}{1+\gamma}V = X + V$, and $\frac{1+r-(1+\gamma)}{1+\gamma}V = X$ and $(r-\gamma)V = X$ and

$$V = \frac{X}{r-\gamma} \tag{6}$$

If $r < \gamma$, however, the equation does not apply. The value of the asset is infinite, so it is a false step to say that it rises at γ per year.

It's interesting to apply this growth idea to the Klein-Leffler reputation model. If the market is not growing, and marginal cost of high quality is c and $r = .10$, the price is $p = (1+r)c$ ten percent above marginal cost. If the market is growing at 8 percent per year, however, the price is only $p = (1+r-\gamma)c$, just 2 percent above marginal cost.