**G751: Old Test Questions Relevant to Test I**

January 24, 2011

Here are two ideas to imbibe fully:

1. Writing down the actions in the game tree is a matter of translating a story into a series of decisions and branches, and isn't that hard. Just ask yourself what happens first, and then what happens next depending on the first move. Writing down the payoffs is a lot harder, but for drawing the actions, think of the story you are telling.

2. The idea of testing Nash equilibrium is central to everything. A lot of brainwork is sidestepped by asking: "If player A chooses strategy X, how will player B respond?" If he doesn't respond with strategy Y, then (X,Y) is not an equilibrium.

1. (3 points) Prove that if one player has a strictly dominant strategy it must be part of any Nash equilibrium, even if there are multiple equilibria.

ANSWER. Suppose not, and player $i$ uses strategy $s'$ instead in Nash equilibrium A. If $s*$ is is strictly dominant for player $i$, it has a higher payoff when used in response to any profile of strategies of the other players, including the profiles in profile A. But the definition of Nash equilibrium is that player $i$'s payoff using his equilibrium strategy $s'$ in response to the other players' strategies is greater than if he switched to any other strategy, including $s*$. This yields a contradiction; $s'$ cannot be a Nash equilibrium strategy if the player has a strictly dominant strategy that is different.

Note that it is not enough just to show that if $s_i^*$ is a dominant strategy it is a Nash equilibrium strategy. You need to show that it is part of *any* Nash equilibrium— that no other Nash strategies exist for $i$. You need to show that there would be deviation from any other $s_i$.

2.

| | | **Column** | | |
|---|---|---|---|---|
| | | *Left* | *Middle* | *Right* |
| | *Up* | 10,10 | $0, 0$ | $-1, 9$ |
| **Row:** | *Sideways* | $9, 1$ | 8, 8 | $-1, -1$ |
| | *Down* | 3,1 | 8,$-1$ | 4,1 |

*Payoffs to: (Row, Column).*

(a) (3 points) Find any pure-strategy Nash equilibria of the game above or say that there are none.

ANSWER. $(Up, Left)$, $(Sideways, Middle)$, and $(Down, Right)$. Most of you realized that I was asking for any equilibria that might exist, but since I think some of you just answered with one, I gave 2 points credit for finding either one or two.

(b) (3 points) Show that there is no mixed-strategy equilibrium in which Column mixes between $Left$ and *Right* with probability $\gamma : 0 < \gamma < 1$ and Row mixes between *Up* and *Down* with probability $\theta : 0 < \theta < 1$.

ANSWER. If there were, then Column would have to equate his payoffs from his two pure strategies. He can't do that, though, because if Column is choosing only *Up* or *Down* then *Right* is weakly dominated. This means he can't be indifferent between *Right* and $Left$. Trying to equate his payoffs shows this:

$$\pi_c(Left) = 10\theta + 1(1-\theta) = \pi_c(Right) = 9\theta + 1(1-\theta)$$

That equation has to be false, because $10\theta > 9\theta$. So in the proposed equilibrium, Column would deviate to the pure strategy $Left$.

3. Two employees, Smith and Jones, are deciding on High or Low effort. The task is either Hard or Easy, with equal probability. The payoffs are

**Hard Task**

| | | **Jones** | |
|---|---|---|---|
| | | *High* | *Low* |
| **Smith:** | *High* | 0,0 | -2,2 |
| | *Low* | 2,-2 | 1,1 |

*Payoffs to: (Smith, Jones).*

**Easy Task**

| | | **Jones** | |
|---|---|---|---|
| | | *High* | *Low* |
| **Smith:** | *High* | 4,10 | 2,2 |
| | *Low* | 3,2 | 1,1 |

*Payoffs to: (Smith, Jones).*

(a) (3 points) If both players know whether the task is hard or easy, and moves are simultaneous, what is the equilibrium? (you don't have to explain why in detail– you can just write down the equilibrium strategies)

ANSWER. Smith: $(Low|Hard, High|Easy)$. Jones: $(Low|Hard, High|Easy)$. Note that each strategy has two parts, since there are two information sets for each player's move.

(b) (3 points) If Smith knows whether the task is hard or easy, and moves are simultaneous, what is the equilibrium? (you don't have to explain why in detail– you can just write down the equilibrium strategies)

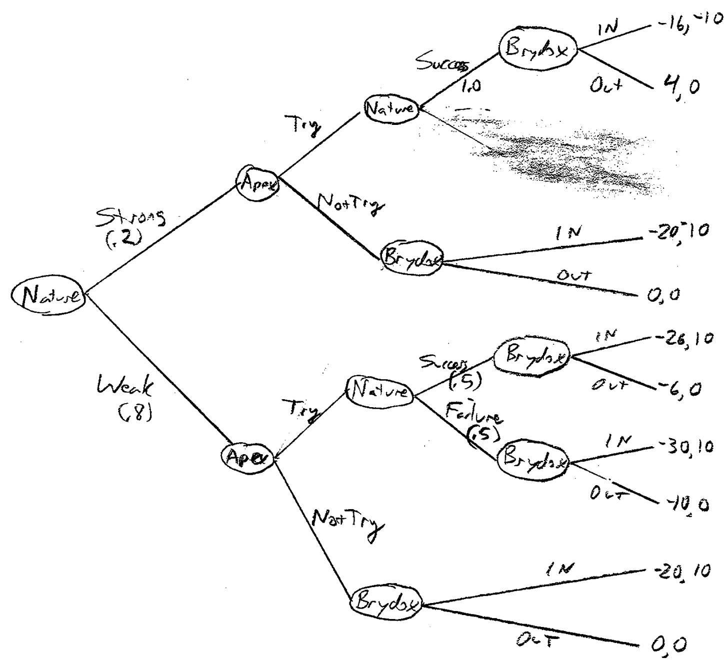
ANSWER. Smith $(Low|Hard, High|Easy)$. Jones: $High$.

4. (long version– see the last part) Apex is currently the only company making widgets, but Brydox is thinking about entering the industry. Initially, Brydox thinks that Apex is a Weak company with probability .8 and a Strong company with probability .2. With no new product or entry, Apex's payoff is 0. Apex must decide whether to $Try$ or $NotTry$ to introduce a new product, the superwidget, which would add 4 to its payoff. If Apex is strong, trying costs 0 and it always succeeds. If Apex is weak, trying costs 10, and it succeeds with probability .5. Brydox must decide whether to be $In$ or $Out$ of the industry after observing whether Apex starts selling superwidgets

($Success$) or not ($Failure$). If Brydox chooses $In$, that reduces Apex's payoff by 20. Brydox receives a payoff of 0 if it chooses $Out$. If Brydox chooses $In$ and Apex is strong, Brydox's payoff is -10, but if Apex is weak, Brydox's payoff is +10.

(a) (3 points) Draw the extensive form of this game if both companies observe every move.

ANSWER. A couple of special notes:
1. Even if Apex chooses $NotTry$, Brydox chooses $In$ or $Out$.
2. The weak Apex's expected gain from the superwidget product is -8 = .5(4) -10, but in the extensive form it shows up as -10 or -6, depending on Success or Failure.




(b) (3 points) If Brydox observes Apex's type, what is the Nash equilibrium?

ANSWER. Apex: $Try|Strong, NotTry|Weak$.
Brydox:$Enter|(Weak, NotTry), Enter|(Weak, Try, Success), Enter|(Weak, Try, Failure)$,

$Out|(Strong, NotTry), Out|(Strong, Try, Success)$.

(c) (3 points) For the rest of the question, assume Brydox does not observe Apex's type, but does see if Apex sells the superwidget (Success) or not (Failure).

What is the strong Apex's strategy in any Nash equilibrium? Explain why there is no equilibrium in which the strong Apex chooses $NotTry$ and Brydox chooses $In|Success$? (or $In|Try$).

ANSWER. Apex's Nash strategy is $Try|Strong$. If we disregard Brydox's response to $Success$, $Try$ yields 4 in extra payoff to Apex. Thus, the only reason for Apex not to choose $Try$ would be if $Try$ or $Success$ made Brydox enter but $NotTry$ or $Failure$ did not.

That is a conceivable reason. Consider the strategy profile,

$(NotTry|Strong, Try|Weak, In|Try, Out|NotTry)$.

The strong Apex would not deviate, because his equilibrium payoff is 0 and his deviation payoff is $-20 + 4$. This shows that $Try|Strong$ is not a dominant or weakly dominant strategy. The profile is not a Nash equilibrium, however, because the weak Apex would deviate to $NotTry$ in order to prevent Brydox from choosing $In$. In any equilibrium in which the strong Apex would choose $NotTry$, so would the weak Apex, since the weak Apex gets an expected 8 less in payoff from trying to invent the superwidget.

(d) (3 points) Why is it not an equilibrium for Apex to use a pooling strategy of always choosing $Try$?

ANSWER. We must look at Brydox's response. Bayes's Rule gives us Brydox's posterior probability.

$$Prob(Weak|Success) = \frac{Prob(Success|Weak)Prob(Weak)}{Prob(Success)}$$

$$= \frac{.5(.8)}{.5(.8)+(1)(.2)} = \frac{.40}{.60} = 2/3. \quad (1)$$

Thus, in the conjectured pooling equilibrium, Brydox would think Apex was weak with probability greater than .5 and Brydox would choose $In|Success$ as well as $In|Failure$ ($Failure$ would be a sure sign of a weak Apex). So the weak Apex gains no advantage from choosing $Try$ and just has a payoff reduced by 8 (=.5(4)-10). Apex would deviate to $NotTry|Weak$.

(e) (3 points)Show why in equilibrium Apex will not use a separating strategy of $(Try|Strong, NotTry|Weak)$.

ANSWER. If Apex does, then Brydox will respond with $In|Failure, Out|Success$, because he knows that $Success$ is a perfect indicator of $Strong$. The weak Apex wants to deter entry if possible, because that hurts Apex by 20. Apex's equilibrium payoff is $-20$. If Apex deviates to $Try|Weak$, given Brydox's strategy, Apex's payoff is $-10 + .5(4) + .5(-20) = -18$. Thus, Apex gains by deviating.

4. Firms Apex and Brydox are thinking of entering the China market, which is either strong or weak, with equal probability. Both firms are so small that their actions do not affect each other's profits. Apex knows whether the market is strong, but Brydox does not.

Apex has high costs with probability .6, and low costs otherwise. Brydox has high costs. The payoff from not entering the China market is always 0. Payoffs from entering are shown in the table below.

Clarification: Apex knows its cost before it enters, but Brydox does not. Brydox observes Apex's entry decision before it decides whether to enter.

| | Strong Market | Weak Market |
|---|---|---|
| Low-Cost Apex | 10 | 5 |
| High-Cost Apex | 4 | -4 |
| Brydox | 4 | -4 |

(a) What does Apex do?

Answer: Apex will $Enter|Low$ and $Enter|High, Strong$ but $StayOut|High, Weak$.

(b) In equilibrium, what probability does Brydox assign to the market being strong if Apex enters it? What is probability does he estimate for the market being strong if Apex does not enter?

Answer:

$$Pr(Strong|Enter) = \frac{Pr(Enter|Strong)Pr(Strong)}{Pr(Enter)} = (1)(.5)(1)(.5) + (.4)(.5) = \frac{5}{7}.$$

$$Pr(Strong|StayOut) = \frac{Pr(StayOut|Strong)Pr(Strong)}{Pr(StayOut)} = (0)(.5)(0)(.5) + (.6)(.5) = 0.$$

(c) What is Brydox's optimal strategy?

Answer: $Enter|Enter, StayOut|StayOut$. If Apex stays out, that is a sure sign the market is weak. If Apex enters, Brydox's expected payoff is $(5/7)4 - (2/7)4 > 0$.