Bertrand Competition Under Uncertainty

June 5, 2001

Maarten Janssen and Eric Rasmusen

Abstract

Consider a Bertrand model in which each firm may be inactive with a known probability, so the number of active firms is uncertain. This activity level can be endogenized in several ways– whether to incur a fixed cost of activity, for example, or what level of output to choose. Our model has a mixed-strategy equilibrium, in which industry profits are positive and decline with the number of firms, the same features which make the Cournot model attractive. Unlike in a Cournot model with similar incomplete information, Bertrand profits always increase in the probability other firms are inactive. Profits decline more sharply than in the Cournot model, and the pattern is similar to that found empirically by Bresnahan and Reiss (1991).

Janssen: Department of Economics, H7-22, Erasmus University, 3000 DR Rotterdam, The Netherlands. Fax: 31-10-4081949. Janssen@few.eur.nl.

Rasmusen: Visiting Professor, CIRJE, Dept. of Economics, University of Tokyo (2001), Professor of Business Economics and Public Policy and Sanjay Subhedar Faculty Fellow, Indiana University, Kelley School of Business, BU 456, 1309 East 10th Street, Bloomington, Indiana, 47405-1701. (812) 855-9219. Fax: 812-855-3354. Erasmuse@indiana.edu. Http: //Php.indiana.edu/∼erasmuse. Copies of this paper can be found at
Php.indiana.edu/∼erasmuse/papers/bertrand.pdf.

We thank David Schmidt, two anonymous referees, and participants in seminars at the Indiana University Dept. of Economics, Erasmus University, and CIRANO in Montreal for their comments. Rasmusen thanks Harvard Law School's

Olin Center and the University of Tokyo's Center for International Research on the Japanese Economy for their hospitality.

1. Introduction

Consider a carpenter who is asked by a homeowner to submit a tender for renovating a house. He considers it very likely that if the homeowner has asked for tenders from other carpenters then the lowest price will win the job. He also knows there is a chance that the homeowner has not found any other carpenter free to do the work this month and will give the job to him even if his tender is rather high. What price will the carpenter offer the homeowner?

The price will certainly be above marginal cost. The carpenter knows that with some probability he is a monopolist who can charge the monopoly price, even though with some probability he does face competition. We will model the situation and show that there exists an equilibrium in mixed strategies and that expected industry profits are positive for any number of firms. Moreover, not only do expected profits rise with seller concentration, but the model does reasonably well in explaining the empirical results of Bresnahan and Reiss (1991) on *how* industry profits increase.

The model allows for a number of interpretations. First, uncertainty about the existence of competitors may arise from uncertainty on the demand side, with respect to consumer information. It may be unclear whether consumers regard rival commodities as perfect substitutes, consumer search costs may be uncertain from the firm's perspective, or consumers may vary in their sophistication. Examples of this range from grocery shopping to buying clothing from mail order companies depending on what catalogs have been received to buying beers that to some consumers all taste alike but to other consumers do not. Any of these things might result in a given consumer not knowing the prices every firm is charging.

Second, uncertainty about the vigor of competition may arise from uncertainty about the supply side. It may be unclear whether rivals have hit their capacity constraints (in which case they cannot compete for additional consumers), whether rivals have entered yet, whether rivals have grossly overpriced by mistake or ignorance, or whether rivals have temporarily high costs. It may be unclear whether other competitors have also discovered a new market, or in black markets it may be difficult to know the number of firms operating in that market (cf. Janssen and Van Reeven, 1998). Examples of these range from wholesale distribution of candy bars (where in periods of peak demand first one and then another manufacturer may hit capacity) to airline ticket pricing to sales of unusual but not rare antiques or used books. Any of these situations can be modelled as uncertainty over the number of active rivals.

The demand and supply interpretations of the previous two paragraphs allow us to treat the probability a firm is actively competing for a given customer as an exogenous probability, independent of the number of potentially active firms. This probability is beyond a firm's control and can be regarded in Bayesian fashion as a decision by Nature. We also will show how for a given number of firms the probability of being active can arise from a firm's decision taken in response to uncertainty about previous decisions by other firms. The two-stage models in Section 3 will endogenize the probability that a given number of firms are active in competing for a given consumer. We will show how in two different settings entry is a random decision in equilibrium, in an auction-like setting, as in the carpenter example, and when firms set output or capacity An important feature of both settings will be that a firm's decision whether to compete for a given consumer is not observed before other firms decide their prices. A disadvantage of endogenizing the probability of inactivity, however, is that the probability will change with the number of potentially active firms and the size of market demand. Thus, when we come to compare our results with the empirical findings of Bresnahan and Reiss, it is the exogenous interpretation of firm activity that we will apply.

The paper is related to several different literatures. A variety of models, of which Salop and Stiglitz (1977) and Varian (1980) are early examples, have shown that competitive markets can have price dispersion even in equilibrium. Different firms charge different prices for an identical good because of heterogeneous consumer search, some consumers observing more prices than others. The closest of these to the present model is Burdett and Judd (1983), in which some consumers might observe one price, some two prices, some three, and so forth. The number of searches is endogenous, and in equilibrium a given consumer observes only one or two prices. Our model differs in a number of respects. First, while one way to look at our model is that consumers differ with respect to the number of prices they observe, our model allows for the other interpretations mentioned earlier, interpretations inappropriate for search models. Indeed, since since the number of prices observed by a consumer is exogenous in our model, endogenous search is not a good interpretation. Second, the firms in our model are strategic, not competitive. This allows us to study the impact on pricing behavior of the number of firms, a variable not relevant in Burdett and Judd (1983). Finally and most important, we treat uncertainty differently. In our model, a firm believes there is a fixed probability that any of its competitors is active, whereas in Burdett and Judd it is the probability that a consumer observes a certain number of prices that is exogenous. The difference lies in what happens as the number of sellers increases. In our model, a seller knows that the probability that at least one other firm is actively competing with it has become closer to one. This drives prices closer to marginal cost, and in the limit we obtain the standard Bertrand outcome. Burdett and Judd still have price dispersion and positive industry profits as the number of firms becomes infinite, because each firm may still be visited by a non-negligible number of consumers who do not search for other prices. In their model, the reason a consumer pays a high price is not that low prices are not available, but that he does not know where to find them.

Also related is Elberfeld and Wolfstetter (1999). They consider a two-stage model in which firms first decide whether to enter and then compete in prices. The outcome of the first stage is known before the firms set their prices in the second stage. Thus, the outcome in the second stage is standard: a firm charges the monopoly price if it is the only firm in the market, otherwise prices are equal to marginal cost. Their main result is that the probability that no firm enters the market increases with the number of potential competitors. Their analysis is closely related to our two-stage game, the important difference being that in our model the entry decision itself is not oberved before firms compete in prices.

Spulber (1995) analyzes a model of Bertrand competition when firms' cost functions are private information. He shows that the model has a unique pure strategy equilibrium in which firms set prices above marginal cost and have positive expected profits. In contrast, the firms in our model do not know how many competitors they have, but assume that any competitor that does exist has the same cost structure. Even though the type of uncertainty varies between Spulber's model and ours, the properties of the market equilibrium are similar: firms set prices above marginal cost and receive positive expected profits.

Another approach to positive profits under Bertrand competition can be found in the epsilon-equilibrium Bertrand model of Baye and Morgan (1999). They show that if firms only choose prices to reach within epsilon of their maximal profit, then a mixed-strategy equilibrium exists, in which profits are positive and large compared to the value of epsilon. Thus, if, due to satisficing or managerial slack, firms do not maximize profits completely, the Bertrand model generates more realistic outcomes. The model in our paper also introduces noise which generates a mixed strategy equilibrium, but our noise is the possibility that a customer does not have alternative sellers from whom to buy.

Finally, our model is also of interest for students of auctions. The similarities between Bertrand price competition and first-price sealed-bid auctions is well-known, as, e.g., Baye and Morgan (1997a, b) explain. Our paper can be regarded as answering the question what is the optimal bid if the number of participantsin a sealed-bid auction is unknown, as is often the case in procurement bids, adding to the literature of which McAfee and Macmillan (1987) is an example.

Section 2 of the paper lays out the basic model and solves for the mixed strategy equilibrium. Section 3 shows how the entry decision can be endogenized in three different types of two-stage games. These three models also show some of the alternative ways our basic model may be interpretated. Section 4 compares the outcome in the model with that of a Cournot model, and compares the expected industry profits in our model for different numbers of potentially active firms with the empirical findings of Bresnahan and Reiss (1991). Section 5 concludes.

2. The Model

Let there be N firms that might produce a homogeneous good. Before deciding price, a firm does not know how many other firms are active in market. The probability a given firm is active is α, where $0 \leq \alpha \leq 1$. If $\alpha = 1$, the market is described by the Bertrand model of price competition, and the equilibrium price equals marginal cost. If $\alpha = 0$ so our one firm is assured of being a monopolist, it will charge the monopoly price. For simplicity, we will assume that there is one consumer, who buys at most one unit, and his maximum willingness to pay for the good is v. In case of tied prices, the consumer picks a firm randomly. Marginal cost is normalized to 0.

First, let us establish that there is no symmetric Nash equilibrium with any firm putting positive probability on choosing any particular price on the continuum. Suppose Firm 1 (without loss of generality) charges price p' with positive probability θ, rather than mixing over a continuous range of prices and putting infinitesimal probability on each. Putting positive probability on $p' = 0$ is not profit maximizing, because if the firm charged the monopoly price of v instead on those occasions it would have an expected payoff of $(1-\alpha)^{N-1}v$, so let us focus on $p' > 0$.

If $p' > 0$, and two firms are putting positive probability θ on p', then with positive probability θ^2 they will both charge p' and will each have a contribution proportional to $(\theta^2/2)(p' - 0)$ towards their expected profits. Firm 1 could increase its expected profit, however, by deviating to putting zero weight on p' and positive weight on $p' - \epsilon$, for sufficiently small ϵ. This would replace the expected profit of $(\theta^2/2)(p' - 0)$ with the larger expected profit of $(\theta^2)(p' - \epsilon)$. Thus, it cannot be that both firms put positive probability on any p' in equilibrium.

Let us then consider a situation in which only Firm 1 chooses p' with positive probability mass. There then exists a neighborhood around p' where prices are not chosen with a strictly positive probability mass. We distinguish two possibilities. First, there exists a neighborhood $[p', x)$ with $x > p'$ such that the probability that any firm charges a price in the neighborhood equals 0. This cannot be an equilibrium, as Firm 1 can increase p' without reducing its chance of winning the customer. Second, there exists a neighborhood (p', x) with $p' < x$ such that the probability that at least one other firm charges a price in the whole neighborhood is strictly positive. This can also not be part of an equilibrium, however, as one of the other firms has incentive to shift probability mass from prices just above p' to prices just below it. Hence, there cannot be any equilibrium in which only one firm puts strictly positive probability on any single price. In conjunction with the previous paragraph, this means that there is no equilibrium in which any firm chooses any price with positive probability mass.

Second, the support over which a firm mixes in equilibrium is connected. Consider hat would happen if Firm 1 randomized over an unconnected support, which would include at least two intervals, denoted by $[\beta_1, \gamma_1]$ and $[\beta_2, \gamma_2]$. It is easy to see that an optimal (mixed strategy) response of Firm 2 does not include prices in the interval $[\gamma_1, \beta_2]$. In this case, there exists, however, an $\epsilon > 0$ such that Firm 1 will not be indifferent between setting a price of $\gamma_1 - \epsilon$ and setting a price of $\beta_2 + \epsilon$. Thus, a necessary condition for Firm 1 randomizing over $[\beta_1, \gamma_1]$ and $[\beta_2, \gamma_2]$ is violated.

Let us therefore construct an equilibrium in mixed strategies with the strategies having a continuous and compact support. Let $F(p_i)$ be the probability that firm i charges a price smaller than p_i. The expected payoff to firm i of charging a price p_i when all other firms choose a mixed strategy according to $F(p_i)$ is

$$\pi_i(p_i, F_i(p)) = \Sigma_{k=0}^{N-1} \left(\begin{array}{c} N-1 \\ k \end{array} \right) (1-\alpha)^k [\alpha(1-F(p_i))]^{N-k-1} p_i. \qquad (1)$$

This expression can be explained in the following way. The probability that exactly $N-k-1$ out of the other $N-1$ firms besides Firm i are active is equal to

$$\left(\begin{array}{c} N-1 \\ k \end{array} \right) (1-\alpha)^k \alpha^{N-k-1}. \qquad (2)$$

The expected payoff to firm i when exactly $N-k-1$ firms are active and when it charges a price of p_i is equal to p_i times the probability that each of these $N-k-1$ firms charges a price that is larger than p_i, which is $(1-F(p_i))^{N-k-1} p_i$. Multiplying these two terms and summing up over all k gives the expression above.

Expression (1) is, of course, nothing but an application of the Binomial Theorem, and a standard result says that

$$\Sigma_{k=0}^{N-1} \left(\begin{array}{c} N-1 \\ k \end{array} \right) a^k b^{N-k-1} = (a+b)^{N-1}. \qquad (3)$$

Applying equation (3) to the profit equation (1), we obtain

$$\pi(p_i, F(p_i)) = [1 - \alpha F(p_i)]^{N-1} p_i. \tag{4}$$

In equilibrium, firm i must be indifferent between all pure strategies that are in the support of the mixed strategy distribution. Hence, it must be that on some interval of prices the derivative of expression (4) with respect to p_i equals zero. Thus, a necessary condition for any equilibrium in continuous mixed strategies is

$$[1 - \alpha F(p_i)]^{N-1} - (N-1)[1 - \alpha F(p_i)]^{N-2} \alpha f(p_i) p_i = 0, \tag{5}$$

or

$$1 - \alpha F(p_i) - \alpha(N-1) f(p_i) p_i = 0, \tag{6}$$

where f is the density function associated with cumulative distribution function F.

It is a matter of straightforward calculations to show that the solution to differential equation (6) is[1]

$$F(p_i) = \frac{1 - (1-\alpha)\left(\sqrt[N-1]{\frac{v}{p_i}}\right)}{\alpha}, \tag{7}$$

for $(1-\alpha)^{N-1} v \leq p_i \leq v$.

Result (7) implies that there is a unique symmetric equilibrium with compact support, and we have shown earlier that an equilibrium in pure strategies does not exist. These results are stated in Proposition 1.

Proposition 1. The unique symmetric equilibrium of the Bertrand model with an uncertain number of competitors is in mixed strategies and the distribution function of a player's strategy is

$$F(p_i) = \begin{cases} 0 & for \quad p_i \leq (1-\alpha)^{N-1} v \\ \frac{1-(1-\alpha)\left(\sqrt[N-1]{\frac{v}{p_i}}\right)}{\alpha} & for \quad (1-\alpha)^{N-1} v \leq p_i \leq v \\ 1 & for \quad p_i \geq v \end{cases} \tag{8}$$

[1]Note that when there are m identical consumers, the profit in equation (4) is simply multiplied by m and and the equilibrium price distribution remains the same.

Price dispersion is a well-known outcome in real-world markets. Warner and Barsky (1995), for example, sampled prices at various stores in Michigan for a number of identical single products and found considerable dispersion.[2] Thus, the mixed strategy we found is quite consistent with reality.

Figure 1 shows the cumulative density for different values of N using equation (8) with $\alpha = .2$ and $v = 100$ (prices are at intervals of 1, connected). As N increases, each firm chooses relatively low prices with higher probability. As N becomes large, the cumulative density function approaches 1 for all values of p that are strictly positive. Of course, the equilibrium price under perfect competition is also equal to 0. The perfectly competitive outcome can be regarded as the limit case of the present model when the number of firms becomes very large.

[2]See Tables I and III of Warner and Barsky (1995). They found, for example, that a GI Joe had prices of 3.88, 2.93, 2.69, 2.96, 2.84, 2.96, and 2.69, while and a Huffy Vortex unassembled boy's bicycle had prices of 73.47, 99.99, 112.63, 119.99, 119.99, and 18.70.

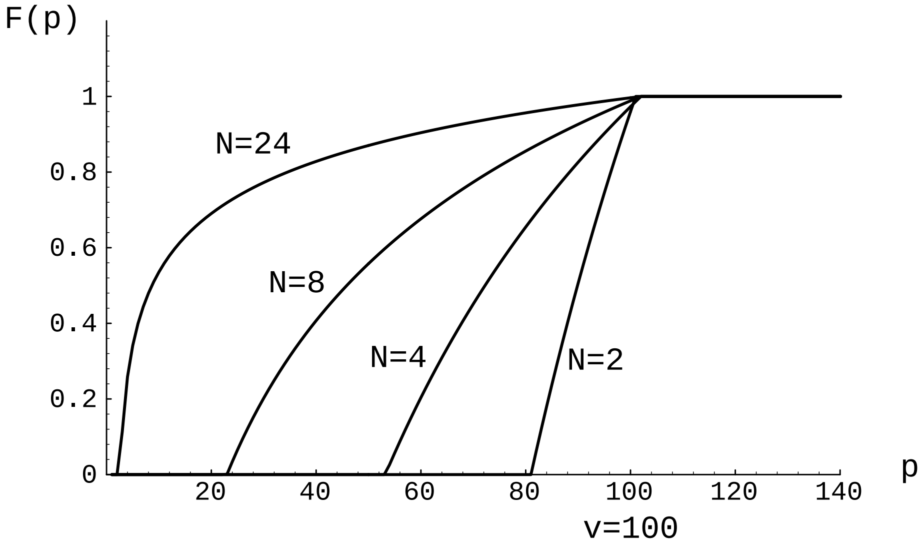
F(p)
1
0.8
0.6
0.4
0.2
0
20
40
60
80
100
120
140
p
v=100
N=24
N=8
N=4
N=2

Figure 1: Equilibrium Price Distributions as Industry Concentration Rises ($\alpha = .2, v = 100$)

The intuition is straightforward. As the number of potential competitors increases, the probability of at least one other firm actively producing the same product rises. With greater probability of competition, the firm reduces its prices. In the limit, a firm is extremely likely to have at least one active competitor. Standard Bertrand competition comes into effect and each firm charges a price equal to marginal cost.

Expected profit for one firm can be found using the pure strategy profit from charging $p = v$. Since the firm is active with probability α, that profit is

$$\pi_i = \alpha(1-\alpha)^{N-1}v. \tag{9}$$

Note that individual profit is declining in N and its sum, industry profit, is equal to[3]

$$N\alpha(1-\alpha)^{N-1}v. \tag{10}$$

Let Π_b denote expected industry profit under Bertrand competition of this kind given that at least one firm is active. The profit in equation (10) can be written as

$$\sum_{i=1}^{N}\pi_i = N\alpha(1-\alpha)^{N-1}v = (1-\alpha)^N(0) + [1-(1-\alpha)^N]\Pi_b, \tag{11}$$

yielding

$$\Pi_b = \frac{N\alpha(1-\alpha)^{N-1}v}{1-(1-\alpha)^N}. \tag{12}$$

To see how industry profit changes with N, note that after some manipulation,

$$\frac{d\Pi_b}{dN} = \left[\frac{(1-(1-\alpha)^N) + Nlog(1-\alpha)}{(1-(1-\alpha)^N)^2}\right]\left[\alpha(1-\alpha)^{N-1}v\right] \tag{13}$$

[3]Note that although the profits of the different firms are not independent, the expected profits are, so this summation is legitimate.

Derivative (13) is well-defined, even though only integer values of N have an economic interpretation. Its sign is the same as the sign of

$$1 - (1-\alpha)^N + N log(1-\alpha). \tag{14}$$

For $N = 1$, expression (14) becomes $\alpha + log(1-\alpha)$, which is negative because $\alpha < 1$. For larger N, expression (14) becomes even more negative, because its derivative with respect to N is $-(1-\alpha)^N log(1-\alpha) + log(1-\alpha) = log(1-\alpha)[1-(1-\alpha)^N] < 0$. Thus,

$$\frac{d\Pi_b}{dN} < 0, \tag{15}$$

and profits fall as the number of firms increases.

In the appendix it is shown that we can say more, namely

$$\frac{d^2\Pi_b}{dN^2} > 0. \tag{16}$$

This means that profits are convexly decreasing in the number of firms in the industry, so the shape shown in the numerical examples graphed in Figure 2 in Section 4 would be found for any example.

More General Demand Structures

So far the assumption has been that the quantity demanded is one unit for all prices smaller than v and zero otherwise. Here, we will consider a more general demand function, which we denote by $D(p)$. For simplicity we will restrict ourselves to the case $N = 2$. We will impose one condition on this demand function, namely that $pD(p)$ is increasing in p for $p < p_m$, where p_m is the monopoly price. Most demand function that are commonly employed satisfy this condition. It is satisfied, for example, if $pD(p)$ is concave in p.

Assumption 4.1. The function $pD(p)$ is increasing and differentiable on $[0, p_m)$.

For general demand functions, the expected profit of firm 1 when firm 2 chooses a price according to the cumulative mixed strategy distribution $F_2(p)$ is given by

$$\pi_1(p_1, F_2(p_1)) = (1-\alpha)p_1 D(p_1) + \alpha(1 - F_2(p_1))p_1 D(p_1). \tag{17}$$

A necessary condition for an equilibrium in mixed strategies with continuous support to exist is that on a certain domain of prices

$$[(1-\alpha) + \alpha(1 - F_2(p_1))][D(p_1) + p_1 D'(p_1)] - \alpha f_2(p_1)p_1 D(p_1) = 0. \tag{18}$$

One can show that the solution to differential equation (18) is given by

$$F_2(p) = \begin{cases} 0 & if \quad p \leq \underline{p} \\ \frac{1}{\alpha}\left[1 - \frac{(1-\alpha)p_m D(p_m)}{pD(p)}\right] & if \quad \underline{p} < p \leq p_m \\ 1 & if \quad p > p_m \end{cases} \tag{19}$$

A similar solution holds for Firm 1. It is clear that equation (19) is similar to equation (12) and the results of the basic model generalize to more general demand functions. Note that from the solution for $F_i(p)$ it is clear why we have to impose a condition on demand: a necessary and sufficient condition for $F_i(p)$ to be increasing in p is that $pD(p)$ is increasing in p for all values of p smaller than p_m. In the present case it is impossible to provide an explicit solution for the domain of prices over which a firm randomizes. It is clear that the upper bound is given by p_m. This is because even if the other firm does not exist, it is not optimal to set a higher price. The lower bound of the domain, denoted by $\underline{p}$, is defined implicitly by the condition $\underline{p}D(\underline{p}) = (1-\alpha)p_m D(p_m)$ As $pD(p)$ is increasing in p for $p < p_m$, $\underline{p}$ is uniquely defined in this way.

Industry profits may be calculated as in Section 2 and equal

$$\Pi_b = \frac{N\alpha(1-\alpha)^{N-1}p_m D(p_m)}{1-(1-\alpha)^N}. \tag{20}$$

3. Endogenizing Entry

One way to think about the probability of a firm being active, α, is in a Bayesian way. A given firm and a given customer either make contact or not, in a way determined by Nature and independent of the number, N, of potential firms. Contact depends on such things as whether a customer notices the firm's existence in the course of his daily activities (and vice versa), whether the firm's equipment and labor are working that day, and whether he knows what potential firms are available. In this case, when there are N firms, the probability that they all know about the customer and compete for his business is α^N. There are two types of firms: those that are in active competition for the customer and those that are not. Those not competing are unimportant; the equilibrium strategy of those that are indeed actively competing is what we have analyzed in Section 2.

Another way to think about the probability of active competition is that it stems from a previous decision of the firm (or consumer) which is observable to other firms setting prices. In the present Section we use this approach, and consider two "front-end" games that endogenize whether a firm is active. We will concentrate on games with just two potential firms since our aim is to illustrate how the probability α in the previous model might arise, but we will briefly consider the general case of N firms. Section (i) is a standard model of entry that requires a fixed cost. A firm does not know whether the other firm has entered when it must choose its price. Section (ii) is a model of output or capacity. Two firms choose how much to produce before they set their prices. When setting prices they do not know the quantity chosen by the other firm. In both models, whether a given firm is active is random in the symmetric equilibrium.

(i) A Model with a Fixed Entry Cost

Consider the following two-stage extension of the basic model. Suppose there are two potential firms. In the first stage, both firms decide whether or not they enter the industry. There is a fixed entry cost denoted by F with F less than v, the consumer's reservation price. At the beginning of the second stage the firms have not observed whether the other firm has entered or not. In the second stage, the firms set a price if they entered in the first stage.

One example is a sealed bid auction with an entry fee, a common situation in government procurement: it is costly to prepare a bid, and when sending in their bids firms do not know how many competing bidders there are. As the outcome of the first stage is not observed, we can analyze the game as a simultaneous move game.

There are three equilibria. In the two asymmetric equilibria, one firm enters the market and sets a price equal to v, while the other firm stays out. In the third, symmetric, equilibrium, both firms are indifferent between entering the market or staying out and they enter the market with a certain probability γ. Given this probability of entering, each firm chooses a price according to the mixed strategy distribution calculated in Section 2, with γ replacing α. The expected payoff in the second stage is $(1-\gamma)v$. The only way in which the firms can be indifferent between staying out and entering the market is if $(1-\gamma)v$ equals the fixed entry cost F. Thus, γ equals $1-F/v$, and expected profits are zero.

This model can easily be extended to N potential firms. Then, as with two firms, there will be a symmetric mixed-strategy equilibrium and a number of asymmetric equilibria. The only novelty is that mixed-strategy asymmetric equilibria can exist if there are more than 2 potential entrants; with 3 firms, for example, in equilibrium one firm might enter with probability 1 and the other two firms would mix. For general values of N, the endogenous parameter γ, does, however, ordinarily depend on market size. If we increase N while maintaining the assumption that there is only one consumer, it is easy to see that the γ will decrease. A more reasonable comparison, however, is to increase the market size at the same time as N, in which case what happens depends on how fast N increases with market size. If we denote m_N as the size of demand when there are N firms, then γ does not depend on market size only if $m_{N+1} = \frac{1}{1-\gamma} m_N$, a special case. This should be kept in mind while reading our discussion of empirical profit-concentration relationships below, because there as N increases we keep constant the probability a given firm is active, rather than either increasing or decreasing it.

(ii) A Model of Output Choice

Kreps and Scheinkman (1983) describe a model in which firms compete first in outputs and then in prices, something we can do here also. Consider a market with two consumers, each buying up to one unit each and with a reservation price of v. There are two firms, each of whom can decide in the first stage whether to produce 1 or 2 units of a homogeneous output. In the second stage, firms compete in prices not knowing the decision of the other firm in the first stage. The cost of producing 1 unit is normalized to 0 and the cost of producing 2 units is K, where $0 < K < v$.

There does not exist an equilibrium in pure strategies. It is not an equilibrium for Firm 1 to produce 1 unit and charge a high price, because Firm 2 would respond with 2 units and a slightly lower price, driving Firm 1's profits to zero. There cannot be one in which Firm 1 chooses to produce 1 unit and charge a low price, because Firm 2 would respond with 1 unit and a price of v, making it profitable for Firm 1 to deviate and produce 2 units and charge slightly less than v. Nor can it be an equilibrium for Firm 1 to produce 2 units and charge a price greater than K, since Firm 2 would produce 2 units also and charge a lower price, making Firm 1 unprofitable. Finally, it cannot be an equilibrium for Firm 1 to produce 2 units and charge a price of K or less, since Firm 2's best response would be to produce 1 unit and undercut Firm 1's price, in which case Firm 1 would do better to produce 1 unit and charge a price of v.

Consider an equilibrium in mixed strategies: Each firm chooses with probability μ to produce 2 units, and otherwise produces 1 unit. If a firm happens to choose 1 unit, it will charge a price equal to v. The profit of any firm producing 2 units is then

$$\pi_i(p_i^2, F_{-i}^1(p_i), F_{-i}^2(p_i^2)) = \begin{cases} 2(1-\mu)p_i^2 + 2\mu(1 - F_{-i}^2(p_i))p_i^2 - K & if \quad p_i^2 < v \\ \\ \frac{4}{3}(1-\mu)v + 2\mu(1 - F_{-i}^2(p_i^2))v - K & if \quad p_i^2 = v \end{cases} \tag{21}$$

where p_i^2 is the price charged by firm i if it has produced 2 units and $F_{-i}^j(p_i^2)$ is the distribution function with which Firm $-i$ chooses prices if it has produced j units.

It is easy to see that it cannot be optimal to set $p_i^2 = v$. A necessary condition for an equilibrium in mixed strategies is

$$(1-\mu) + \mu(1 - F_{-i}^2(p_i^2)) - \mu f_{-i}^2(p_i^2)p_i = 0, \tag{22}$$

or

$$1 - \mu F^2_{-i}(p^2_i) - \mu f^2_{-i}(p^2_i)p_i = 0, \tag{23}$$

This equation has the same form as equation (5) for $N = 2$. Hence, the solution is given by

$$F^2_{-i}(p^2_i) = \frac{1 - (1-\mu)\frac{v}{p^2_i}}{\mu} \tag{24}$$

for $(1-\mu)v < p_i < v$.

The profit to either firm of producing 1 unit is

$$\pi_i(p^1_i, F^1_{-i}(p), F^2_{-i}(p)) = [1 - \mu + \mu(1 - F^2_{-i}(p^1_i)]p^1_i, \tag{25}$$

where p^1_i is the price charged by firm i if it has produced 1 unit.

It is easy to see that given equation (24) the right-hand-side of equation (25) reduces to $(1-\mu)v$, which is independent of p^1_i. Hence, given (24), it is optimal to set $p^1_i = v$.

The profits of producing 1 and 2 units are $(1-\mu)v$ and $2(1-\mu)v - K$. In the mixed strategy equilibrium, these two expressions have to be equal to each other, which implies that $\mu = (v-K)/v$.

So, the entry decision– here, with how many units to enter the Bertrand competition phase– is random, and the uncertainty about the existence of a fierce competitor has been derived endogeneously.[4]

4. Comparing Bertrand and Cournot

Cournot (1838) proposed a model in which N firms simultaneously choose quantities and let the market determine the price. Bertrand (1883) pointed out that entirely different conclusions result if the firms choose prices simultaneously instead. Even though the assumptions of price competition seem more realistic, the quantity model yields more realistic outcomes, because profits are positive, but fall gradually as the number of firms increases. We have shown that this objectionable feature of the Bertrand model disappears when uncertainty about the presence of competitors is taken into

[4]Note that there exists a continuum of equilibria,all in mixed strategies of the form derived in Section 2, indexed by the price a firm sets when producing only 1 unit.

account. We will now see what happens to the Cournot model when uncertainty is added, and compare the Bertrand and Cournot models under uncertainty. To make the comparison clearer, we will use linear demand,

$$p\left(\sum_{i=1}^{N} q_i\right) = a - b\sum_{i=1}^{N} q_i. \tag{26}$$

Let us define $q(p)$ as the demand facing a monopolist at a price of p, so

$$q(p) = \frac{a}{b} - \frac{p}{b}. \tag{27}$$

The monopoly price then equals $a/2$ and the quantity demanded is $a/2b$ at that price.

We will compute the expected profits from Cournot and Bertrand for different levels of N to obtain some idea of the effects of concentration in each.

Bertrand equilibrium

Applying equation (20) to the case of linear demand, the industry profits in the Bertrand model with uncertainty are

$$\Pi_{bertrand} = \frac{N\alpha(1-\alpha)^{N-1}p_m D(p_m)}{1-(1-\alpha)^N} = \frac{N\alpha(1-\alpha)^{N-1}\frac{a^2}{4b}}{1-(1-\alpha)^N}. \tag{28}$$

Adding uncertainty eliminates the discontinuous behavior of the original Bertrand model. Uncertainty makes a big difference, and the comparative statics become consistent and intuitive. Profits are always positive, but they fall whenever the number of firms or the probability of more firms being active increases. Figure 2 shows this for a particular numerical example with $a = 100$, $b = 1$, N from 0 to 7, and α from 0 to 1.[5]

Figure 2: Bertrand Profits For Different Probabilities of Activity, α, and Numbers of Firms, N
(from Equation (28 (conditional on at least one firm being active)

Cournot Equilibrium

The Cournot equilibrium is calculated the same way as in the standard linear Cournot model except that we must account for the possibility that the number of active firms might be anywhere from 1 to N. Let q^* be the

[5] In every case, expected industry profits are conditional upon at least one firm being active. When $\alpha = 0$, this is to be interpreted as the probability zero (but nonetheless possible) event that one firm is active and the expected number of other firms is zero.

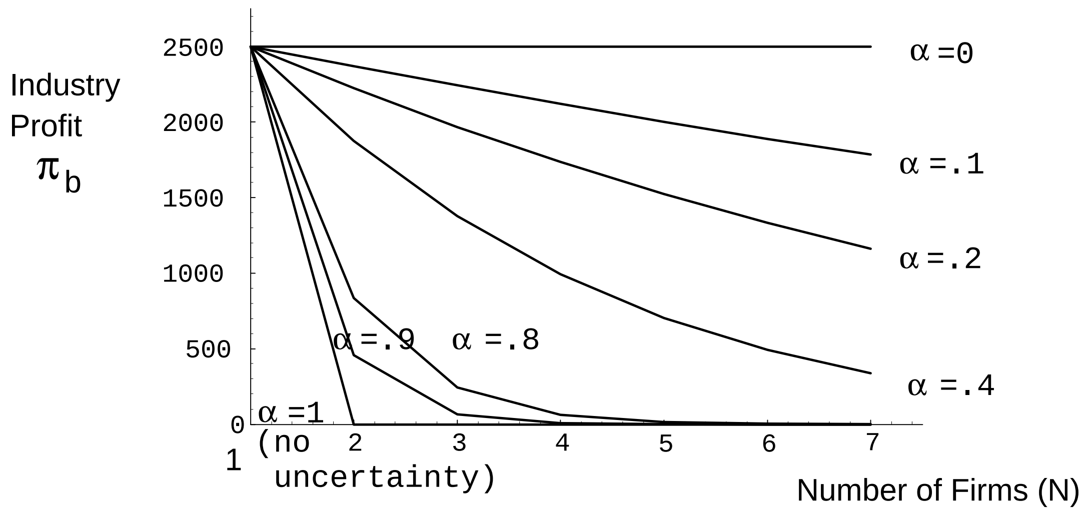

Figure 1:

Cournot output we are trying to determine. Firm i's expected profit if all other firms choose q^* is the sum of his profits for each possible number of active firms times the probability exactly that many firms are active,

$$\pi_i(q_i, q^*) \quad = \sum_{j=0}^{N-1} \begin{pmatrix} N-1 \\ j \end{pmatrix} (1-\alpha)^j \alpha^{N-1-j} [p(q_i + (N-1-j)q^*)]q_i. \tag{29}$$

Substituting in the linear demand function, differentiating with respect to q_i, setting $q^* = q_i$, and solving for q^* yields the equilibrium expected Cournot industry profit conditional upon one firm being active,[6]

$$\Pi_{Cournot} = \frac{a^2 \alpha N}{b[1-(1-\alpha)^N][2+\alpha(N-1)]^2}. \tag{30}$$

[6]Equation (30) is conditional upon Nq^* being not so large as to drive the price to zero, which might rationally happen, since a firm would be willing to accept a price of zero occasionally as the result of all N firms coincidentally being active and producing a large amount.

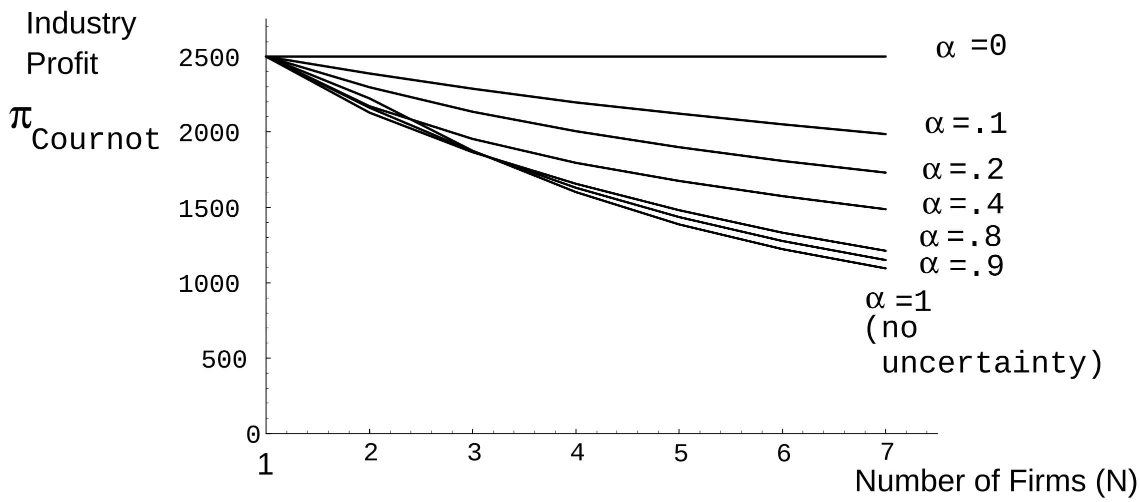

Figure 2:

Figure 3 depicts Cournot profits for different degrees of activity and concentration, using the same numerical parameters as the Bertrand profits in Figure 2.

Figure 3: Cournot Profits For Different Probabilities of Activity, α, and Numbers of Firms, N
(from Equation (30), conditional on at least one firm being active)

Figure 3 shows that depending on the number of firms in the industry, the presence of uncertainty to the Cournot model can either increase or reduce industry profits, but it does not radically change the equilibrium. Under Cournot competition, a firm expands its output when it expects fewer rivals to be helping push down the price and the next effect on expected industry output is unclear. Conflicting forces are at work in Cournot equilibrium, and the net result is sensitive to particular values of the parameters underlying the model.[7]

Profits and Concentration in Bertrand, Cournot, and the Bresnahan-Reiss Study

Let us now compare Bertrand and Cournot. Using profit equations (28) and (30), the ratio of industry profits under Bertrand and Cournot competition is

$$\frac{\Pi_{Bertrand}}{\Pi_{Cournot}} = (1-\alpha)^{N-1}\left[1+\frac{\alpha}{2}(N-1)\right]^2, \tag{31}$$

which is decreasing in both N and α.

Table 1 and Figure 4 show the outcomes of our numerical example for different degrees of concentration under Cournot and Bertrand behavior with certainty and with $\alpha = .8$. (Figure 4 also illustrates the Bresnahan-Reiss empirical result, of which more will be said later.) As we have seen, uncertainty changes the Bertrand model in a crucial way, because profits do become positive and monotonic in the number of firms. The sharp fall in profits moving from monopoly to duopoly under certainty in the Bertrand model is perhaps not so unreasonable as it looks. It is extreme, but it is a limiting result as α goes to one, as Figures 2 and 4 illustrate.

Number of Firms N	1	2	3	4	5	6	7
Bertrand, $\alpha = 1$	2500	0	0	0	0	0	0
Bertrand, $\alpha = .8$ (eq. (28))	2500	833	242	64	16	4	1
Cournot, $\alpha = 1$	2500	2222	1875	1600	1388	1224	1093
Cournot, $\alpha = .8$ (eq. (30))	2500	2125	1867	1650	1480	1333	1211

[7] The result is reminiscient of the peculiarities of profit per firm in the Cournot model, which can (but do not always) give rise to an incentive for a Cournot firm to split in two to increase its profits, as noted by Salant, Switzer and Reynolds (1983).

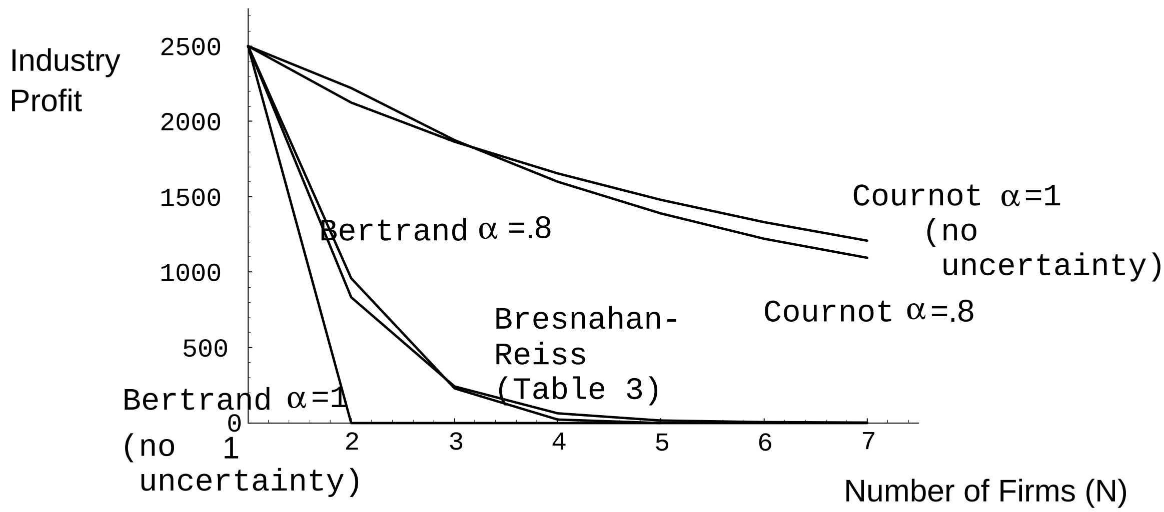

Figure 3:

Table 1: Industry Profits for Different Concentration Levels[8]

Figure 4: Bertrand and Cournot Profits

Consider the shape of the profit-concentration paths. All the curves in Figures 2 through 4 have convex shapes, if only weakly in the limiting cases, but the curvatures, and therefore the empirical implications, are different. As Figure 4 and Table 1, in particular, show, profits decline much more rapidly in Bertrand than in Cournot. For the parameters chosen, industry profits fall from the monopoly level of 2500 to duopoly profits of 833, triopoly profits of 242, and low levels thereafter. Cournot profits show a much more uniform decline as concentration falls.

Comparison of Figures 2 and 3 shows that for smaller values of the activity probability α the Bertrand profit path becomes flatter and the Cournot

[8]Numerical calculations and figures use *Mathematica*. Values are rounded.

path, perhaps more curved, but even at extreme values Cournot does not generate such sharp differences from the addition of one firm to the market.

For most modelling purposes, these models are building blocks, and such subtle differences in the profit-concentration path are unimportant. They are interesting, however, if one wishes to consider Bertrand and Cournot as serious oligopoly models in their own right. Empirically, then, how do profits react to the number of firms? Do they decline to zero with duopoly and then stay constant, as in the original Bertrand model? Do they decline smoothly, as either version of the Cournot model would suggest? Or do they decline rapidly, as the Bertrand model with uncertainty would suggest?

Measuring the relationship between profits and concentration is an old exercise now in some disrepute.[9] The difficulty is that the usual unit of observation has been the industry. This is natural enough, since one needs a measurement of concentration for each observation. Comparing accounting profits across industries is fraught with danger, however, since accounting profits differ from economic profits in ways that depend on the industry chosen and which are very likely to be correlated with technology, and hence with concentration. Moreover, it is not clear that the concentration-profits path is even the same across industries.

Bresnahan and Reiss (1991) took a clever empirical approach to the same problem. They took the unit of observation to be the market for a particular product in a particular small town, rather than for many products over the entire United States, and they looked at market size rather than directly at profits. They collected data on the size of a town and the number of dentists there, for example. If a town is very small—say, 500 people—- it will have no dentist, since a dentist incurs a fixed cost and could not make any profit there even as a monopoly. If it grows to 800 people, it will have one dentist, since the profits are enough for monopoly, but entry by a second dentist would drive them negative. If the town grows to 1,600 people, however, it may still have only one dentist— if entry by the second dentist would not just split the industry profits, but reduce them.

[9]See pp. 349-366 of Carlton and Perloff 's 1994 industrial organization text for a good discussion of the problems of the profits-concentration literature.

Number of Firms N	1	2	3	4	5
Doctors	0.88	1.75	1.93	1.93	1.83
Dentists	0.71	1.27	1.39	1.36	1.28
Druggists	0.53	1.06	1.68	1.92	1.88
Plumbers	1.43	1.51	1.51	1.55	1.49
Tire Dealers	0.49	0.89	1.14	1.19	1.22

Table 2: Bresnahan-Reiss Entry Thresholds s_i: Original (1,000's of inhabitants)[10]

Bresnahan and Reiss used this approach to estimate the thresholds s_i for entry in small markets for a number of industries. Table 2 shows these thresholds in thousands of inhabitants per firm. Table 3 rescales the same numbers to be very roughly comparable with the numerical example used earlier in this paper.[11] The rescaling is somewhat arbitrary, since the theory of Bresnahan and Reiss is that some quasi-rents remain to cover fixed cost even when the minimum scale for entry flattens out, but it creates a comparison measure for how the intensity of competition changes with the number of firms.

[10]Calculated from Table 5A of Bresnahan and Reiss (1991). Note that the entry of .79 in the second row of their original paper is a mistake and should be 1.09, and their Figure 4 illustrates s_i/s_5, not the s_5/s_i in the legend.

[11]Table 3's rescaling uses the following procedure.

Define the monopoly level of profits in an industry to be 2500, and the competitive level to be 0. Assume that when s_i reaches its maximum level s_m over $[1, 5]$, the competitive level of profits is reached and any further changes are measurement error. Apply the conversion formula $s_i^* = \frac{25(s_m - s_i)}{(s_m - s_1)}$, and Table 3 results.

Number of Firms, N	1	2	3	4	5
Doctors	2500	430	0	0	0
Dentists	2500	440	0	0	0
Druggists	2500	1550	430	0	0
Plumbers	2500	830	830	0	0
Tire Dealers	2500	1130	270	100	0
Average	2500	960	230	20	0

Table 3: Bresnahan-Reiss Entry Thresholds: Rescaled and Rounded ($\frac{25(s_m - s_i)}{(s_m - s_1)}$)

What is significant is how profits flatten out, even though the choice of 0 as the flat level in Table 3 is arbitrary. The empirical result that full-fledged competition kicks in quickly, so going from one firm to two is much more important than going from two to three, matches the Bertrand model with uncertainty very well but is inconsistent with the Cournot model. And this is true even though we have maintained our assumption that the probability a firm is active is constant as N varies. If we took that probability to be a free variable, we could of course fit the data even better. As it is, we see a correspondence between the Bresnahan-Reiss findings and one version of our model: the simple version in which the probability a given firm is active is exogenous and hence neither rises nor falls with the size of the market.

We do not want to argue that the Bertrand model with uncertainty is the only model that may explain the data presented. Other models in the industrial organization literature may also explain the data. By no means all models do, however. For example, consider the model of Salop (1979) as set out in Tirole (1988, pp.282-4) with linear and/or quadratic cost for an exogenously determined value of N. With linear (quadratic) cost, the relation between price and N is given by $p = c + t/N$, respectively $p = c + t/N^2$, where t is a transportation cost parameter and c is marginal cost. As toptal demand is constant in the Salop model, industry profits are given by t/N, and t/N^2, respectively Confronting these equations with the Bresnahan and Reiss results reveals that profits in the Salop model do not decline exponentially, as our model and the empirical results suggests.

5. Concluding Remarks

The Bertrand model with uncertainty about the number of competitors is simple, but its properties are both interesting and useful, and, in particular, the extreme transition from monopoly to competition found in the standard Bertrand model disappears. Expected profits are positive, but decline with the number of firms in the industry, and decline in a way that empirical work suggest is more realistic than the way they decline in the Cournot model. We have tried to show that the model is useful both as a simple description of oligopoly and as a building block for other topics in industrial organization, and this usefulness has already been illustrated in Gwin (1997), and Janssen and Van Reeven (1998).

Appendix on Convexity

This appendix shows that Bertrand industry profits are convex in N.

The second derivative $\frac{d^2\Pi_b}{dN^2}$ is derived from the first derivative in (13), which can be rewritten as

$$\frac{d\Pi_b}{dN} = \alpha v \left\{ \frac{(1-\alpha)^{N-1}}{1-(1-\alpha)^N} + \frac{(1-\alpha)^{N-1} N log(1-\alpha)}{[1-(1-\alpha)^N]^2} \right\}. \tag{32}$$

The derivative of this is

$$\begin{aligned}
\frac{d^2\Pi_b}{dN^2} &= \alpha v \left\{ \frac{[1-(1-\alpha)^N](1-\alpha)^{N-1}log(1-\alpha)+(1-\alpha)^N(1-\alpha)^{N-1}log(1-\alpha)}{[1-(1-\alpha)^N]^2} + \right. \\
&\left. \frac{[(1-\alpha)^{N-1}log(1-\alpha)+(1-\alpha)^{N-1}Nlog^2(1-\alpha)][1-(1-\alpha)^N]^2+2[1-(1-\alpha)^N](1-\alpha)^N log(1-\alpha)[N(1-\alpha)^{N-1}log(1-\alpha)]}{[1-(1-\alpha)^N]^4} \right\}. \\
&= \alpha v \left\{ 2\frac{(1-\alpha)^{N-1}log(1-\alpha)}{[1-(1-\alpha)^N]^2} + \frac{(1-\alpha)^{N-1}Nlog^2(1-\alpha)[1-2(1-\alpha)^N]+(1-\alpha)^{2N}+2(1-\alpha)^N-2(1-\alpha)^{2N}}{[1-(1-\alpha)^N]^4} \right\} \\
&= \alpha v \left\{ 2\frac{(1-\alpha)^{N-1}log(1-\alpha)}{[1-(1-\alpha)^N]^2} + \frac{(1-\alpha)^{N-1}Nlog^2(1-\alpha)[1-(1-\alpha)^{2N}]}{[1-(1-\alpha)^N]^4} \right\} \\
&= \frac{(1-\alpha)^{N-1}log(1-\alpha)}{[1-(1-\alpha)^N]^2} \left\{ 2 + \frac{Nlog(1-\alpha)[1+(1-\alpha)^N]}{1-(1-\alpha)^N} \right\} \alpha v.
\end{aligned} \tag{33}$$

The first term of this expression is negative because $log(1-\alpha)$ is negative.

The second term has the same sign as

$$2 - 2(1-\alpha)^N + Nlog(1-\alpha)[1+(1-\alpha)^N]. \tag{34}$$

We will show that expression (34) is also negative for all N and all $\alpha \in (0,1)$. We will first show that it is negative for $N=1$. In this case we can define $f(\alpha) = 2\alpha + (2-\alpha)log(1-\alpha)$. It is easy to see that $f(0) = f'(0) = 0$ and that $f''(\alpha) = -\frac{\alpha}{(1-\alpha)^2}$, which is strictly negative for all $\alpha > 0$. Hence, for all $\alpha \in (0,1)$, $f(\alpha) < 0$.

Let us then consider for fixed α,

$$g(N) = 2 - 2(1-\alpha)^N + Nlog(1-\alpha)[1+(1-\alpha)^N]. \tag{35}$$

It can be shown that $g'(N)$ has the sign of

$$(1-\alpha)^N - 1 - (1-\alpha)^N N log(1-\alpha) \tag{36}$$

and that $g''(N)$ has the sign of

$$-N(1-\alpha)^N log^2(1-\alpha). \tag{37}$$

As $g(1)$,$g'(1)$, and $g''(N)$ are strictly negative, we can conclude that expression (34) is negative, so that

$$\frac{d^2\Pi_b}{dN^2} > 0. \tag{38}$$

Appendix on Comparison of Bertrand and Cournot profits

This appendix shows that the ratio (31) is decreasing in N and α. To see the first, take the derivative with respect to N, which is

$$\begin{aligned}&log(1-\alpha)(1-\alpha)^{N-1}\left[1+\tfrac{\alpha}{2}(N-1)\right]^2+\alpha(1-\alpha)^{N-1}\left[[1+\tfrac{\alpha}{2}(N-1)\right]\\&=\{log(1-\alpha)\left[[1+\tfrac{\alpha}{2}(N-1)\right]+\alpha\}(1-\alpha)^{N-1}\left[[1+\tfrac{\alpha}{2}(N-1)\right]\end{aligned} \tag{39}$$

The sign of is derivative (39) is determined by the sign of the first term. Since

$$\left[1+\frac{\alpha}{2}(N-1)\right]\geq 1>\frac{-\alpha}{log(1-\alpha)}, \tag{40}$$

the derivative is negative.

To see that ratio (31) is decreasing in α, take the derivative with respect to α, which is

$$\begin{aligned}&-(N-1)(1-\alpha)^{N-2}\left[1+\tfrac{\alpha}{2}(N-1)\right]^2+(N-1)(1-\alpha)^{N-1}\left[1+\tfrac{\alpha}{2}(N-1)\right]\\&=-(N-1)(1-\alpha)^{N-2}\left[1+\tfrac{\alpha}{2}(N-1)\right]\left[1+\tfrac{\alpha}{2}(N-1)-(1-\alpha)\right]\\&=-(N-1)(1-\alpha)^{N-2}\left[1+\tfrac{\alpha}{2}(N-1)\right]\left[\tfrac{\alpha}{2}(N+1)\right]\end{aligned} \tag{41}$$

which is negative.

References

Baye, Michael & John Morgan (1997a) "Information Transmission, Information Acquisition, and Price Dispersion in 'Thin' Markets," Working paper, Indiana University, Kelley School, Dept. of Business Economics and Public Policy.

Baye, Michael & John Morgan (1997b) "Necessary and Sufficient Conditions for Existence and Uniqueness of Bertrand Paradox Outcomes, " Working paper, Indiana University, Kelley School, Dept. of Business Economics and Public Policy. Http://php.indiana.edu/∼mbaye/bert.pdf.

Baye, Michael & John Morgan (1999) "Epsilon Equilibria in Bertrand Games: Theory and Experimental Evidence," Working paper, Indiana University, Kelley School, Dept. of Business Economics and Public Policy, January 1999.

Bertrand, Joseph (1883) "Rechercher sur la Theorie Mathematique de la Richesse," *Journal des Savants*, September 1883. 48: 499-508.

Bresnahan, Timothy & Peter Reiss (1991) "Entry and Competition in Concentrated Markets," *Journal of Political Economy,* October 1991. 99: 977-1009.

Burdett, Kenneth & Kenneth Judd (1983) "Equilibrium Price Dispersion," *Econometrica*, July 1983, 51: 981-996.

Carlton, Dennis & Jeff Perloff (1994) *Modern Industrial Organization*, second edition, New York: HarperCollins College Publishers, 1994.

Cournot, Augustin (1838) *Recherches sur les Principes Mathematiques de la Theorie des Richesses*, Paris: M. Riviere & C., 1938. Translated in *Researches into the Mathematical Principles of Wealth.* New York: A.M. Kelly, 1960.

Edgeworth, Francis (1897) "La teoria pura del monopolio," *Giornale degli economisti,* 1925. 40: 13-31. Translated in Edgeworth, Francis, *Papers Relating to Political Economy*, Vol. I. London: Macmillan, 1925, 111-42.

Elberfeld, Walter & Elmar Wofstetter (1999) "A Dynamic Model of

Bertrand Competition with Entry," *International Journal of Industrial Organization*, 17: 513-525 (1999).

Gwin, Carl (1997) "Special Pricing Agreements as Vertical Restraints," in *Three Common Business Practices as Reponses to Asymmetric Information Problems*, Ph.D. Dissertation, Indiana University School of Business, July 1997.

Janssen, Maarten C.W. and Peran Van Reeven (1998) "Market Prices and Illegal Practices," *International Review of Law and Economics* , 18: 51-60.

Kreps, David & Jose Scheinkman (1983) "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes," *Bell Journal of Economics*, 14: 326-37 (Autumn 1983).

McAfee, R. Preston & John Macmillan (1987) "Auctions with a Stochastic Number of Bidders," *Journal of Economic Theory*, 43: 1-19.

Rasmusen, Eric (1994) *Games and Information*. 2nd edition. Oxford: Basil Blackwell, 1994.

Salant, Stephen, Sheldon Switzer & Robert Reynolds (1983) "Losses from Horizontal Merger: The Effects of an Exogenous Change in Industry Structure on Cournot-Nash Equilibrium," *Quarterly Journal of Economics,* May 1983. 98: 185-99.

Salop, Steven & Joseph Stiglitz (1977) "Bargains and Ripoffs; A Model of Monopolistically Competitive Price Dispersion," *Review of Economic Studies*, 44: 493-510 (October 1977).

Spulber, Daniel (1995) "Bertrand Competition When Rivals' Costs are Unknown," *Journal of Industrial Economics*, March 1995. 43: 1-11.

Varian, Hal (1980) "A Model of Sales," *American Economic Review*, September 1980. 70: 651-659. "Erratum," June 1981.

Warner, Elizabeth & Robert Barsky (1995) "The Timing and Magnitude of Retail Store Markdowns: Evidence from Weekends and Holidays," *Quarterly Journal of Economics*, May 1995. 110: 321-52.