## Cedars-Style Exponents, October 12, 2022

In class, someone got confused about  $2^3$  and thought it meant 2 + 2 + 2 = 6, whereas it really means  $2 \cdot 2 \cdot 2 = 8$ . That got us going as to how we might use exponential-like notation for *all* the arithmetic operations, not just multiplication. We might say that

$$2^3 \equiv 2 \cdot 2 \cdot 2 \qquad \qquad = 8 \tag{1}$$

$$^{3}2 \equiv 2 + 2 + 2 = 6$$
 (2)

$$_{3}2 \equiv 2 - 2 - 2$$
 = -2 (3)

$$2_3 \equiv 2 \div 2 \div 2 \qquad \qquad = 1/2 \tag{4}$$

Note that " $\equiv$ " means "equals by definition", where "=" means "equals" without saying whether it's by definition or by use of previous definitions.

One thing we need to do with our new definitions is figure out what words to use for them. For  $5^6$ , we would say "5 to the power 6". For 65, we could say "5 to the plus power 6". That has nice alliteration, and it's short and punchy. For  $_65$ , we could say "5 to the minus power 6". For  $5_6$  we could say "5 to the division power 6".

Another thing we need to do is define what happens if there is more than one exponent on a number. Someone in a class asked what would happen if we wrote:

$$^{2}_{2}2^{2}_{2}$$
 (5)

That was near the end of class, so I added to the homework a new question: figure out a sensible way to say what  ${}^{2}_{2}2^{2}_{2}$  means.

Before, we had to make one decision, where to put the 4 different power exponents. It was natural to put the multiplication one at the top right, since that's the ordinary definition, but we had choices of where to put the others.

Now we have to make a second definitional decision: the order of operations. We'd probably want to start with the familiar top right exponent, multiplying 2 by 2, but what happens next? Do we go clockwise, counterclockwise, or criss-cross?

And there is a third definitional decision: do we keep the starting number as the one multiplied, divided, etc., or do we use the modified one as we go along? That is, we start with  $2 \cdot 2 = 4$ , but if we go clockwise, do we go on to  $4 \div 2 = 2$  or to  $4 \div 4 = 1$ ?

In the next class, I showed them my method, which was to go counterclockwise and keep using 2. We talked about definitions. Why is it good for everyone to have the same definition? To communicate and avoid chaos. Why are some definitions better than others? Because we want a definition to be clear and useful. The students persuaded me that my definitions weren't the best ones. It's better to go clockwise, because that keeps the standard grade school order of operations-multiplication and division come before addition and subtraction. It's better to modify the base number as we go along, because then we continue to have 2 copies of the processed number, e.g.  $4 \div 4 = 1$  as the second step instead of  $4 \div 2 = 2$ , and 1 - 1 = 0 as the third step instead of 1 - 2 = -1.

Thus, we define:

$${}^{2}_{2}2{}^{2}_{2} \equiv {}_{2}((2^{2})_{2}) + {}_{2}((2^{2})_{2}) \tag{6}$$

We can build up the number progressively, which will teach you the definition in equation (6) better and quicker than reading the equation:

$$2^2 = 2 \cdot 2 = 4 \tag{7}$$

$$2_2^2 = 4 \div 4 = 1$$
 (8)

$$_{2}2_{2}^{2} = 1 - 1 \qquad \qquad = 0 \tag{9}$$

$${}^{2}_{2}2{}^{2}_{2} = 0 + 0 \qquad = 0 \tag{10}$$

"What if we didn't choose 2 as the base number and the exponent?" asked the class. Perhaps we'll continue to delve into this. My friend and co-author Professor Connell says, "I am guessing you tried all the different orders of operations to see what you get. This one probably has nice arithmetic functorial properties....One could try to do something like this in any division algebra, or the three \*,+,- operations in arbitrary rings... there is probably at least one interesting research paper hiding in all that!" He doesn't like the fact that if you have two's all around as in our example you always get 0 out, but I kind of like that result. He notes that if you put b for the base and k for the exponent, you get  $(1 - k)b^2$  in the end.