Theorem. The square root of 2 is irrational.

Proof: Suppose not. If the square root of 2 is rational,

$$\sqrt{2} = \frac{a}{b}$$

for some integers a and b.

First, note that this implies that

$$\sqrt{2} = \frac{x}{y}$$

for some integers x and y that are not both even. That's because if $\sqrt{2} = \frac{a}{b}$ and both a and b are even, then it is also true that a and b are divisible by 2, so

$$\sqrt{2} = \frac{a/2}{b/2}$$

with both $a/2$ and $b/2$ being integers. If $a/2$ and $b/2$ are both even, we can repeat the division by 2, and we can keep doing this till either the numerator or denominator is odd (or both are). So suppose $\sqrt{2} = \frac{x}{y}$ where x and y are not both even. We will show that this implies that both x and y are even, a contradiction.

First, square both sides so

$$2 = x^2/y^2.$$

Then

$$2y^2 = x^2.$$

This implies that x^2 is even. But then x must be even too, since if x is odd, $x(x-1)$ is even because it multiplies an odd number (x) times an even one $(x-1)$; and from thiswe know that $x(x-1)+x$ is odd because it adds an even number $(x(x-1))$ to an odd number (x).

If x is even, we can write it as

$$x = 2m$$

for some integer m. But then

$$2y^2 = (2m)^2 = 4m^2,$$

so

$$y^2 = 2m^2.$$

But then y^2 is even, so by our earlier argument, y is even too. But that means both x and y are even, which contradicts our starting assumption! So it must be that $sqrt2 \neq \frac{a}{b}$; there are no integers a and b such that $\sqrt{2} = \frac{a}{b}$, so $\sqrt{2}$ is irrational. *Quod erat demonstrandum.*

Is there a proof that $\sqrt{2}$ is irrational that does not use contradiction? I don't know.